Cyclic space isomorphism of unitary operators

KRZYSZTOF FRACZEK

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Abstract
We introduce a new equivalence relation between unitary operators on separable Hilbert spaces and discuss a possibility to have in each equivalence class a measure-preserving transformation.

Introduction
Let \( U \) be a unitary operator on a separable Hilbert space \( H \). For any \( x \in H \) we define the cyclic space generated by \( x \) as \( Z(x) = \text{span}\{U^n x : n \in \mathbb{Z}\} \). By the spectral measure \( \mu_x \) of \( x \) we mean a Borel measure on the circle determined by the equalities

\[
\hat{\mu}_x(n) = \int_T z^n d\mu_x(z) = (U^n x, x)
\]

for every \( n \in \mathbb{Z} \).

**Theorem 0.1 (spectral theorem).** (see [9]) There exists in \( H \) a sequence \( x_1, x_2, \ldots \) such that

\[
H = \bigoplus_{n=1}^{\infty} Z(x_n) \quad \text{and} \quad \mu_{x_1} \gg \mu_{x_2} \gg \ldots .
\]

Moreover, for any sequence \( y_1, y_2, \ldots \) in \( H \) satisfying (1) we have \( \mu_{x_1} \equiv \mu_{y_1}, \mu_{x_2} \equiv \mu_{y_2}, \ldots \).

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One of the most important problems (still open) in ergodic theory is a classification of ergodic dynamical systems with respect to the spectral equivalence, i.e. given a sequence
\[ \mu_1 \gg \mu_2 \gg ... \]  
(2)
of positive finite measures on the circle we ask if there exists an ergodic dynamical system \( T : (X, \mathcal{B}, \rho) \to (X, \mathcal{B}, \rho) \) such that a spectral sequence (1) for \( U = U_T (U_T : L^2(X, \rho) \to L^2(X, \rho), U_T f = f \circ T) \) coincides with (2).

The spectral type of \( \mu_{x_1} \) (the equivalence class of measures) is called the maximal spectral type of \( U \). By the multiplicity function \( M_U \) of \( U \) we mean the function \( M_U : \mathbb{T} \to \mathbb{N} \cup \{+\infty\} \) given by:
\[ M_U(z) = \sum_{n=1}^{\infty} \chi A_n(z) \]
where \( A_1 = \mathbb{T} \) and \( A_n = A_n(U) = \{ z \in \mathbb{T} : \frac{d \mu_{x_n}}{d \mu_{x_1}}(z) > 0 \} \) (it is well-defined up to a \( \mu_{x_1} - \)nullset). Then we have
\[ \mathbb{T} = A_1 \supset A_2 \supset A_3 \supset .... \]
The set
\[ E(U) = \{ n \in \mathbb{N} \cup \{+\infty\} : \mu_{x_1} \{ z \in \mathbb{T} : M_U(z) = n \} > 0 \} \]
is called the set of essential values of the multiplicity function \( M_U \).
For the background on spectral theory we refer to [3].

In the last few years, problems concerning spectral multiplicity have become of a renewed interest (see [1], [2], [4], [6], [7], [8], [10], [11]). In [5], M. Lemańczyk and J. Kwiatkowski (jr.) show that for an arbitrary set \( A \subseteq \mathbb{N}^+ \) containing 1, an ergodic automorphism \( T \) whose set of essential values of the multiplicity function is equal to \( A \) is constructed. The aim of this paper is a new viewpoint on spectral classification stated to me by Professor Lemańczyk.

Every measure \( \mu \) can be uniquely decomposed into a sum \( \mu = \mu^c + \mu^d \) where \( \mu^c \) is continuous and \( \mu^d \) is discret. For a spectral sequence \( \mu_{x_1} \gg \mu_{x_2} \gg ... \) we have \( \mu^c_{x_1} \gg \mu^c_{x_2} \gg .... \) By the c-multiplicity function \( M^c_U \) we mean the function \( M^c_U : \mathbb{T} \to \mathbb{N} \cup \{+\infty\} \) given by
\[ M^c_U(z) = \sum_{n=1}^{\infty} \chi c_n(z) \]
where $C_1 = \mathbf{T}$ and $C_n = \{z \in \mathbf{T} : \frac{d\mu_{x_1}}{dz_1}(z) > 0\}$. The set
\[
E^c(U) = \{n \in \mathbf{N} \cup \{+\infty\} : \mu_{x_1}\{z \in \mathbf{T} : M_U(z) = n\} > 0\}
\]
is called the set of essential values of c-multiplicity function $M_U$.

