Cyclic space isomorphism of unitary operators

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Abstract

We introduce a new equivalence relation between unitary operators on separable Hilbert spaces and discuss a possibility to have in each equivalence class a measure–preserving transformation.

Introduction

Let U be a unitary operator on a separable Hilbert space H. For any $x \in H$ we define the *cyclic space* generated by x as $Z(x) = span\{U^n x : n \in \mathbf{Z}\}$. By the *spectral measure* μ_x of x we mean a Borel measure on the circle determined by the equalities

$$\hat{\mu}_x(n) = \int_T z^n d\mu_x(z) = (U^n x, x)$$

for every $n \in \mathbf{Z}$.

Theorem 0.1 (spectral theorem). (see [9]) *There exists in H a sequence $x_1, x_2, \ldots$ such that*

$$H = \bigoplus_{n=1}^{\infty} Z(x_n) \quad \text{and} \quad \mu_{x_1} \gg \mu_{x_2} \gg \ldots \, . \tag{1}$$

Moreover, for any sequence $y_1, y_2, \ldots$ in H satisfying (1) we have $\mu_{x_1} \equiv \mu_{y_1}, \mu_{x_2} \equiv \mu_{y_2}, \ldots$.

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One of the most important problems (still open) in ergodic theory is a classification of ergodic dynamical systems with respect to the spectral equivalence, i.e. given a sequence

$$\mu_1 \gg \mu_2 \gg ... \tag{2}$$

of positive finite measures on the circle we ask if there exists an ergodic dynamical system $T : (X, \mathcal{B}, \varrho) \to (\mathcal{X}, \mathcal{B}, \varrho)$ such that a spectral sequence (1) for $U = U_T$ ($U_T : L^2(X, \varrho) \to L^2(X, \varrho)$, $U_T f = f \circ T$) coincides with (2).

The spectral type of μ_{x_1} (the equivalence class of measures) is called the *maximal spectral type* of U. By the *multiplicity function* M_U of U we mean the function $M_U : \mathbf{T} \to \mathbf{N} \cup \{+\infty\}$ given by:

$$M_U(z) = \sum_{n=1}^{\infty} \chi_{A_n}(z)$$

where $A_1 = \mathbf{T}$ and $A_n = A_n(U) = \{z \in \mathbf{T} : \frac{d\mu_{x_n}}{d\mu_{x_1}}(z) > 0\}$ (it is well-defined up to a $\mu_{x_1} - -nullset$). Then we have

$$\mathbf{T} = A_1 \supset A_2 \supset A_3 \supset ....$$

The set

$$E(U) = \{n \in \mathbf{N} \cup \{+\infty\} : \mu_{x_1}\{z \in \mathbf{T} : M_U(z) = n\} > 0\}$$

is called the set of *essential values* of the multiplicity function M_U.
For the background on spectral theory we refer to [3].

In the last few years, problems concerning spectral multiplicity have become of a renewed interest (see [1], [2], [4], [6], [7], [8], [10], [11]). In [5], M. Lemańczyk and J. Kwiatkowski (jr.) show that for an arbitrary set $A \subseteq \mathbf{N}^+$ containing 1, an ergodic automorphism T whose set of essential values of the multiplicity function is equal to A is constructed. The aim of this paper is a new viewpoint on spectral classification stated to me by Professor Lemańczyk.

Every measure μ can be uniquely decomposed into a sum $\mu = \mu^c + \mu^d$ where μ^c is continuous and μ^d is discret. For a spectral sequence $\mu_{x_1} \gg \mu_{x_2} \gg ...$ we have $\mu^c_{x_1} \gg \mu^c_{x_2} \gg ....$ By the *c-multiplicity function* M^c_U we mean the function $M^c_U : \mathbf{T} \to \mathbf{N} \cup \{+\infty\}$ given by

$$M^c_U(z) = \sum_{n=1}^{\infty} \chi_{C_n}(z)$$

where $C_1 = \mathbf{T}$ and $C_n = \{z \in \mathbf{T} : \frac{d\mu^c_{x_n}}{d\mu^c_{x_1}}(z) > 0\}$. The set

