Linear growth of the derivative for measure–preserving diffeomorphisms

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Abstract

In this paper we consider measure–preserving diffeomorphisms of the torus with zero entropy. We prove that every ergodic C^1–diffeomorphism with linear growth of the derivative is algebraically conjugated to a skew product of an irrational rotation on the circle and a circle C^1–cocycle. We also show that for no positive real $\beta \neq 1$ does an ergodic C^2-diffeomorphism of polynomial growth of the derivative with degree β exist.

1 Introduction

Let M be a compact Riemannian C^1–manifold, $\mathcal{B}$ its Borel σ–algebra and μ its probability Lebesgue measure. Assume that $f : (M, \mathcal{B}, \mu) \to (M, \mathcal{B}, \mu)$ is a measure–preserving C^1–diffeomorphism of the manifold M.

Definition 1. We say that the derivative of f has *linear growth* if the sequence

$$\frac{1}{n} Df^n : M \to \bigcup_{x \in M} \mathcal{L}(T_x M)$$

converges μ–a.e. to a measurable μ–nonzero function $g : M \to \bigcup_{x \in M} \mathcal{L}(T_x M)$, i.e. there exists a set $A \in \mathcal{B}$ such that $\mu(A) > 0$ and $g(x) \neq 0$ for all $x \in A$.

Our purpose is to study ergodic diffeomorphisms of torus with linear growth of the derivative.

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By $\mathbb{T}^2$ ($\mathbb{T}$) we will mean the torus $\mathbb{R}^2/\mathbb{Z}^2$ (the circle $\mathbb{R}/\mathbb{Z}$) which most often will be treated as the square $[0,1)\times[0,1)$ (the interval $[0,1)$) with addition mod 1; λ will denoted Lebesgue measure on $\mathbb{T}^2$. One of the examples of the ergodic diffeomorphisms with linear growth of the derivative is a skew product of an irrational rotation on the circle and a circle C^1-cocycle with nonzero topological degree. Let $\alpha\in\mathbb{T}$ be an irrational number and let $\varphi:\mathbb{T}\to\mathbb{T}$ be a C^1–cocycle. We denote by $d(\varphi)$ the topological degree of φ. Consider the skew product $T_\varphi:(\mathbb{T}^2,\lambda)\to(\mathbb{T}^2,\lambda)$ defined by

$$T_\varphi(x_1,x_2)=(x_1+\alpha,x_2+\varphi(x_1)).$$

Lemma 1. *The sequence $\frac{1}{n}DT_\varphi^n$ converges uniformly to the matrix $\begin{bmatrix} 0 & 0 \\ d(\varphi) & 0 \end{bmatrix}$.*

Proof. Observe that

$$\frac{1}{n}DT_\varphi^n(x_1,x_2)=\begin{bmatrix} \frac{1}{n} & 0 \\ \frac{1}{n}\sum_{k=0}^{n-1}D\varphi(x_1+k\alpha) & \frac{1}{n} \end{bmatrix}.$$

By Ergodic Theorem, the sequence $\frac{1}{n}\sum_{k=0}^{n-1}D\varphi(\cdot\,+k\alpha)$ converges uniformly to $\int_{\mathbb{T}}D\varphi(x)dx=d(\varphi)$. ■

It follows that if $d(\varphi)\neq 0$, then T_φ is the ergodic (see [3]) diffeomorphism with linear growth of the derivative.

We will say that diffeomorphisms f_1 and f_2 of $\mathbb{T}^2$ are algebraically conjugated if there exists a group automorphism $\psi:\mathbb{T}^2\to\mathbb{T}^2$ such that $f_1\circ\psi=\psi\circ f_2$. It is clear that if f_1 has linear growth of the derivative, f_1 and f_2 are algebraically conjugated, then f_2 has linear growth of the derivative. Therefore every C^1–diffeomorphism of $\mathbb{T}^2$ algebraically conjugated to a skew product T_φ with $d(\varphi)\neq 0$ has linear growth of the derivative.

The aim of this paper is to prove that every ergodic measure–preserving C^1–diffeomorphism of the torus with linear growth of the derivative is algebraically conjugated to a skew product of an irrational rotation on the circle and a circle C^1–cocycle with nonzero degree. In [3], A. Iwanik, M. Lema'nczyk, D. Rudolph have proved that if φ is C^2–cocycle with $d(\varphi)\neq 0$, then the skew product T_φ has countable Lebesgue spectrum on the orthocomplement of the space of functions depending only on first variable. Therefore every ergodic measure–preserving C^2–diffeomorphism of the torus with linear growth of the derivative has countable Lebesgue spectrum on the orthocomplement of its eigenfunctions.