Let $D(U) : \mathbf{N} \cup \{+\infty\} \to \mathbf{N} \cup \{+\infty\}$ be a function given by $D(U)(n) = card D_n$ where
\[
D_n = \left\{ \begin{array}{ll}
\{z \in A_n \setminus A_{n+1} : \mu_{x_1}((z)) > 0\} & \text{for } n = 1, 2, \ldots \\
\{z \in \bigcap_{n=1}^{\infty} A_n : \mu_{x_1}((z)) > 0\} & \text{for } n = +\infty.
\end{array} \right.
\]

In Section 1 we define a cyclic space (s.c.) isomorphism of unitary operators on separable Hilbert space and we try to find a complete set of invariants for a c.s. isomorphism. Using results from Section 1 and those from [5], we show that in the c.s. equivalence class of an operator $U : H \to H$ whose maximal spectral type is continuous and $1 \in E^c(U)$ we can find a weakly mixing automorphism.

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1 Cyclic space isomorphism and its invariants

Lemma 1.1. Let $U_1 : H_1 \to H_1$ and $U_2 : H_2 \to H_2$ be unitary operators. Then, for every unitary operator $V : H_1 \to H_2$ the following conditions are equivalent.

(i) For every $x \in H_1$, $Z(Vx) = VZ(x)$.

(ii) If $H$ is a $U_1$-invariant closed subspace of $H_1$, then $VH$ is $U_2$-invariant and if $H$ is a $U_2$-invariant closed subspace of $H_2$, then $V^{-1}H$ is $U_1$-invariant.

Proof. $(i) \Rightarrow (ii)$. Suppose that $H$ is a $U_1$-invariant closed subspace of $H_1$ and $y \in VH$. There exists $x \in H$ such that $y = Vx$. Since $Z(y) = VZ(x)$,
\[
U_2^{-1}y, U_2y \in Z(y) = VZ(x) \subset VH
\]
and finally that $VH$ is $U_2$-invariant. Similarly, we can get the remaining part of $(ii)$. 

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(ii) ⇒ (i). Let $x \in H_1$. Since $Z(x)$ is $U_1$-invariant, $VZ(x)$ is $U_2$-invariant. Since $Vx \in VZ(x)$, $Z(Vx) \subset VZ(x)$. Similarly, if $y = Vx$ then $Z(x) = Z(V^{-1}y) \subset V^{-1}Z(y) = V^{-1}Z(Vx)$. This gives $VZ(x) \subset Z(Vx)$ and finally $Z(Vx) = VZ(x)$.

**Definition 1.1.** We call a unitary operator $V : H_1 \rightarrow H_2$ is a cyclic space isomorphism of $U_1$ and $U_2$ if it satisfies (i) from Lemma 1.1 or equivalently (ii).

**Lemma 1.2.** Let $\mu$ and $\nu$ be positive finite Borel measures on the circle. Assume $U_1 : L^2(T, \mu) \rightarrow L^2(T, \mu)$, $U_2 : L^2(T, \nu) \rightarrow L^2(T, \nu)$ are unitary operators given by

$$U_1 f(z) = U_2 f(z) = zf(z).$$

If $V : L^2(T, \mu) \rightarrow L^2(T, \nu)$ is a c.s. isomorphism of $U_1$ and $U_2$ then there exists a nonsingular invertible map $S : (T, B, \nu) \rightarrow (T, B, \mu)$ and $h \in L^2(T, \nu)$ such that

$$Vf = h \cdot f \circ S$$

for every $f \in L^2(T, \mu)$.