$$E^c(U) = \{n \in \mathbf{N} \cup \{+\infty\} : \mu_{x_1}\{z \in \mathbf{T} : M^c_U(z) = n\} > 0\}$$

is called the set of *essential values* of c-multiplicity function M^c_U.
Let $D(U) : \mathbf{N} \cup \{+\infty\} \to \mathbf{N} \cup \{+\infty\}$ be a function given by $D(U)(n) = card\, D_n$ where

$$D_n = \begin{cases} \{z \in A_n \setminus A_{n+1} : \mu_{x_1}(\{z\}) > 0\} & \text{for} \quad n = 1, 2, ... \\ \{z \in \bigcap_{n=1}^{\infty} A_n : \mu_{x_1}(\{z\}) > 0\} & \text{for} \quad n = +\infty. \end{cases}$$

In Section 1 we define a cyclic space (s.c.) isomorphism of unitary operators on separable Hilbert space and we try to find a complete set of invariants for a c.s. isomorphism. Using results from Section 1 and those from [5], we show that in the c.s. equivalence class of an operator $U : H \to H$ whose maximal spectral type is continuous and $1 \in E^c(U)$ we can find a weakly mixing automorphism.
The author would like to thank Professor Lemańczyk for some valuable discussions.

1 Cyclic space isomorphism and its invariants

Lemma 1.1. *Let $U_1 : H_1 \to H_1$ and $U_2 : H_2 \to H_2$ be unitary operators. Then, for every unitary operator $V : H_1 \to H_2$ the following conditions are equivalent.*

(i) *For every $x \in H_1$, $Z(Vx) = VZ(x)$.*

(ii) *If H is a U_1-invariant closed subspace of H_1, then VH is U_2-invariant and if H is a U_2-invariant closed subspace of H_2, then $V^{-1}H$ is U_1-invariant.*

Proof. $(i) \Rightarrow (ii)$. Suppose that H is a U_1-invariant closed subspace of H_1 and $y \in VH$. There exists $x \in H$ such that $y = Vx$. Since $Z(y) = VZ(x)$,

$$U_2^{-1}y, U_2 y \in Z(y) = VZ(x) \subset VH$$

and finally that VH is U_2-invariant. Similarly, we can get the remaining part of (ii).

$(ii) \Rightarrow (i)$. Let $x \in H_1$. Since $Z(x)$ is U_1-invariant, $VZ(x)$ is U_2-invariant. Since $Vx \in VZ(x)$, $Z(Vx) \subset VZ(x)$. Similarly, if $y = Vx$ then $Z(x) = Z(V^{-1}y) \subset V^{-1}Z(y) = V^{-1}Z(Vx)$. This gives $VZ(x) \subset Z(Vx)$ and finally $Z(Vx) = VZ(x)$.■

Definition 1.1. We call a unitary operator $V : H_1 \to H_2$ is a *cyclic space isomorphism* of U_1 and U_2 if it satisfies (i) from Lemma 1.1 or equivalently (ii).

Lemma 1.2. *Let μ and ν be positive finite Borel measures on the circle. Assume $U_1 : L^2(\mathbf{T}, \mu) \to L^2(\mathbf{T}, \mu)$, $U_2 : L^2(\mathbf{T}, \nu) \to L^2(\mathbf{T}, \nu)$ are unitary operators given by*

$$U_1 f(z) = U_2 f(z) = z f(z).$$

If $V : L^2(\mathbf{T}, \mu) \to L^2(\mathbf{T}, \nu)$ is a c.s. isomorphism of U_1 and U_2 then there exists a nonsingular invertible map $S : (\mathbf{T}, \mathcal{B}, \nu) \to (\mathbf{T}, \mathcal{B}, \mu)$ and $h \in L^2(\mathbf{T}, \nu)$ such that

$$Vf = h \cdot f \circ S$$

for every $f \in L^2(\mathbf{T}, \mu)$.

