

SOLVING THE COHOMOLOGICAL EQUATION FOR LOCALLY HAMILTONIAN FLOWS, PART II - GLOBAL OBSTRUCTIONS

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ABSTRACT. Continuing the research initiated in [12], we study the existence of solutions and their regularity for the cohomological equations $Xu = f$ for locally Hamiltonian flows (determined by the vector field X) on a compact surface M of genus $g \geq 1$. We move beyond the case studied so far by Forni in [6, 8], when the flow is minimal over the entire surface and the function f satisfies some Sobolev regularity conditions. We deal with the flow restricted to any its minimal component and any smooth function f whenever the flow satisfies the Full Filtration Diophantine Condition (FFDC) (this is a full measure condition).

The main goal of this article is to quantify optimal regularity of solutions. For this purpose we construct a family of invariant distributions $\mathfrak{F}_{\bar{t}}$, $\bar{t} \in \mathcal{TF}^*$ that play the roles of the Forni's invariant distributions introduced in [6, 8] by using the language of translation surfaces. The distributions $\mathfrak{F}_{\bar{t}}$ are global in nature (as emphasized in the title of the article), unlike the distributions $\mathfrak{d}_{\sigma,j}^k$, $(\sigma, k, j) \in \mathcal{TD}$ and $\mathfrak{C}_{\sigma,l}^k$, $(\sigma, k, l) \in \mathcal{TC}$ introduced in [12], which are defined locally. All three families are used to determine the optimal regularity of the solutions for the cohomological equation, see Theorem 1.1 and 1.2. As a by-product, we also obtained, interesting in itself, a spectral result (Theorem 1.3) for the Kontsevich-Zorich cocycle acting on functional spaces arising naturally at the transition to the first-return map.

1. INTRODUCTION

Let M be a smooth compact connected orientable surface of genus $g \geq 1$. Our primary focus is on smooth flows $\psi_{\mathbb{R}} = (\psi_t)_{t \in \mathbb{R}}$ on M preserving a smooth positive measure μ , i.e. such that for any (orientable) local coordinates (x, y) we have $d\mu = V(x, y)dx \wedge dy$ with V positive and smooth. Denote by $X : M \rightarrow TM$ the associated vector field. Then for (orientable) local coordinates (x, y) such that $d\mu = V(x, y)dx \wedge dy$, the flow $\psi_{\mathbb{R}}$ is (locally) a solution to the Hamiltonian equation

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}(x, y), \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x}(x, y)$$

for a smooth real-valued locally defined function H . The flows $\psi_{\mathbb{R}}$ are usually called *locally Hamiltonian flows* or *multivalued Hamiltonian flows*. For general introduction to locally Hamiltonian flows on surfaces, we refer readers to [14, 11, 22, 24].

The main goal of the article is to fully understand the problem of existence of the solution $u : M \rightarrow \mathbb{R}$ and its regularity for the cohomological equation $Xu = f$, if $f : M \rightarrow \mathbb{R}$ is any smooth observable (recall that $Xu(x) = \frac{d}{dt}u(\psi_t x)|_{t=0}$). We always assume that all fixed points of $\psi_{\mathbb{R}}$ are isolated. Then the set of fixed points $\text{Fix}(\psi_{\mathbb{R}})$ is finite. As $\psi_{\mathbb{R}}$ is area-preserving, every fixed point is either a center or

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a saddle. In what follows, we deal only with *perfect (harmonic)* saddles defined as follows: a fixed point $\sigma \in \text{Fix}(\psi_{\mathbb{R}})$ is a perfect saddle of multiplicity $m_{\sigma} \geq 2$ if there exists a chart (x, y) in a neighborhood U_{σ} of σ such that $d\mu = V(x, y)dx \wedge dy$ and $H(x, y) = \Im(x + iy)^{m_{\sigma}}$. We call (x, y) a *singular chart*. We denote by $\text{Sd}(\psi_{\mathbb{R}})$ the set of perfect saddles of $\psi_{\mathbb{R}}$.

We call a *saddle connection* an orbit of $\psi_{\mathbb{R}}$ running from a saddle to a saddle. A *saddle loop* is a saddle connection joining the same saddle. We deal only with flows such that all their saddle connections are loops. If every fixed point is isolated then M splits into a finite number of components ($\psi_{\mathbb{R}}$ -invariant surfaces with boundary) so that every component is either a *minimal component* (every orbit, except of fixed points and saddle loops, is dense in the component) or is a *periodic component* (filled by periodic orbits, fixed points and saddle loops).