**Proof.** For a set $A \in B$ put $H = \chi_A L^2(T, \mu)$. Then $H$ is a $U_1$-invariant subspace of $L^2(T, \mu)$. By Wiener Lemma (e.g. [9] Appendix) there exists a Borel set $\Phi(A)$ such that $VH = \chi_{\Phi(A)} L^2(T, \nu)$. From $V(\{0\}) = \{0\}$ and $V^{-1}(\{0\}) = \{0\}$ we obtain that $\mu(A) = 0$ iff $\nu(\Phi(A)) = 0$. If $A \cap B = \emptyset$ then $\chi_A L^2(T, \mu) \perp \chi_B L^2(T, \mu)$ hence

$$\chi_{\Phi(A)} L^2(T, \nu) \perp \chi_{\Phi(B)} L^2(T, \nu)$$

and finally $\Phi(A) \cap \Phi(B) = \emptyset$. If $A = \bigcup_{n=1}^{\infty} A_n$ with $\{A_n\}$ pair wise disjoint then

$$\chi_{\Phi(A)} L^2(T, \nu) = V(\chi_{\bigcup_{n=1}^{\infty} A_n} L^2(T, \mu)) = V(\bigoplus_{n=1}^{\infty} \chi_{A_n} L^2(T, \mu)) =$$

$$= \bigoplus_{n=1}^{\infty} V(\chi_{A_n} L^2(T, \mu)) = \bigoplus_{n=1}^{\infty} \chi_{\Phi(A_n)} L^2(T, \nu) = \chi_{\bigcup_{n=1}^{\infty} \Phi(A_n)} L^2(T, \nu)$$

hence $\Phi(A) = \bigcup_{n=1}^{\infty} \Phi(A_n)$ and by a standard argument the equality holds if $\{A_n\}$ are not pair wise disjoint. Since $V(L^2(T, \mu)) = L^2(T, \nu)$ we have $\Phi(T) = T$. Hence $T = \Phi(A) \cup \Phi(A^c)$ and therefore $\Phi(A)^c = \Phi(A^c)$.
Consequently \( \Phi : (B, \mu) \to (B, \nu) \) is a \( \sigma \)-Boolean isomorphism. Therefore there exists a nonsingular invertible map \( S : (T, B, \nu) \to (T, B, \mu) \) such that \( \Phi(A) = S^{-1}(A) \) for every \( A \in B \).

Set \( h = V(1) \). For \( A \in B \) we have \( 1 = \chi_A + \chi_{A'} \), hence \( h = V(\chi_A) + V(\chi_{A'}) \).

But the functions \( V(\chi_A) \) and \( V(\chi_{A'}) \) have disjoint supports, so \( V(\chi_A) \) must be equal to \( h \) on its support and the same remark can be applied to \( V(\chi_{A'}) \) hence
\[
V(\chi_A) = h \cdot \chi_{\Phi(A)} = h \cdot \chi_A \circ S.
\]

Since this is true for any characteristic function, it is also true for linear combinations of such functions and finally for all \( f \in L^2(T, \mu) \).

Since \( V \) is unitary, for every \( A \in B \)
\[
\mu(SA) = \int_T |\chi_A S^{-1}|^2 d\mu = ||\chi_A S^{-1}||^2_{L^2(\mu)} =
\]
\[
= ||V(\chi_A S^{-1})||^2_{L^2(\nu)} = ||h \cdot \chi_A||^2_{L^2(\nu)} = \int_A |h|^2 d\nu.
\]
Hence \( |h|^2 = \frac{d\mu S}{d\nu} \).

**Lemma 1.3.** Assume that \( U_1 : H_1 \to H_1 \) and \( U_2 : H_2 \to H_2 \) are unitary operators and \( V : H_1 \to H_2 \) a c.s. isomorphism of \( U_1 \) and \( U_2 \). Let
\[
H_1 = \bigoplus_{n=1}^{\infty} Z(x_n) \quad \text{and} \quad \mu_{x_1} \gg \mu_{x_2} \gg ...
\]
be a spectral decomposition of \( U_1 \). Then we have
\[
H_2 = \bigoplus_{n=1}^{\infty} Z(V x_n) \quad \text{and} \quad \mu_{V x_1} \gg \mu_{V x_2} \gg ...
\]
Moreover, \( \mu_{x_n} \equiv \mu_{x_{n+1}} \) iff \( \mu_{V x_n} \equiv \mu_{V x_{n+1}} \) and hence \( E(U_1) = E(U_2) \).