Proof. For a set $A \in \mathcal{B}$ put $H = \chi_A L^2(\mathbf{T}, \mu)$. Then H is a U_1-invariant subspace of $L^2(\mathbf{T}, \mu)$. By Wiener Lemma (e.g. [9] Appendix) there exists a Borel set $\Phi(A)$ such that $VH = \chi_{\Phi(A)} L^2(\mathbf{T}, \nu)$. From $V(\{0\}) = \{0\}$ and $V^{-1}(\{0\}) = \{0\}$ we obtain that $\mu(A) = 0$ iff $\nu(\Phi(A)) = 0$. If $A \cap B = \emptyset$ then $\chi_A L^2(\mathbf{T}, \mu) \perp \chi_B L^2(\mathbf{T}, \mu)$ hence

$$\chi_{\Phi(A)} L^2(\mathbf{T}, \nu) \perp \chi_{\Phi(B)} L^2(\mathbf{T}, \nu)$$

and finally $\Phi(A) \cap \Phi(B) = \emptyset$. If $A = \bigcup_{n=1}^{\infty} A_n$ with $\{A_n\}$ pair wise disjoint then

$$\chi_{\Phi(A)} L^2(\mathbf{T}, \nu) = V(\chi_{\bigcup_{n=1}^{\infty} A_n} L^2(\mathbf{T}, \mu)) = V(\bigoplus_{n=1}^{\infty} \chi_{A_n} L^2(\mathbf{T}, \mu)) =$$

$$= \bigoplus_{n=1}^{\infty} V(\chi_{A_n} L^2(\mathbf{T}, \mu)) = \bigoplus_{n=1}^{\infty} \chi_{\Phi(A_n)} L^2(\mathbf{T}, \nu) = \chi_{\bigcup_{n=1}^{\infty} \Phi(A_n)} L^2(\mathbf{T}, \nu)$$

hence $\Phi(A) = \bigcup_{n=1}^{\infty} \Phi(A_n)$ and by a standard argument the equality holds if $\{A_n\}$ are not pair wise disjoint. Since $V(L^2(\mathbf{T}, \mu)) = L^2(\mathbf{T}, \nu)$ we have $\Phi(\mathbf{T}) = \mathbf{T}$. Hence $\mathbf{T} = \Phi(A) \cup \Phi(A^c)$ and therefore $\Phi(A)^c = \Phi(A^c)$.

Consequently $\Phi : (\mathcal{B}, \mu) \to (\mathcal{B}, \nu)$ is a σ-Boolean isomorphism. Therefore there exists a nonsingular invertible map $S : (\mathbf{T}, \mathcal{B}, \nu) \to (\mathbf{T}, \mathcal{B}, \mu)$ such that $\Phi(A) = S^{-1}(A)$ for every $A \in \mathcal{B}$.
Set $h = V(1)$. For $A \in \mathcal{B}$ we have $1 = \chi_A + \chi_{A^c}$, hence $h = V(\chi_A) + V(\chi_{A^c})$. But the functions $V(\chi_A)$ and $V(\chi_{A^c})$ have disjoint supports, so $V(\chi_A)$ must be equal to h on its support and the same remark can be applied to $V(\chi_{A^c})$ hence

$$V(\chi_A) = h \cdot \chi_{\Phi(A)} = h \cdot \chi_A \circ S.$$

Since this is true for any characteristic function, it is also true for linear combinations of such functions and finally for all $f \in L^2(\mathbf{T}, \mu)$.
Since V is unitary, for every $A \in \mathcal{B}$

$$\mu(SA) = \int_{\mathbf{T}} \mid \chi_A S^{-1} \mid^2 d\mu = \mid\mid \chi_A S^{-1} \mid\mid^2_{L^2(\mu)} =$$

$$= \mid\mid V(\chi_A S^{-1}) \mid\mid^2_{L^2(\nu)} = \mid\mid h \cdot \chi_A \mid\mid^2_{L^2(\nu)} = \int_A \mid h \mid^2 d\nu.$$