It would be interesting to change the definition of linear growth of the derivative. For example, one could study a weaker property that there exist positive real constants a, b such that

$$0 < a \leq \|Df^n\|/n \leq b$$

for every natural n. Of course, if a diffeomorphism is C^1–conjugated to a skew product of an irrational rotation on the circle and a circle C^1–cocycle with nonzero degree, then it satisfies this weaker property and is ergodic. The inverted fact might be true, too.

2 Linear growth

Assume that $f : (\mathbb{T}^2, \lambda) \to (\mathbb{T}^2, \lambda)$ is a measure–preserving C^1–diffeomorphism. Then there exists the matrix $\{a_{ij}\}_{i,j=1,2} \in M_2(\mathbb{Z})$ and $\mathbb{Z}^2$–periodic (i.e. periodic of period 1 in each coordinates) C^1–functions $\widetilde{f}_1, \widetilde{f}_2 : \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(x_1, x_2) = (a_{11}x_1 + a_{12}x_2 + \widetilde{f}_1(x_1, x_2), a_{21}x_1 + a_{22}x_2 + \widetilde{f}_2(x_1, x_2)).$$

Denote by $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ the functions given by

$$f_1(x_1, x_2) = a_{11}x_1 + a_{12}x_2 + \widetilde{f}_1(x_1, x_2), \quad f_2(x_1, x_2) = a_{21}x_1 + a_{22}x_2 + \widetilde{f}_2(x_1, x_2)).$$

Then

$$\left| \det \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\bar{x}) & \frac{\partial f_1}{\partial x_2}(\bar{x}) \\ \frac{\partial f_2}{\partial x_1}(\bar{x}) & \frac{\partial f_2}{\partial x_2}(\bar{x}) \end{bmatrix} \right| = 1$$

for all $\bar{x} \in \mathbb{R}^2$.

Suppose that the diffeomorphism f is ergodic. We will need the following lemmas.

Lemma 2. *If the sequence $\frac{1}{n}Df^n : \mathbb{T}^2 \to M_2(\mathbb{R})$ converges λ–a.e. to a measurable function $g : \mathbb{T}^2 \to M_2(\mathbb{R})$, then*

$$g(\bar{x}) = g(f^n\bar{x})\, Df^n(\bar{x})$$

for λ–a.e. $\bar{x} \in \mathbb{T}^2$ and all natural n.

Proof. Let $A \subset \mathbb{T}^2$ be a full measure f–invariant set such that if $\bar{x} \in A$, then $\lim_{n\to\infty} \frac{1}{n}Df^n(\bar{x}) = g(\bar{x})$. Assume that $\bar{x} \in A$. Then for any naturals m, n we have

$$\frac{m+n}{m}\frac{1}{m+n}Df^{m+n}(\bar{x}) = \frac{1}{m}Df^m(f^n\bar{x})\, Df^n(\bar{x})$$

and $f^n\bar{x} \in A$. Letting $m \to \infty$, we obtain

$$g(\bar{x}) = g(f^n\bar{x})\, Df^n(\bar{x})$$

for $\bar{x} \in A$ and $n \in \mathbb{N}$. ■

Lemma 3.

$$\lambda \otimes \lambda(\{(\bar{x}, \bar{y}) \in \mathbb{T}^2 \times \mathbb{T}^2;\ g(\bar{y})\, g(\bar{x}) = 0\}) = 1$$

and

$$\lambda(\{\bar{x} \in \mathbb{T}^2;\ g(\bar{x})^2 = 0\}) = 1.$$

Proof. Choose a sequence $\{A_k\}_{k\in\mathbb{N}}$ of measurable subsets of A (see proof of Lemma 2) such that the function $g : A_k \to M_2(\mathbb{R})$ is continuous and $\lambda(A_k) > 1 - 1/k$ for any natural k. Since the transformation $f_{A_k} : (A_k, \lambda_{A_k}) \to (A_k, \lambda_{A_k})$ induced by f on A_k is ergodic, for every natural k we can find a measurable subset $B_k \subset A_k$ such that for any $\bar{x} \in B_k$ the sequence $\{f_{A_k}^n \bar{x}\}_{n\in\mathbb{N}}$ is dense in A_k in the induced topology and $\lambda(B_k) = \lambda(A_k)$.
Let $\bar{x}, \bar{y} \in B_k$. Then there exists an increasing sequence $\{m_i\}_{i\in\mathbb{N}}$ of naturals such that $f_{A_k}^{m_i} \bar{x} \to \bar{y}$. Hence there exists an increasing sequence $\{n_i\}_{i\in\mathbb{N}}$ of naturals such that $f^{n_i}\bar{x} \to \bar{y}$ and $f^{n_i}\bar{x} \in A_k$ for all $i \in \mathbb{N}$. Since $g : A_k \to M_2(\mathbb{R})$ is continuous, we get $g(f_{n_i}\bar{x}) \to g(\bar{y})$. Since