**Proof.** Since \( V \) is a unitary operator,
\[
H_2 = V(H_1) = V \left( \bigoplus_{n=1}^{\infty} Z(x_n) \right) = \bigoplus_{n=1}^{\infty} V Z(x_n) = \bigoplus_{n=1}^{\infty} Z(V x_n).
\]
We first show that \( Z(V x_1) \) is a maximal cyclic space. Suppose there exists \( y \in H_2 \) such that \( Z(V x_1) \subseteq Z(y) \). Then we have \( Z(x_1) \subseteq Z(V^{-1} y) \). This contradicts the fact that \( Z(x_1) \) is maximal. This gives us that \( \mu_{V x_1} \) is the maximal spectral type of \( U_2 \).
Similarly, since $V |_{Z(x_1)}$ is a c.s. isomorphism, $\mu_{Vx_2}$ is the maximal spectral type of $U_2 |_{Z(Vx_2)}$. In this way we conclude that $\mu_{Vx_n}$ is the maximal spectral type of $U_2$ restricted to $Z(Vx_n) \oplus Z(Vx_{n+1}) \oplus ...$ for every $n \geq 1$ and finally that $\mu_{Vx_1} \gg \mu_{Vx_2} \gg ...$.

If $\mu_{x_n} \gg \mu_{x_{n+1}}$ but they are not equivalent then we can write

$$Z(x_n) \oplus Z(x_{n+1}) = Z(x'_n) \oplus Z(x''_n) \oplus Z(x_{n+1})$$

where $\mu_{x''_n} \perp \mu_{x_{n+1}}$ and $\mu_{x'_n} \ll \mu_{x_{n+1}}$ (in fact these latter measures are equivalent). Now

$$V(Z(x_n) \oplus Z(x_{n+1})) = Z(Vx'_n) \oplus Z(Vx''_n) \oplus Z(Vx_{n+1})$$

but $Z(x''_n) \oplus Z(x_{n+1})$ is a cyclic space, hence so must be

$$V(Z(x'_n) \oplus Z(x_{n+1})) = Z(Vx'_n) \oplus Z(Vx_{n+1}).$$

This shows that the spectral measures $\mu_{Vx'_n}$ and $\mu_{Vx_{n+1}}$ are orthogonal so $\mu_{Vx_n} \gg \mu_{Vx_{n+1}}$ and they are not equivalent. ☐

**Remark.** It follows from this lemma that $E(U)$ is an invariant of a c.s. isomorphism. Notice that if $x$ is an eigenvector of $U_1$, the $Z(x)$ is a one-dimensional space. Therefore its image via a c.s. isomorphism $V$ must be also one-dimensional, hence $Vx$ is also eigenvector (though corresponding to possibly different eigenvalue). This gives rise to a second invariant of a c.s. isomorphism. The theorem below explains how a combination of these two invariants gives rise to a complete set of invariants for a c.s. isomorphism.

**Theorem 1.4.** Let $U_i : H_i \rightarrow H_i$ be a unitary operator on a separable Hilbert space, $i = 1, 2$. Then the following conditions are equivalent.

(i) $U_1$ and $U_2$ are cyclic space equivalent.

(ii) There are spectral sequences $\mu_1 \gg \mu_2 \gg ...$ of $U_1$ and $\nu_1 \gg \nu_2 \gg ...$ of $U_2$ and measure space isomorphism $S : (T, \nu_1) \rightarrow (T, \mu_1)$ such that

$$\nu_n = \mu_n \circ S \quad \text{for all} \quad n \geq 1.$$

(iii) $E^c(U_1) = E^c(U_2)$ and $D(U_1) = D(U_2)$. 

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Proof. \((i) \Rightarrow (ii)\). Suppose \(V : H_1 \to H_2\) is a c.s. isomorphism of \(U_1\) and \(U_2\). Fix a spectral decomposition \(H_1 = \bigoplus_{n=1}^{\infty} Z(x_n)\) of \(U_1\) and put \(\mu_n := \mu_{x_n}\) for each \(n \geq 1\). By Lemma 1.3 we have a spectral decomposition \(H_2 = \bigoplus_{n=1}^{\infty} Z(V x_n)\) of \(U_2\) and \(\nu_n := \mu_{V x_n}\) for each \(n \geq 1\). There exists a unitary isomorphism \(V_1 : \bigoplus_{n=1}^{\infty} L^2(T, \mu_n) \to H_1\) of operators \(U\) and \(U_1\) and a unitary isomorphism \(V_2 : H_2 \to \bigoplus_{n=1}^{\infty} L^2(T, \nu_n)\) of operators \(U_2\) and \(U\) such that \(V_1(L^2(T, \mu_n)) = Z(x_n)\) and \(V_2 Z(x_n) = L^2(T, \nu_n)\) for \(n \geq 1\), where

\[
U \left( \sum_{n=1}^{\infty} f_n(z_n) \right) = \sum_{n=1}^{\infty} z_n f_n(z_n).
\]

Hence the operator \(V' = V_2 V V_1\), is a c.s. isomorphism of the operator \(U\) on \(\bigoplus_{n=1}^{\infty} L^2(T, \mu_n)\) and \(U\) on \(\bigoplus_{n=1}^{\infty} L^2(T, \nu_n)\) and \(V'(L^2(T, \mu_n)) = L^2(T, \nu_n)\) (so \(V'\) restricted establishes a c.s. isomorphism) for \(n \geq 1\).