Hence $\mid h \mid^2 = \frac{d\mu \circ S}{d\nu}$. ■

Lemma 1.3. *Assume that $U_1 : H_1 \to H_1$ and $U_2 : H_2 \to H_2$ are unitary operators and $V : H_1 \to H_2$ a c.s. isomorphism of U_1 and U_2. Let*

$$H_1 = \bigoplus_{n=1}^{\infty} Z(x_n) \quad and \quad \mu_{x_1} \gg \mu_{x_2} \gg ...$$

be a spectral decomposition of U_1. Then we have

$$H_2 = \bigoplus_{n=1}^{\infty} Z(Vx_n) \quad and \quad \mu_{Vx_1} \gg \mu_{Vx_2} \gg ... \ .$$

Moreover, $\mu_{x_n} \equiv \mu_{x_{n+1}}$ iff $\mu_{Vx_n} \equiv \mu_{Vx_{n+1}}$ and hence $E(U_1) = E(U_2)$

Proof. Since V is a unitary operator,

$$H_2 = V(H_1) = V(\bigoplus_{n=1}^{\infty} Z(x_n)) = \bigoplus_{n=1}^{\infty} VZ(x_n) = \bigoplus_{n=1}^{\infty} Z(Vx_n).$$

We first show that $Z(Vx_1)$ is a maximal cyclic space. Suppose there exists $y \in H_2$ such that $Z(Vx_1) \subsetneq Z(y)$. Then we have $Z(x_1) \subsetneq Z(V^{-1}y)$. This contradicts the fact that $Z(x_1)$ is maximal. This gives us that μ_{Vx_1} is the maximal spectral type of U_2.

Similarly, since $V \mid_{Z(x_1)^\perp}$ is a c.s. isomorphism, μ_{Vx_2} is the maximal spectral type of $U_2 \mid_{Z(Vx_2)^\perp}$. In this way we conclude that μ_{Vx_n} is the maximal spectral type of U_2 restricted to $Z(Vx_n) \oplus Z(Vx_{n+1}) \oplus ...$ for every $n \geq 1$ and finally that $\mu_{Vx_1} \gg \mu_{Vx_2} \gg ...$.
If $\mu_{x_n} \gg \mu_{x_{n+1}}$ but they are not equivalent then we can write

$$Z(x_n) \oplus Z(x_{n+1}) = Z(x_n') \oplus Z(x_n'') \oplus Z(x_{n+1})$$

where $\mu_{x_n''} \perp \mu_{x_{n+1}}$ and $\mu_{x_n'} \ll \mu_{x_{n+1}}$ (in fact these latter measures are equivalent). Now

$$V(Z(x_n) \oplus Z(x_{n+1})) = Z(Vx_n') \oplus Z(Vx_n'') \oplus Z(Vx_{n+1})$$

but $Z(x_n'') \oplus Z(x_{n+1})$ is a cyclic space, hence so must be

$$V(Z(x_n'') \oplus Z(x_{n+1})) = Z(Vx_n'') \oplus Z(Vx_{n+1}).$$

This shows that the spectral measures $\mu_{Vx_n''}$ and $\mu_{Vx_{n+1}}$ are orthogonal so $\mu_{Vx_n} \gg \mu_{Vx_{n+1}}$ and they are not equivalent. ■

Remark. It follows from this lemma that $E(U)$ is an invariant of a c.s. isomorphism. Notice that if x is an eigenvector of U_1, the $Z(x)$ is a one–dimensional space. Therefore its image via a c.s. isomorphism V must be also one–dimensional, hence Vx is also eigenvector (though corresponding to possibly different eigenvalue). This gives rise to a second invariant of a c.s. isomorphism. The theorem below explains how a combination of these two invariants gives rise to a complete set of invariants for a c.s. isomorphism.