The problem of existence and regularity of solutions for the cohomological equation $Xu = f$ was essentially solved in two seminal articles [6, 8] by Forni when the flow $\psi_{\mathbb{R}}$ is minimal over the whole surface M (has no saddle connection) and the function f belongs to a certain weighted Sobolev space $H_W^s(M)$, $s \geq 1$. Let us mention that being an element of a weighted Sobolev space enforces significant constraints on the behavior of the function f around saddles, even for smooth functions, as described in [8]. In [6, 8], for a.e. locally Hamiltonian flows, Forni proved the existence of fundamental invariant distributions on $H_W^s(M)$ which are responsible for the degree of smoothness of the solution of $Xu = f$ for $f \in H_W^s(M)$. If all Forni's distributions at $f \in H_W^s(M)$ are zero then there exists a solution $u \in H_{\omega}^{s'}(M)$ for some $0 < s' < s$.

The problem of solving cohomological equations for other classes of smooth dynamical systems of parabolic nature and the regularity of solutions using invariant distributions were studied also in [1, 2, 4, 5, 9, 15, 16, 23, 27].

1.1. Invariant distributions and the main results when saddle loops exist.

The main goal of this article is to go beyond the case of a minimal flow on the whole surface M and beyond the case of the function f belonging to weighted Sobolev spaces. We deal with locally Hamiltonian flows restricted to any minimal component $M' \subset M$ and $f : M \rightarrow \mathbb{R}$ is any smooth function. The main novelty of the proposed approach is that it is used to study the regularity of the solution u when the flow has saddle loops, which has not been systematically studied before. The study of locally Hamiltonian flows in such a context gives rise to new invariant distributions, which, unlike Forni's distributions, are local in nature. Two families of such local functionals $\mathfrak{C}_{\sigma, l}^k$ and $\mathfrak{d}_{\sigma, j}^k$ were introduced by the authors in [12]. As it was shown in [12] both families play an important role in understanding the regularity of solution for cohomological equation if f is any smooth function.

Throughout the article we use the notation $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$ for any pair of real numbers x, y . Denote by $\mathcal{T}\mathcal{D}$ the set of triples $(\sigma, k, j) \in (\text{Sd}(\psi_{\mathbb{R}}) \cap M') \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ such that $0 \leq j \leq k \wedge (m_{\sigma} - 2)$ and $j \neq k - (m_{\sigma} - 1) \bmod m_{\sigma}$. For every $(\sigma, k, j) \in \mathcal{T}\mathcal{D}$ we define the functional $\mathfrak{d}_{\sigma, j}^k : C^k(M) \rightarrow \mathbb{C}$ as follows:

$$(1.1) \quad \mathfrak{d}_{\sigma, j}^k(f) = \sum_{0 \leq n \leq \frac{k-j}{m_{\sigma}}} \frac{\binom{k}{j+nm_{\sigma}} \binom{(m_{\sigma}-1)-j-1}{m_{\sigma}n}}{\binom{(k-j)-(m_{\sigma}-1)}{m_{\sigma}n}} \frac{\partial^k(f \cdot V)}{\partial z^{j+nm_{\sigma}} \partial \bar{z}^{k-j-nm_{\sigma}}}(0, 0).$$

The real number $\widehat{\mathfrak{d}}(\mathfrak{d}_{\sigma, j}^k) = \widehat{\mathfrak{d}}(\sigma, k) = k - (m_{\sigma} - 2)$ we call the *hat-order* of $\mathfrak{d}_{\sigma, j}^k$.

For any $\sigma \in \text{Sd}(\psi_{\mathbb{R}}) \cap M'$ its neighbourhood U_σ splits into $2m_\sigma$ angular sectors bounded by separatrices. In singular coordinates $z = (x, y)$ they are of the form

$$U_{\sigma,l} := \{z \in U_\sigma : \text{Arg } z \in (\frac{\pi l}{m_\sigma}, \frac{\pi(l+1)}{m_\sigma})\} \text{ for } 0 \leq l < 2m_\sigma.$$