By Lemma 1.2 there exist nonsingular invertible maps \(S_n : (T, B, \nu) \to (T, B, \mu)\) and \(h_n \in L^2(T, \nu_n)\) such that \(V' |_{L^2(T, \mu_n)} f = h_n \cdot f \circ S_n\) for every \(n \geq 1\). Hence we have

\[
V' \left( \sum_{n=1}^{\infty} f_n(z_n) \right) = \sum_{n=1}^{\infty} h_n(z_n) \cdot f_n(S_n z_n)
\]

for \(\sum_{n=1}^{\infty} f_n \in \bigoplus_{n=1}^{\infty} L^2(T, \mu_n)\).

For every \(n \neq m\), consider

\[
H = \{ f(z_n) + f(z_m) : f \in L^2(T, \mu_1) \}.
\]

This is a closed \(U\)-invariant subspace of \(\bigoplus_{k=1}^{\infty} L^2(T, \mu_k)\). Without loss of generality, we can assume that \(\mu_n = \mu_1 \mid_{A_n}\) (i.e. that \(\frac{d\mu_n}{d\mu_1} = \chi_{A_n}\)). Then

\[
V' H = \{ h_n(z_n)f(S_n z_n) + h_m(z_m)f(S_m z_m) : f \in L^2(T, \mu_1) \}.
\]

Since \(V' H\) is \(U\)-invariant, for every \(f \in L^2(T, \mu_1)\) there exists \(g \in L^2(T, \mu_1)\) such that

\[
z_n h_n(z_n)f(S_n z_n) + z_m h_m(z_m)f(S_m z_m) = h_n(z_n)g(S_n z_n) + h_m(z_m)g(S_m z_m).
\]

By the orthogonality of the natural embedding of \(L^2(T, \mu_n)\) and \(L^2(T, \mu_m)\) in the space under consideration

\[
zh_n(z)f(S_n z) = h_n(z)g(S_n z), \quad z \in T, \quad \mu_n - \text{a.e.}
\]

\[
zh_m(z)f(S_m z) = h_m(z)g(S_m z), \quad z \in T, \quad \mu_m - \text{a.e.}
\]
hence $S_n^{-1}(z)f(z) = g(z)$ and $S_m^{-1}(z)f(z) = g(z)$ a.e., because $h_n \neq 0$ $\mu_n$-a.e. and $h_m \neq 0 \mu_m$-a.e. by Lemma 1.2. If $f = 1$ then $S_n^{-1}(z) = g(z) = S_m^{-1}(z)$ hence $S = S_n = S_m$ for every $n \neq m$ and we get $\nu_n \equiv \mu_n \circ S$ so by replacing $\nu_n$ by $\mu_n \circ S$ the result follows.

(ii) $\Rightarrow$ (i). Suppose there are spectral sequence $\mu_1 \gg \mu_2 \gg \ldots$ of $U_1$ and $\nu_1 \gg \nu_2 \gg \ldots$ of $U_2$ and an isomorphism $S : (T, \nu_1) \to (T, \mu_1)$ such that $\nu_n = \mu_n \circ S$ for all $n \geq 1$. We will consider the unitary operator $V' : \bigoplus_{n=1}^{\infty} L^2(T, \mu_n) \to \bigoplus_{n=1}^{\infty} L^2(T, \nu_n)$ given by

$$V'(\sum_{n=1}^{\infty} f_n(z_n)) = \sum_{n=1}^{\infty} f_n(Sz_n).$$

We first prove that $V'$ is a cyclic space isomorphism of $U$ on $\bigoplus_{n=1}^{\infty} L^2(T, \mu_n)$ and $U$ on $\bigoplus_{n=1}^{\infty} L^2(T, \nu_n)$. Let $H$ be a closed $U$-invariant subspace of $\bigoplus_{n=1}^{\infty} L^2(T, \mu_n)$. We show that $V'H$ is $U$-invariant. We have that $H$ is $\psi(U)$-invariant for every $\psi \in L^\infty(T, \mu_1)$. Hence if $\sum_{n=1}^{\infty} f_n(z_n) \in H$ then $\sum_{n=1}^{\infty} \psi(z_n)f_n(z_n) \in H$.