Theorem 1.4. *Let $U_i : H_i \to H_i$ be a unitary operator on a separable Hilbert space, $i = 1, 2$. Then the following conditions are equivalent.*

(i) *U_1 and U_2 are cyclic space equivalent.*

(ii) *There are spectral sequences $\mu_1 \gg \mu_2 \gg ...$ of U_1 and $\nu_1 \gg \nu_2 \gg ...$ of U_2 and measure space isomorphism $S : (\mathbf{T}, \nu_1) \to (\mathbf{T}, \mu_1)$ such that*

$$\nu_n = \mu_n \circ S \quad \text{for all} \quad n \geq 1.$$

(iii) *$E^c(U_1) = E^c(U_2)$ and $D(U_1) = D(U_2)$.*

Proof. $(i) \Rightarrow (ii)$. Suppose $V : H_1 \to H_2$ is a c.s. isomorphism of U_1 and U_2. Fix a spectral decomposition $H_1 = \bigoplus_{n=1}^{\infty} Z(x_n)$ of U_1 and put $\mu_n := \mu_{x_n}$ for each $n \geq 1$. By Lemma 1.3 we have a spectral decomposition $H_2 = \bigoplus_{n=1}^{\infty} Z(Vx_n)$ of U_2 and $\nu_n := \mu_{Vx_n}$ for each $n \geq 1$. There exists a unitary isomorphism $V_1 : \bigoplus_{n=1}^{\infty} L^2(\mathbf{T}, \mu_n) \to H_1$ of operators U and U_1 and a unitary isomorphism $V_2 : H_2 \to \bigoplus_{n=1}^{\infty} L^2(\mathbf{T}, \nu_n)$ of operators U_2 and U such that $V_1(L^2(\mathbf{T}, \mu_n)) = Z(x_n)$ and $V_2 Z(x_n) = L^2(\mathbf{T}, \nu_n))$ for $n \geq 1$, where

$$U(\sum_{n=1}^{\infty} f_n(z_n)) = \sum_{n=1}^{\infty} z_n f_n(z_n).$$

Hence the operator $V' = V_2 V V_1$ is a c.s. isomorphism of the operator U on $\bigoplus_{n=1}^{\infty} L^2(\mathbf{T}, \mu_n)$ and U on $\bigoplus_{n=1}^{\infty} L^2(\mathbf{T}, \nu_n)$ and $V'(L^2(\mathbf{T}, \mu_n)) = L^2(\mathbf{T}, \nu_n)$ (so V' restricted establishes a c.s. isomorphism) for $n \geq 1$.
By Lemma 1.2 there exist nonsingular invertible maps $S_n : (\mathbf{T}, \mathcal{B}, \nu_\backslash) \to (\mathbf{T}, \mathcal{B}, \mu_\backslash)$ and $h_n \in L^2(\mathbf{T}, \nu_n)$ such that $V' |_{L^2(\mathbf{T}, \mu_n)} f = h_n \cdot f \circ S_n$ for every $n \geq 1$. Hence we have

$$V'(\sum_{n=1}^{\infty} f_n(z_n)) = \sum_{n=1}^{\infty} h_n(z_n) \cdot f_n(S_n z_n)$$

for $\sum_{n=1}^{\infty} f_n \in \bigoplus_{n=1}^{\infty} L^2(\mathbf{T}, \mu_n)$.
For every $n \neq m$, consider

$$H = \{f(z_n) + f(z_m) : f \in L^2(\mathbf{T}, \mu_1)\}.$$

This is a closed U-invariant subspace of $\bigoplus_{k=1}^{\infty} L^2(\mathbf{T}, \mu_k)$. Without loss of generality, we can assume that $\mu_n = \mu_1 |_{A_n}$ (i.e. that $\frac{d\mu_n}{d\mu_1} = \chi_{A_n}$). Then

$$V'H = \{h_n(z_n) f(S_n z_n) + h_m(z_m) f(S_m z_m) : f \in L^2(\mathbf{T}, \mu_1)\}.$$

Since $V'H$ is U-invariant, for every $f \in L^2(\mathbf{T}, \mu_1)$ there exists $g \in L^2(\mathbf{T}, \mu_1)$ such that

$$z_n h_n(z_n) f(S_n z_n) + z_m h_m(z_m) f(S_m z_m) = h_n(z_n) g(S_n z_n) + h_m(z_m) g(S_m z_m).$$