Denote by \mathcal{TC} the set of triples $(\sigma, k, l) \in (\text{Sd}(\psi_{\mathbb{R}}) \cap M') \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ such that $0 \leq l < 2m_\sigma$ and $U_{\sigma,l} \subset M'$. For every $(\sigma, k, l) \in \mathcal{TC}$ we define the functional $\mathfrak{C}_{\sigma,l}^k : C^k(M) \rightarrow \mathbb{C}$ as follows:

$$\mathfrak{C}_{\sigma,l}^k(f) := \sum_{\substack{0 \leq i \leq k \\ i \neq m_\sigma - 1 \pmod{m_\sigma} \\ i \neq k - (m_\sigma - 1) \pmod{m_\sigma}}} \theta_\sigma^{l(2i-k)} \binom{k}{i} \mathfrak{B}\left(\frac{(m_\sigma-1)-i}{m_\sigma}, \frac{(m_\sigma-1)-k+i}{m_\sigma}\right) \frac{\partial^k(f \cdot V)}{\partial z^i \partial \bar{z}^{k-i}}(0, 0),$$

where θ_σ is the principal $2m_\sigma$ -th root of unity. The (beta-like) function $\mathfrak{B}(x, y)$ is defined for any pair x, y of real numbers such that $x, y \notin \mathbb{Z}$ as follows:

$$\mathfrak{B}(x, y) = \frac{\pi e^{i\frac{\pi}{2}(y-x)} \Gamma(x+y-1)}{2^{x+y-2} \Gamma(x)\Gamma(y)},$$

where we adopt the convention $\Gamma(0) = 1$ and $\Gamma(-n) = 1/(-1)^n n!$. For the real number $\mathfrak{o}(\mathfrak{C}_{\sigma,l}^k) = \mathfrak{o}(\sigma, k) = \frac{k-(m_\sigma-2)}{m_\sigma}$, we call it the *order* of $\mathfrak{C}_{\sigma,l}^k$.

In this paper, for a.e. locally Hamiltonian flow (satisfying the Full Filtration Diophantine Condition (FFDC) defined in Section 3.2), we define the third family of distributions $\mathfrak{F}_{\bar{t}}$ which have global nature and are smooth version of Forni's invariant distributions introduced in [6, 8]. We should emphasize that the definition of $\mathfrak{F}_{\bar{t}}$ (unlike Forni's approach) does not use tools from translational surface theory. Such techniques cannot be used due to the existence of saddle loops. Our approach is based on the use of a (modified by us) correction operator invented by Marmi-Moussa-Yoccoz in [18] (see also [19] and [20]) in its simplest version and later extended in [13] and [11].

Let $g \geq 1$ be the genus of M' and let γ be the number of saddles in M' . Denote by \mathcal{TF}^* the set of triples of the form $(k, +, i)$, $(k, 0, s)$ or $(k, -, j)$ for $k \geq 0$, $1 \leq i, j \leq g$ and $1 \leq s < \gamma$. Let \mathcal{TF} be the subset of triples in \mathcal{TF}^* after removing all triples of the form $(k, -, 1)$ for $k \geq 0$. Denote by $0 < \lambda_g < \dots < \lambda_2 < \lambda_1$ the positive Lyapunov exponents associated to a flow satisfying FFDC (see again Section 3.2). In Section 7.1, for every triple $\bar{t} \in \mathcal{TF}^*$ we define a corresponding functional $\mathfrak{F}_{\bar{t}}$. For the real number

$$\mathfrak{o}(\mathfrak{F}_{\bar{t}}) = \mathfrak{o}(\bar{t}) = \begin{cases} k - \frac{\lambda_i}{\lambda_1} & \text{if } \bar{t} = (k, +, i) \\ k & \text{if } \bar{t} = (k, 0, s) \\ k + \frac{\lambda_j}{\lambda_1} & \text{if } \bar{t} = (k, -, j), \end{cases}$$

we call the *order* of $\mathfrak{F}_{\bar{t}}$.