Let $\sum_{n=1}^{\infty} g_n(z_n) \in V'H$. There exists $\sum_{n=1}^{\infty} f_n(z_n) \in H$ such that $g_n = f_n \circ S$. From $|S^{-1}(z)| = 1$, it follows that

$$\sum_{n=1}^{\infty} S^{-1}(z_n)f_n(z_n) \in H$$

and hence

$$U(\sum_{n=1}^{\infty} g_n(z_n)) = \sum_{n=1}^{\infty} z_nf_n(Sz_n) \in V'H.$$  

In the same manner we can see that if $H$ is a $U$-invariant subspace of $\bigoplus_{n=1}^{\infty} L^2(T, \nu_n)$ then $V'^{-1}H$ is $U$-invariant. Consequently the operator $V = V_2^{-1}V'V_1^{-1}$ is a c.s. isomorphism of $U_1$ and $U_2$.

(ii) $\Rightarrow$ (iii). If there are spectral sequence $\mu_1 \gg \mu_2 \gg \ldots$ of $U_1$ and $\nu_1 \gg \nu_2 \gg \ldots$ of $U_2$ and an isomorphism $S : (T, \nu_1) \to (T, \mu_1)$ such that $\nu_n = \mu_n \circ S$ for all $n \geq 1$ then

$$A_n(U_2) = S^{-1}A_n(U_1), \ C_n(U_2) = S^{-1}C_n(U_1), \ \nu_1^d = \mu_1^d \circ S.$$  

Hence

$$C_n(U_2) \setminus C_{n+1}(U_2) = S^{-1}(C_n(U_1) \setminus C_{n+1}(U_1)),$$

$$\nu_1^d |_{A_n(U_2) \setminus A_{n+1}(U_2)} = \mu_1^d |_{A_n(U_1) \setminus A_{n+1}(U_1)} \circ S.$$
for \( n \geq 1 \) and
\[
\bigcap_{n=1}^{\infty} C_n(U_2) = S^{-1}\big( \bigcap_{n=1}^{\infty} C_n(U_1) \big),
\]
\[
\nu^d(x | \bigcap_{n=1}^{\infty} A_n(U_2)) = \mu^d(x | \bigcap_{n=1}^{\infty} A_n(U_1) \circ S)
\]
and finally \( E^c(U_1) = E^c(U_2) \) and \( D(U_1) = D(U_2) \).

(iii) \( \Rightarrow \) (ii). Let \( \mu \) and \( \nu \) be the maximal spectral type of \( U_1 \) and \( U_2 \). If \( E^c(U_1) = E^c(U_2) \) and \( D(U_1) = D(U_2) \) then
\[
\nu(C_n(U_2) \setminus C_{n+1}(U_2)) > 0 \hspace{0.5cm} \text{iff} \hspace{0.5cm} \mu(C_n(U_1) \setminus C_{n+1}(U_1)) > 0
\]
for \( n \geq 1 \) and
\[
\nu(\bigcap_{n=1}^{\infty} C_n(U_2)) > 0 \hspace{0.5cm} \text{iff} \hspace{0.5cm} \mu(\bigcap_{n=1}^{\infty} C_n(U_1)) > 0
\]
and \( \text{card } D_n(U_1) = \text{card } D_n(U_2) \) for \( n \in \mathbb{N} \cup \{+\infty\} \).