By the orthogonality of the natural embedding of $L^2(\mathbf{T}, \mu_n)$ and $L^2(\mathbf{T}, \mu_m)$ in the space under consideration

$$\begin{aligned} z h_n(z) f(S_n z) &= h_n(z) g(S_n z), \quad z \in \mathbf{T} \quad \mu_n - \text{a.e.} \\ z h_m(z) f(S_m z) &= h_m(z) g(S_m z), \quad z \in \mathbf{T} \quad \mu_m - \text{a.e.} \end{aligned}$$

hence $S_n^{-1}(z)f(z) = g(z)$ and $S_m^{-1}(z)f(z) = g(z)$ a.e., because $h_n \neq 0$ μ_n-a.e. and $h_m \neq 0$ μ_m-a.e. by Lemma 1.2. If $f = 1$ then $S_n^{-1}(z) = g(z) = S_m^{-1}(z)$ hence $S = S_n = S_m$ for every $n \neq m$ and we get $\nu_n \equiv \mu_n \circ S$ so by replacing ν_n by $\mu_n \circ S$ the result follows.
$(ii) \Rightarrow (i)$. Suppose there are spectral sequence $\mu_1 \gg \mu_2 \gg ...$ of U_1 and $\nu_1 \gg \nu_2 \gg ...$ of U_2 and an isomorphism $S : (\mathbf{T}, \nu_1) \to (\mathbf{T}, \mu_1)$ such that $\nu_n = \mu_n \circ S$ for all $n \geq 1$. We will consider the unitary operator $V' : \bigoplus_{n=1}^{\infty} L^2(\mathbf{T}, \mu_n) \to \bigoplus_{n=1}^{\infty} L^2(\mathbf{T}, \nu_n)$ given by

$$V'(\sum_{n=1}^{\infty} f_n(z_n)) = \sum_{n=1}^{\infty} f_n(Sz_n).$$

We first prove that V' is a cyclic space isomorphism of U on $\bigoplus_{n=1}^{\infty} L^2(\mathbf{T}, \mu_n)$ and U on $\bigoplus_{n=1}^{\infty} L^2(\mathbf{T}, \nu_n)$. Let H be a closed U-invariant subspace of $\bigoplus_{n=1}^{\infty} L^2(\mathbf{T}, \mu_n)$. We show that $V'H$ is U-invariant. We have that H is $\psi(U)$-invariant for every $\psi \in L^{\infty}(\mathbf{T}, \mu_1)$. Hence if $\sum_{n=1}^{\infty} f_n(z_n) \in H$ then $\sum_{n=1}^{\infty} \psi(z_n) f_n(z_n) \in H$.
Let $\sum_{n=1}^{\infty} g_n(z_n) \in V'H$. There exists $\sum_{n=1}^{\infty} f_n(z_n) \in H$ such that $g_n = f_n \circ S$. From $\mid S^{-1}(z) \mid = 1$, it follows that

$$\sum_{n=1}^{\infty} S^{-1}(z_n) f_n(z_n) \in H$$

and hence

$$U(\sum_{n=1}^{\infty} g_n(z_n)) = \sum_{n=1}^{\infty} z_n f_n(Sz_n) \in V'H.$$

In the same manner we can see that if H is a U-invariant subspace of $\bigoplus_{n=1}^{\infty} L^2(\mathbf{T}, \nu_n)$ then $V'^{-1}H$ is U-invariant. Consequently the operator $V = V_2^{-1} V' V_1^{-1}$ is a c.s. isomorphism of U_1 and U_2.
$(ii) \Rightarrow (iii)$. If there are spectral sequence $\mu_1 \gg \mu_2 \gg ...$ of U_1 and $\nu_1 \gg \nu_2 \gg ...$ of U_2 and an isomorphism $S : (\mathbf{T}, \nu_1) \to (\mathbf{T}, \mu_1)$ such that $\nu_n = \mu_n \circ S$ for all $n \geq 1$ then

$$A_n(U_2) = S^{-1} A_n(U_1),\ C_n(U_2) = S^{-1} C_n(U_1),\ \nu_1^d = \mu_1^d \circ S.$$