$$\frac{1}{n_i} g(\bar{x}) = g(f^{n_i}\bar{x})\, \frac{1}{n_i} Df^{n_i}(\bar{x})$$

letting $i \to \infty$, we obtain $g(\bar{y})\, g(\bar{x}) = 0$. Therefore

$$B_k \times B_k \subset \{(\bar{x}, \bar{y}) \in \mathbb{T}^2 \times \mathbb{T}^2;\ g(\bar{y})\, g(\bar{x}) = 0\} \text{ and } B_k \subset \{\bar{x} \in \mathbb{T}^2;\ g(\bar{x})^2 = 0\}$$

for any natural k. It follows that

$$\lambda \otimes \lambda(\{(\bar{x}, \bar{y}) \in \mathbb{T}^2 \times \mathbb{T}^2;\ g(\bar{y})\, g(\bar{x}) = 0\}) > (1 - \frac{1}{k})^2$$

and

$$\lambda(\{\bar{x} \in \mathbb{T}^2;\ g(\bar{x})^2 = 0\}) > 1 - \frac{1}{k}$$

for any natural k, which proves the lemma. ■

Lemma 4. *Let $A, B \in M_2(\mathbb{R})$ be nonzero matrixes. Suppose that*

$$A^2 = B^2 = AB = 0.$$

Then there exist real numbers $a, b \neq 0$ and c such that

$$A = \begin{bmatrix} ac & -ac^2 \\ a & -ac \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} bc & -bc^2 \\ b & -bc \end{bmatrix}$$

or

$$A = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}.$$

Proof. Since $A^2 = 0$ and $A \neq 0$, we immediately see that the Jordan form of the matrix A is $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. It follows that there exist matrixes $C = \{c_{ij}\}_{i,j=1,2}$, $C' = \{c'_{ij}\}_{i,j=1,2} \in M_2(\mathbb{C})$ such that $\det C = \det C' = 1$ and

$$\begin{aligned} A &= C \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} C^{-1} &= \begin{bmatrix} c_{12}c_{22} & -c_{12}^2 \\ c_{22}^2 & -c_{12}c_{22} \end{bmatrix}, \\ B &= C' \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (C')^{-1} &= \begin{bmatrix} c'_{12}c'_{22} & -{c'_{12}}^2 \\ {c'_{22}}^2 & -c'_{12}c'_{22} \end{bmatrix}. \end{aligned}$$

Since matrixes A and B commute, their eigenvectors belonging to 0, i.e. (c_{12}, c_{22}) and (c'_{12}, c'_{22}) generate the same subspace. Therefore there exist real numbers $a, b \neq 0$ and c such that

$$A = \begin{bmatrix} ac & -ac^2 \\ a & -ac \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} bc & -bc^2 \\ b & -bc \end{bmatrix}$$

or

$$A = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}. \quad \blacksquare$$

Lemma 5. *Suppose that $f : (\mathbb{T}^2, \lambda) \to (\mathbb{T}^2, \lambda)$ is an ergodic measure–preserving C^1–diffeomorphism such that the sequence $\frac{1}{n} Df^n$ converges λ–a.e. to a nonzero measurable function $g : \mathbb{T}^2 \to M_2(\mathbb{R})$. Then there exist a measurable function $h : \mathbb{T}^2 \to \mathbb{R}$ and $c \in \mathbb{R}$ such that*

$$g(\bar{x}) = h(\bar{x}) \begin{bmatrix} c & -c^2 \\ 1 & -c \end{bmatrix} \qquad \text{for } \lambda\text{–a.e. } \bar{x} \in \mathbb{T}^2$$

or

$$g(\bar{x}) = h(\bar{x}) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{for } \lambda\text{–a.e. } \bar{x} \in \mathbb{T}^2.$$

Moreover, $h(\bar{x}) \neq 0$ for λ–a.e. $\bar{x} \in \mathbb{T}^2$.