Since \( A_n \setminus A_{n+1} = (C_n \setminus C_{n+1}) \setminus D_n \) and \( \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} C_n \setminus D_\infty \), there exist nonsingular invertible maps \( S_n : (A_n(U_2) \setminus A_{n+1}(U_2), \nu) \to (A_n(U_1) \setminus A_{n+1}(U_1), \mu) \) for \( n \geq 1 \) and \( S_\infty : (\bigcap_{n=1}^{\infty} A_n(U_2), \nu) \to (\bigcap_{n=1}^{\infty} A_n(U_1), \mu) \). We define a nonsingular invertible map \( S : (T, \nu) \to (T, \mu) \) by
\[
S(x) = \begin{cases} 
S_n(x) & \text{for } x \in A_n(U_2) \setminus A_{n+1}(U_2) \\
S_\infty(x) & \text{for } x \in \bigcap_{n=1}^{\infty} A_n(U_2).
\end{cases}
\]
Then we have
\[
\nu |_{A_n(U_2)} \equiv \mu |_{A_n(U_1) \circ S}.
\]

Let \( \mu_n := \mu |_{A_n(U_1)} \) and \( \nu_n := \mu |_{A_n(U_1) \circ S} \) then \( \mu_1 \gg \mu_2 \gg \ldots \) and \( \nu_1 \gg \nu_2 \gg \ldots \) is a spectral sequence of \( U_1 \) and \( U_2 \) and \( \nu_n = \mu_n \circ S \) for all \( n \geq 1 \).

### 2 Cyclic space isomorphism of unitary operators in the case where an operator corresponds to an ergodic dynamical system

Given a dynamical system \( T : (X, \mathcal{B}, \varrho) \to (X, \mathcal{B}, \varrho) \), set \( \text{Sp}(T) = \{ \lambda \in \mathbb{C} : \exists f \in L^2(X, \varrho) \text{ s.t. } T = \lambda f \} \).

**Corollary 2.1.** Let \( (X_1, \mathcal{B}_1, \varrho_1, T_1) \) and \( (X_2, \mathcal{B}_2, \varrho_2, T_2) \) be invertible, ergodic dynamical systems. Then \( U_{T_1} \) and \( U_{T_2} \) are cyclic space equivalent if and only if \( E^c(U_{T_1}) = E^c(U_{T_2}) \) and \( \text{card } \text{Sp}(T_1) = \text{card } \text{Sp}(T_2) \).
Proof. By the ergodicity, for an arbitrary spectral sequence $\mu_1^{(i)} \geq \mu_2^{(i)} \geq \ldots$ corresponding to $U_{T_i}$, $i = 1, 2$ only the maximal spectral type $\mu_1^{(i)}$ need not be a continuous measure. ■

Without ergodicity the above corollary is not valid as the following example shows.

Example. Let $Tx = x + \alpha$ be an irrational rotation. Then $T$ and $T \times T$ are not s.c. equivalent (because $D(T)(1) = \infty$ and $D(T \times T)(1) = 0$), though $\text{card Sp}(T) = \text{card Sp}(T \times T)$.

Corollary 2.2. Let $T_1$ and $T_2$ be weakly mixing. Then $U_{T_1}$ and $U_{T_2}$ are cyclic space equivalent if and only if $E_c(U_{T_1}) = E_c(U_{T_2})$.

In [5] M. Lemańczyk and J. Kwiatkowski (jr.) proved that

Proposition 1. Given a set $A \subseteq \mathbb{N}^+$, $1 \in A$, there exists an ergodic $T$ such that $E(U_T) = A$. Moreover, $T$ can be constructed to be weakly mixing.

From the proof of Proposition 1 in [5] it follows that for a set $A \subseteq \mathbb{N}^+$, $1 \in A$, there exists a weakly mixing $T$ such that $E_c(U_T) = A$. Since all their examples have singular spectra, by taking a direct product of an example $T$ realizing $A \subset \mathbb{N}^+$ with a $\tau$ having countable Lebesgue spectrum we reach

$$E(T \times \tau) = A \cup \{+\infty\}.$$

Hence

Corollary 2.3. Let $\mathcal{M}_{\infty,c} = \{\mathcal{U}: \text{has continuous spectrum and } \infty \in E(\mathcal{U})\}$. Partition $\mathcal{M}_{\infty,c}$ into the equivalence classes with respect to cyclic space equivalence relation. Then in every equivalence class there exists a unitary operator $U_T : L^2_0(X, \varrho) \rightarrow L^2_0(X, \varrho)$, where $T$ is weakly mixing and $L^2_0(X, \varrho) = \{f \in L^2(X, \varrho) : \int f d\varrho = 0\}$.

References


Department of Mathematics and Computer Science
Nicholas Copernicus University
ul. Chopina 12/18, 87-100 Toruń
Poland
E-mail: fraczek@mat.uni.torun.pl