Hence

$$C_n(U_2) \setminus C_{n+1}(U_2) = S^{-1}(C_n(U_1) \setminus C_{n+1}(U_1)),$$
$$\nu_1^d \mid_{A_n(U_2) \setminus A_{n+1}(U_2)} = \mu_1^d \mid_{A_n(U_1) \setminus A_{n+1}(U_1)} \circ S$$

for $n \geq 1$ and

$$\bigcap_{n=1}^{\infty} C_n(U_2) = S^{-1}(\bigcap_{n=1}^{\infty} C_n(U_1)),$$

$$\nu_1^d |_{\bigcap_{n=1}^{\infty} A_n(U_2)} = \mu_1^d |_{\bigcap_{n=1}^{\infty} A_n(U_1)} \circ S$$

and finally $E^c(U_1) = E^c(U_2)$ and $D(U_1) = D(U_2)$.
$(iii) \Rightarrow (ii)$. Let μ and ν be the maximal spectral type of U_1 and U_2. If $E^c(U_1) = E^c(U_2)$ and $D(U_1) = D(U_2)$ then

$$\nu(C_n(U_2) \setminus C_{n+1}(U_2)) > 0 \quad \text{iff} \quad \mu(C_n(U_1) \setminus C_{n+1}(U_1)) > 0$$

for $n \geq 1$ and

$$\nu(\bigcap_{n=1}^{\infty} C_n(U_2)) > 0 \quad \text{iff} \quad \mu(\bigcap_{n=1}^{\infty} C_n(U_1)) > 0$$

and $card\, D_n(U_1) = card\, D_n(U_2)$ for $n \in \mathbf{N} \cup \{+\infty\}$.
Since $A_n \setminus A_{n+1} = (C_n \setminus C_{n+1}) \cup D_n$ and $\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} C_n \cup D_\infty$, there exist nonsingular invertible maps $S_n : (A_n(U_2) \setminus A_{n+1}(U_2), \nu) \to (A_n(U_1) \setminus A_{n+1}(U_1), \mu)$ for $n \geq 1$ and $S_\infty : (\bigcap_{n=1}^{\infty} A_n(U_2), \nu) \to (\bigcap_{n=1}^{\infty} A_n(U_1), \mu)$. We define a nonsingular invertible map $S : (\mathbf{T}, \nu) \to (\mathbf{T}, \mu)$ by

$$S(x) = \begin{cases} S_n(x) & \text{for} \quad x \in A_n(U_2) \setminus A_{n+1}(U_2) \\ S_\infty(x) & \text{for} \quad x \in \bigcap_{n=1}^{\infty} A_n(U_2). \end{cases}$$

Then we have

$$\nu |_{A_n(U_2)} \equiv \mu |_{A_n(U_1)} \circ S.$$

Let $\mu_n := \mu |_{A_n(U_1)}$ and $\nu_n := \mu |_{A_n(U_1)} \circ S$ then $\mu_1 \gg \mu_2 \gg \ldots$ and $\nu_1 \gg \nu_2 \gg \ldots$ is a spectral sequence of U_1 and U_2 and $\nu_n = \mu_n \circ S$ for all $n \geq 1$. ■

2 Cyclic space isomorphism of unitary operators in the case where an operator corresponds to an ergodic dynamical system

Given a dynamical system $T : (X, \mathcal{B}, \varrho) \to (\mathcal{X}, \mathcal{B}, \varrho)$, set $Sp(T) = \{\lambda \in \mathbf{C}: \exists_{f \in L^2(X,\varrho)} fT = \lambda f\}$.

Corollary 2.1. *Let $(X_1, \mathcal{B}_\infty, \varrho_\infty, \mathcal{T}_\infty)$ and $(X_2, \mathcal{B}_\in, \varrho_\in, \mathcal{T}_\in)$ be invertible, ergodic dynamical systems. Then U_{T_1} and U_{T_2} are cyclic space equivalent if and only if $E^c(U_{T_1}) = E^c(U_{T_2})$ and $card\, Sp(T_1) = card\, Sp(T_2)$.*

Proof. By the ergodicity, for an arbitrary spectral sequence $\mu_1^{(i)} \geq \mu_2^{(i)} \geq$ ... corresponding to U_{T_i}, $i = 1, 2$ only the maximal spectral type $\mu_1^{(i)}$ need not be a continuous measure. ■

Without ergodicity the above corollary is not valid as the following example shows.