Proof. Denote by $F \subset \mathbb{T}^2$ the set of all points $\bar{x} \in \mathbb{T}^2$ with $g(\bar{x}) \neq 0$. By Lemma 2, the set F is f–invariant. As f is ergodic and $\lambda(F) > 0$ we have $\lambda(F) = 1$. By Lemma 3, we can find $\bar{y} \in \mathbb{T}^2$ such that $g(\bar{y}) \neq 0$, $g(\bar{y})^2 = 0$ and $g(\bar{x}) \neq 0$, $g(\bar{x})^2 = g(\bar{x})g(\bar{y}) = 0$ for λ–a.e. $\bar{x} \in \mathbb{T}^2$. An application of Lemma 4, completes the proof. $\blacksquare$

Assume that $f : (\mathbb{T}^2, \lambda) \to (\mathbb{T}^2, \lambda)$ is an ergodic measure–preserving C^1–diffeomorphism with linear growth of the derivative. Then the sequence

$\frac{1}{n}Df^n$ converges λ–a.e. to a function $g : \mathbb{T}^2 \to M_2(\mathbb{R})$. In the remainder of this section assume that g can be represented as follows

$$g = h \begin{bmatrix} c & -c^2 \\ 1 & -c \end{bmatrix},$$

where $h : \mathbb{T}^2 \to \mathbb{R}$ and $c \in \mathbb{R}$. We can do it because the second case

$$g = h \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

reduces to case $c = 0$ after interchanging the coordinates, which is an algebraic isomorphism.
Now by Lemma 2,

$$h(\bar{x})\, h(f\bar{x})^{-1} \begin{bmatrix} c & -c^2 \\ 1 & -c \end{bmatrix} = \begin{bmatrix} c & -c^2 \\ 1 & -c \end{bmatrix} Df(\bar{x}) \tag{1}$$

for λ–a.e. $\bar{x} \in \mathbb{T}^2$. It follows that

$$\begin{aligned} h(\bar{x})\, h(f\bar{x})^{-1} c &= c\frac{\partial f_1}{\partial x_1}(\bar{x}) - c^2 \frac{\partial f_2}{\partial x_1}(\bar{x}), \\ -h(\bar{x})\, h(f\bar{x})^{-1} c &= \frac{\partial f_1}{\partial x_2}(\bar{x}) - c\frac{\partial f_2}{\partial x_2}(\bar{x}) \end{aligned}$$

for λ–a.e. $\bar{x} \in \mathbb{T}^2$. Therefore

$$-c\frac{\partial}{\partial x_1}(f_1(\bar{x}) - cf_2(\bar{x})) = \frac{\partial}{\partial x_2}(f_1(\bar{x}) - cf_2(\bar{x}))$$

for λ–a.e. $\bar{x} \in \mathbb{T}^2$. Since the functions $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ are of class C^1 the equality holds for every $\bar{x} \in \mathbb{R}^2$. Then there exists a C^1–function $u : \mathbb{R} \to \mathbb{R}$ such that

$$f_1(x_1, x_2) - cf_2(x_1, x_2) = u(x_1 - cx_2). \tag{2}$$

Lemma 6. *If c is irrational, then $f(x_1, x_2) = (x_1 + d, x_2 + e)$, where $d, e \in \mathbb{R}$.*

Proof. Represent the diffeomorphism f as follows

$$\begin{aligned} f_1(x_1, x_2) &= a_{11}x_1 + a_{12}x_2 + \widetilde{f}_1(x_1, x_2), \\ f_2(x_1, x_2) &= a_{21}x_1 + a_{22}x_2 + \widetilde{f}_2(x_1, x_2), \end{aligned}$$

where $\{a_{ij}\}_{i,j=1,2} \in M_2(\mathbb{Z})$ and $\widetilde{f}_1, \widetilde{f}_2 : \mathbb{R}^2 \to \mathbb{R}$ are $\mathbb{Z}^2$–periodic. From (2),

$$u(x_1 - cx_2) = (a_{11} - ca_{21})x_1 + (a_{12} - ca_{22})x_2 + \widetilde{f}_1(x_1, x_2) - c\widetilde{f}_2(x_1, x_2). \tag{3}$$