Example. Let $Tx = x + \alpha$ be an irrational rotation. Then T and $T \times T$ are not s.c. equivalent (because $D(T)(1) = \infty$ and $D(T \times T)(1) = 0$), though $card\, Sp(T) = card\, Sp(T \times T)$.

Corollary 2.2. *Let T_1 and T_2 be weakly mixing. Then U_{T_1} and U_{T_2} are cyclic space equivalent if and only if $E^c(U_{T_1}) = E^c(U_{T_2})$.*

In [5] M. Lemańczyk and J. Kwiatkowski (jr.) proved that

Proposition 1. *Given a set $A \subseteq \mathbf{N}^+$, $1 \in A$, there exists an ergodic T such that $E(U_T) = A$. Moreover, T can be constructed to be weakly mixing.*

From the proof of Proposition 1 in [5] it follows that for a set $A \subseteq \mathbf{N}^+$, $1 \in A$, there exists a weakly mixing T such that $E^c(U_T) = A$. Since all their examples have singular spectra, by taking a direct product of an example T realizing $A \subset \mathbf{N}^+$ with a τ having countable Lebesgue spectrum we reach

$$E(T \times \tau) = A \cup \{+\infty\}.$$

Hence

Corollary 2.3. *Let $\mathcal{M}_{\infty,\mathcal{C}} = \{\mathcal{U}\colon \text{has continuous spectrum and } \infty \in \mathcal{E}^{\rfloor}(\mathcal{U})\}$. Partition $\mathcal{M}_{\infty,\mathcal{C}}$ into the equivalence classes with respect to cyclic space equivalence relation. Then in every equivalence class there exists a unitary operator $U_T : L^2_0(X, \varrho) \to L^2_0(X, \varrho)$, where T is weakly mixing and $L^2_0(X, \varrho) = \{f \in L^2(X, \varrho) : \int f d\varrho = 0\}$.*

References

[1] O.N. Ageev, *Dynamical systems with an even multiplicity Lebesgue component in the spectrum,* Math. USSR Sb. **64(2)**, (1989), 305–317.

[2] F. Blanchard, M. Lemańczyk, *Measure preserving diffeomorphisms with an arbitrary spectral multiplicity,* Topological Methods in Nonlinear Analysis,**1** (1993), 275–294.

[3] N. Dunford, T. Schwartz, *Linear Operators*, Wiley-Interscience, 1971.

[4] G.R. Goodson, J. Kwiatkowski, M. Lemańczyk, P. Liardet *On the multiplicity function of ergodic group extensions of rotations*, Studia Math. **102(2)** (1992), 157–174.

[5] J. Kwiatkowski (jr.), M. Lemańczyk, *On the multiplicity function of ergodic group extensions.II*, Studia Math. **116(3)** (1995), 207-215.

[6] M. Lemańczyk, *Toeplitz Z_2–extensions*, Ann. H. Poincaré **24** (1988), 1-43.

[7] J. Mathew, M.G. Nadkarni, *Measure–preserving transformation whose spectrum has Lebesgue component of multiplicity two,* Bull. London Math. Soc. **16** (1984), 402–406.

[8] M. Quéffelec, *Substitution Dynamical Systems – Spectral Analysis*, Lect. Notes in Math. 1294, Springer–Verlag, Berlin, 1988.

[9] W. Parry, *Topics in Ergodic Theory*, Cambridge Univ. Press., Cambridge, 1981.

[10] E.A. Robinson, *Ergodic measure–preserving transformations with arbitrary finite spectral multiplicity,* Invent. Math. **72** (1983), 299–314.

[11] E.A. Robinson, *Transformations with highly non–homogeneous spectrum of finite multiplicity,* Isr. J. Math. **56** (1986), 75–88.

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