Since the function $\widetilde{f}_1 - c\widetilde{f}_2 : \mathbb{R}^2 \to \mathbb{R}$ is $\mathbb{Z}^2$–periodic, there exists $(\widetilde{x}_1, \widetilde{x}_2) \in \mathbb{R}^2$ such that

$$\frac{\partial \widetilde{f}_1}{\partial x_1}(\widetilde{x}_1, \widetilde{x}_2) - c\frac{\partial \widetilde{f}_2}{\partial x_1}(\widetilde{x}_1, \widetilde{x}_2) = \frac{\partial \widetilde{f}_1}{\partial x_2}(\widetilde{x}_1, \widetilde{x}_2) - c\frac{\partial \widetilde{f}_2}{\partial x_2}(\widetilde{x}_1, \widetilde{x}_2) = 0.$$

From (3) it follows that

$$\begin{aligned} Du(\widetilde{x}_1 - c\widetilde{x}_2) &= a_{11} - ca_{21}, \\ -cDu(\widetilde{x}_1 - c\widetilde{x}_2) &= a_{12} - ca_{22}. \end{aligned}$$

Hence

$$a_{12} - ca_{22} = -c(a_{11} - ca_{21}). \tag{4}$$

Then

$$u(x_1 - cx_2) = (a_{11} - ca_{21})(x_1 - cx_2) + \widetilde{f}_1(x_1, x_2) - c\widetilde{f}_2(x_1, x_2).$$

Let $v : \mathbb{R} \to \mathbb{R}$ be the function given by $v(x) = u(x) - (a_{11} - ca_{21})x$. As $\widetilde{f}_1 - c\widetilde{f}_2$ is $\mathbb{Z}^2$–periodic we have

$$v(x+1) = \widetilde{f}_1(x+1, 0) - c\widetilde{f}_2(x+1, 0) = \widetilde{f}_1(x, 0) - c\widetilde{f}_2(x, 0) = v(x)$$

and

$$v(x+c) = \widetilde{f}_1(x, -1) - c\widetilde{f}_2(x, -1) = \widetilde{f}_1(x, 0) - c\widetilde{f}_2(x, 0) = v(x).$$

Since v is continuous and c is irrational we conclude that the function v is constant and equal to a real number v. Therefore $\widetilde{f}_1 - c\widetilde{f}_2 = v$ and

$$f(x_1, x_2) = (a_{11}x_1 + a_{12}x_2 + c\widetilde{f}_2(x_1, x_2) + v, a_{21}x_1 + a_{22}x_2 + \widetilde{f}_2(x_1, x_2)).$$

As the diffeomorphism f preserves measure λ we have $\det Df = \varepsilon$, where $\varepsilon \in \{-1, 1\}$. Then

$$\begin{aligned} \varepsilon &= (a_{11} + c\frac{\partial \widetilde{f}_2}{\partial x_1})(a_{22} + \frac{\partial \widetilde{f}_2}{\partial x_2}) - (a_{12} + c\frac{\partial \widetilde{f}_2}{\partial x_2})(a_{21} + \frac{\partial \widetilde{f}_2}{\partial x_1}) \\ &= a_{11}a_{22} - a_{12}a_{21} + (ca_{22} - a_{12})\frac{\partial \widetilde{f}_2}{\partial x_1} + (a_{11} - ca_{21})\frac{\partial \widetilde{f}_2}{\partial x_2} \\ &= a_{11}a_{22} - a_{12}a_{21} + (ca_{22} - a_{12})(\frac{\partial \widetilde{f}_2}{\partial x_1} + c\frac{\partial \widetilde{f}_2}{\partial x_2}), \end{aligned}$$

by (4). Since for a certain $\bar{x} \in \mathbb{R}$, $\frac{\partial \widetilde{f}_2}{\partial x_1}(\bar{x}) = \frac{\partial \widetilde{f}_2}{\partial x_2}(\bar{x}) = 0$ we see that $a_{11}a_{22} - a_{12}a_{21} = \varepsilon$ and $\frac{\partial \widetilde{f}_2}{\partial x_1} + c\frac{\partial \widetilde{f}_2}{\partial x_2} = 0$. Therefore there exists a C^1–function $s : \mathbb{R} \to \mathbb{R}$ such that

$$s(x_1 - cx_2) = \widetilde{f}_2(x_1, x_2).$$

Since $\widetilde{f}_2$ is $\mathbb{Z}^2$–periodic and c is irrational, the function s is constant and equal to a real number s. It follows that

$$f(x_1, x_2) = (a_{11}x_1 + a_{12}x_2 + d, a_{21}x_1 + a_{22}x_2 + e),$$

where $d = cs + v$ and $e = s$. Then

$$Df^n = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^n$$

for any natural n. It follows that the function g is constant and finally that h is constant. From (1), we get

$$\begin{bmatrix} c & -c^2 \\ 1 & -c \end{bmatrix} = \begin{bmatrix} c & -c^2 \\ 1 & -c \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Hence $1 = a_{11} - ca_{21}$ and $-c = a_{12} - ca_{22}$. Since c is irrational, we conclude that $a_{11} = 1$, $a_{12} = 0$, $a_{21} = 0$, $a_{22} = 1$. ■

Lemma 7. *If c is rational, then there exist a group automorphism $\psi : \mathbb{T}^2 \to \mathbb{T}^2$, a real number α, a C^1–function $\varphi : \mathbb{T} \to \mathbb{T}$ and $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$ such that*

$$\psi \circ f \circ \psi^{-1}(x_1, x_2) = (\varepsilon_1 x_1 + \alpha, \varepsilon_2 x_2 + \varphi(x_1)).$$

Proof. Denote by p and q the integers such that $q > 0$, $\gcd(p, q) = 1$ and $c = p/q$. Choose $a, b \in \mathbb{Z}$ with $ap + bq = 1$. Consider the group automorphism $\psi : \mathbb{T}^2 \to \mathbb{T}^2$ defined by $\psi(x_1, x_2) = (qx_1 - px_2, ax_1 + bx_2)$. Let $\widehat{f} = \psi \circ f \circ \psi^{-1}$ and let $\pi_i : \mathbb{T}^2 \to \mathbb{T}$ be the projection on i–th coordinate for $i = 1, 2$. From (2),

$$\begin{aligned} \widehat{f}_1(x_1, x_2) &= q\, f_1 \circ \psi^{-1}(x_1, x_2) - p\, f_2 \circ \psi^{-1}(x_1, x_2) \\ &= q\, u(\pi_1 \circ \psi^{-1}(x_1, x_2) - \frac{p}{q}\pi_2 \circ \psi^{-1}(x_1, x_2)) \\ &= q\, u(\frac{1}{q}x_1). \end{aligned}$$

Therefore, $\widehat{f}_1$ depends only on first variable. Then

$$D\widehat{f} = \begin{bmatrix} \frac{\partial \widehat{f}_1}{\partial x_1} & 0 \\ \frac{\partial \widehat{f}_2}{\partial x_1} & \frac{\partial \widehat{f}_2}{\partial x_2} \end{bmatrix}$$

and

$$\frac{\partial \widehat{f}_1}{\partial x_1}\frac{\partial \widehat{f}_2}{\partial x_2} = \det D\widehat{f} = \varepsilon \in \{-1,1\}.$$

Since $\frac{\partial \widehat{f}_2}{\partial x_2}(x_1,x_2) = \varepsilon / \frac{\partial \widehat{f}_1}{\partial x_1}(x_1,0)$, there exists a C^1–function $\varphi : \mathbb{T} \to \mathbb{T}$ such that

$$\widehat{f}_2(x_1,x_2) = \frac{\varepsilon}{\frac{\partial \widehat{f}_1}{\partial x_1}(x_1,0)} x_2 + \varphi(x_1).$$

Hence $\varepsilon / \frac{\partial \widehat{f}_1}{\partial x_1}(x_1,0)$ is an integer constant. As the map

$$\mathbb{T} \ni x \longmapsto \widehat{f}_1(x,0) \in \mathbb{T}$$

is continuous, it follows that $\frac{\partial \widehat{f}_1}{\partial x_1}(x_1,0) = \varepsilon_1 \in \{-1,1\}$. Therefore

$$\widehat{f}(x_1,x_2) = (\varepsilon_1 x_1 + \alpha, \varepsilon_1 \varepsilon x_2 + \varphi(x_1)). \ \blacksquare$$

Theorem 8. *Every ergodic measure–preserving C^1–diffeomorphism of $\mathbb{T}^2$ with linear growth of the derivative is algebraically conjugated to a skew product of an irrational rotation on $\mathbb{T}$ and a circle C^1–cocycle with nonzero degree.*

Proof. Let $f : (\mathbb{T}^2, \lambda) \to (\mathbb{T}^2, \lambda)$ be an ergodic C^1–diffeomorphism with linear growth of the derivative. Then the sequence $\frac{1}{n}Df^n$ converges λ–a.e. to a nonzero measurable function $g : \mathbb{T}^2 \to M_2(\mathbb{R})$. By Lemma 5, there exist a measurable function $h : \mathbb{T}^2 \to \mathbb{R}$ and $c \in \mathbb{R}$ such that

$$g(\bar{x}) = h(\bar{x}) \begin{bmatrix} c & -c^2 \\ 1 & -c \end{bmatrix}$$

for λ–a.e. $\bar{x} \in \mathbb{T}^2$.

First note that c is rational. Suppose, contrary to our claim, that c is irrational. By Lemma 6, $Df^n = \mathbb{I}$ for all natural n. Therefore the sequence $\frac{1}{n}Df^n$ converges uniformly to zero, which is impossible.

By Lemma 7, there exist a group automorphism $\psi : \mathbb{T}^2 \to \mathbb{T}^2$, a real number α, a C^1–function $\varphi : \mathbb{T} \to \mathbb{T}$ and $\varepsilon_1, \varepsilon_2 \in \{-1,1\}$ such that

$$\psi \circ f \circ \psi^{-1}(x_1,x_2) = (\varepsilon_1 x_1 + \alpha, \varepsilon_2 x_2 + \varphi(x_1)).$$

As f is ergodic, the map

$$\mathbb{T} \ni x \longmapsto \varepsilon_1 x + \alpha \in \mathbb{T}$$

is ergodic. It follows immediately that $\varepsilon_1 = 1$ and α is irrational.

Next note that $\varepsilon_2 = 1$. Suppose, contrary to our claim, that $\varepsilon_2 = -1$. Then

$$\frac{1}{2n}D(\psi \circ f^{2n} \circ \psi^{-1})(x_1, x_2)$$
$$= \begin{bmatrix} \frac{1}{2n} & 0 \\ \frac{1}{2n}\sum_{k=0}^{n-1}(D\varphi(x_1+\alpha+2k\alpha) - D\varphi(x_1+2k\alpha)) & \frac{1}{2n} \end{bmatrix}.$$

By Ergodic Theorem,

$$\frac{1}{2n}\sum_{k=0}^{n-1}(D\varphi(x_1+\alpha+2k\alpha) - D\varphi(x_1+2k\alpha)) \to \frac{1}{2}\int_{\mathbb{T}}(D\varphi(x+\alpha) - D\varphi(x))dx = 0$$

uniformly. Therefore the sequence $\frac{1}{2n}Df^{2n}$ converges uniformly to zero, which is impossible. It follows that

$$\psi \circ f \circ \psi^{-1}(x_1, x_2) = (x_1 + \alpha, x_2 + \varphi(x_1)),$$

where α is irrational. By Lemma 1, the sequence $\frac{1}{n}D(\psi \circ f^n \circ \psi^{-1})$ converges uniformly to the matrix $\begin{bmatrix} 0 & 0 \\ d(\varphi) & 0 \end{bmatrix}$. It follows that the topological degree of φ is not equal to zero, which completes the proof. ■

For measure–preserving C^1–diffeomorphisms Lemma 1 and Theorem 8 give the following characterization of the property to be algebraically conjugated to a skew product of an irrational rotation and a C^1–cocycle with nonzero degree.

Corollary 1. *For a measure–preserving C^1–diffeomorphism $f : (\mathbb{T}^2, \lambda) \to (\mathbb{T}^2, \lambda)$ the following are equivalent:*

(i) *f is ergodic and has linear growth of the derivative;*

(ii) *f is algebraically conjugated to a skew product of an irrational rotation on the circle and a circle C^1–cocycle with nonzero degree.*

3 Polynomial growth

Assume that $f : (M, \mathcal{B}, \mu) \to (M, \mathcal{B}, \mu)$ is a measure–preserving C^2–diffeomorphism of a compact Riemannian C^2–manifold M. Let β be a positive real number. We say that the derivative of f has *polynomial growth with degree β* if the sequence $\frac{1}{n^\beta}Df^n$ converges μ–a.e. to a measurable μ–nonzero function.

It is clear that replacing n by n^β in lemmas of previous section we obtain the following property. Every ergodic measure–preserving C^2–diffeomorphism of polynomial growth of the derivative with degree β is algebraically conjugated to a diffeomorphism $\widehat{f}:(\mathbb{T}^2,\lambda)\to(\mathbb{T}^2,\lambda)$ of the form

$$\widehat{f}(x_1,x_2)=(x_1+\alpha,\varepsilon x_2+\varphi(x_1)),$$

where α is irrational, $\varphi:\mathbb{T}\to\mathbb{T}$ is a C^2–cocycle and $\varepsilon\in\{-1,1\}$. Note that $\varepsilon=1$. Suppose, contrary to our claim, that $\varepsilon=-1$. Then

$$\begin{aligned}&\frac{1}{(2n)^\beta}D\widehat{f}^{2n}(x_1,x_2)\\ &=\left[\begin{array}{cc}\frac{1}{(2n)^\beta} & 0\\ \frac{1}{(2n)^\beta}\sum_{k=0}^{n-1}(D\varphi(x_1+\alpha+2k\alpha)-D\varphi(x_1+2k\alpha)) & \frac{1}{(2n)^\beta}\end{array}\right].\end{aligned}$$

Recall (see [2] p.73) that if $\{q_n\}_{n\in\mathbb{N}}$ is the sequence of the denominators of an irrational number γ and $\xi:\mathbb{T}\to\mathbb{R}$ is a function of bounded variation then

$$|\sum_{k=0}^{q_n-1}\xi(x+k\gamma)-q_n\int_{\mathbb{T}}\xi(t)dt|\le \mathrm{Var}\xi$$

for any $x\in\mathbb{T}$ and $n\in\mathbb{N}$.
Denote by $\{q_n\}_{n\in\mathbb{N}}$ the sequence of the denominators of 2α. Since $\int_{\mathbb{T}}(D\varphi(t+\alpha)-D\varphi(t))dt=0$, we obtain

$$|\sum_{k=0}^{q_n-1}(D\varphi(x+\alpha+2k\alpha)-D\varphi(x+2k\alpha))|\le 2\mathrm{Var}D\varphi$$

for any $x\in\mathbb{T}$. Hence the sequence $\frac{1}{(2q_n)^\beta}D\widehat{f}^{2q_n}$ converges uniformly to zero, which is impossible. Therefore $\varepsilon=1$.

Since the derivative of $\widehat{f}$ has polynomial growth with degree β and

$$\frac{1}{n^\beta}D\widehat{f}^{n}(x_1,x_2)=\left[\begin{array}{cc}\frac{1}{n^\beta} & 0\\ \frac{1}{n^\beta}\sum_{k=0}^{n-1}D\varphi(x_1+k\alpha) & \frac{1}{n^\beta}\end{array}\right]$$

it follows that the sequence $\frac{1}{n^\beta}\sum_{k=0}^{n-1}D\varphi(\,\cdot\,+k\alpha)$ converges a.e. to a nonzero measurable function $h:\mathbb{T}\to\mathbb{R}$. Choose $x\in\mathbb{T}$ such that

$$\lim_{n\to\infty}\frac{1}{n^\beta}\sum_{k=0}^{n-1}D\varphi(x+k\alpha)=h(x)\neq 0.$$

Now denote by $\{q_n\}_{n\in\mathbb{N}}$ the sequence of the denominators of α. Since

$$|\sum_{k=0}^{q_n-1} D\varphi(x+k\alpha) - q_n\int_{\mathbb{T}} D\varphi(t)dt| \leq \mathrm{Var}D\varphi$$

we have

$$\lim_{n\to\infty}(\frac{1}{{q_n}^{\beta}}\sum_{k=0}^{q_n-1} D\varphi(x+k\alpha) - {q_n}^{1-\beta}\int_{\mathbb{T}} D\varphi(t)dt = 0.$$

Hence

$$\lim_{n\to\infty} {q_n}^{1-\beta}\int_{\mathbb{T}} D\varphi(t)dt) = h(x) \neq 0.$$

It follows that $\beta = 1$ and $d(\varphi) = \int_{\mathbb{T}} D\varphi(t)dt \neq 0$. From the above we conclude.

Theorem 9. *For no positive real $\beta \neq 1$ does an ergodic measure–preserving C^2–diffeomorphism of polynomial growth of the derivative with degree β exist.*

References

[1] I.P. Cornfeld, S.W. Fomin, J.G. Sinai, *Ergodic Theory*, Springer-Verlag, Berlin, 1982.

[2] M. Herman, *Sur la conjugaison difféomorphismes du cercle ka des rotation*, Publ. Mat. IHES **49** (1979), 5-234.

[3] A. Iwanik, M. Lema‘nczyk, D. Rudolph, *Absolutely continuous cocycles over irrational rotations*, Isr. J. Math. **83** (1993), 73-95.

[4] L. Kuipers, H. Niederreiter, *Uniform Distribution of Sequences*, John Wiley & Sons, New York, 1974.

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