Let m be the maximal multiplicity of saddles in $\text{Sd}(\psi_{\mathbb{R}}) \cap M'$. Following [12], for every $r > 0$ let

$$k_r = \begin{cases} \lceil mr + (m-1) \rceil & \text{if } m = 2 \text{ and } r \leq \frac{1}{2} \\ \lceil mr + (m-2) \rceil & \text{otherwise.} \end{cases}$$

Recall that

$$\begin{aligned} \max\{k \geq 0 : \exists \sigma \in \text{Sd}(\psi_{\mathbb{R}}) \cap M' \mathfrak{o}(\sigma, k) < r\} + 1 &= \lceil mr + (m-2) \rceil \leq k_r \\ \max\{k \geq 0 : \exists \sigma \in \text{Sd}(\psi_{\mathbb{R}}) \cap M' \widehat{\mathfrak{o}}(\sigma, k) < r\} + 1 &= \lceil r + (m-2) \rceil \leq k_r. \end{aligned}$$

Then for every flow $\psi_{\mathbb{R}}$ restricted to its minimal component M' and satisfying FFDC and for every $\bar{t} \in \mathcal{TF}^*$, the corresponding functional $\mathfrak{F}_{\bar{t}}$ is defined on $C^{k_{\circ(\bar{t})}+1}(M)$. The following main two results show how the three families of invariant distributions influence on the regularity of the solution for the cohomological equation $Xu = f$ with smooth u defined on the end compactification M'_e of $M' \setminus \text{Sd}(\psi_{\mathbb{R}})$ considered in [12].

Theorem 1.1. *Let $\psi_{\mathbb{R}}$ be a locally Hamiltonian flow such that its restriction to a minimal component M' satisfies FFDC. Let $r \in \mathbb{R}_{>0} \setminus (\{\circ(\sigma, k) : k \geq 0, \sigma \in \text{Sd}(\psi_{\mathbb{R}}) \cap M'\} \cup \{\circ(\bar{t}) : \bar{t} \in \mathcal{TF}\})$. Suppose that $f \in C^{k_r}(M)$ and*

- $\mathfrak{d}_{\sigma,j}^k(f) = 0$ for all $(\sigma, k, j) \in \mathcal{TD}$ with $\widehat{\circ}(\mathfrak{d}_{\sigma,j}^k) < r$;
- $\mathfrak{C}_{\sigma,l}^k(f) = 0$ for all $(\sigma, k, l) \in \mathcal{TC}$ with $\circ(\mathfrak{C}_{\sigma,l}^k) < r$;
- $\mathfrak{F}_{\bar{t}}(f) = 0$ for all $\bar{t} \in \mathcal{TF}$ with $\circ(\mathfrak{F}_{\bar{t}}) < r$.

Then there exists $u \in C^r(M'_e)$ such that $Xu = f$ on M'_e . Moreover, there exists $C_r > 0$ such that $\|u\|_{C^r(M'_e)} \leq C_r \|f\|_{C^{k_r}(M)}$.

Theorem 1.2 (optimal regularity). *Let $\psi_{\mathbb{R}}$ be a locally Hamiltonian flow such that its restriction to a minimal component M' satisfies FFDC. For any $r > 0$ suppose that $f \in C^{k_r}(M)$ and there exists $u \in C^r(M'_e)$ such that $Xu = f$ on M'_e . Then*

- $\mathfrak{d}_{\sigma,j}^k(f) = 0$ for all $(\sigma, k, j) \in \mathcal{TD}$ with $\widehat{\circ}(\mathfrak{d}_{\sigma,j}^k) < r$;
- $\mathfrak{C}_{\sigma,l}^k(f) = 0$ for all $(\sigma, k, l) \in \mathcal{TC}$ with $\circ(\mathfrak{C}_{\sigma,l}^k) < r$;
- $\mathfrak{F}_{\bar{t}}(f) = 0$ for all $\bar{t} \in \mathcal{TF}$ with $\circ(\mathfrak{F}_{\bar{t}}) < r$.

1.2. Cohomological equations over IETs and a spectral result. Let us consider the restriction of a locally Hamiltonian flow $\psi_{\mathbb{R}}$ on M to its minimal component $M' \subset M$ and let $I \subset M'$ be a transversal smooth curve. We always assume that each end of I is the first meeting point of a separatrix (that is not a saddle connection) emanating by a saddle (incoming or outgoing) with the curve I . By minimality, I is a global transversal and the first return map $T : I \rightarrow I$ is an interval exchange transformation (IET) in so called standard coordinates on I . Denote by I_{α} , $\alpha \in \mathcal{A}$ the intervals exchanged by T and by $\tau : I \rightarrow \mathbb{R}_{>0} \cup \{+\infty\}$ the first return time map to the curve I , called also the roof function. The roof function $\tau : I \rightarrow \mathbb{R}_{>0} \cup \{+\infty\}$ is smooth on the interior of each exchanged interval and has *singularities* at discontinuities of T . For any continuous observable $f : M \rightarrow \mathbb{C}$ we deal with the corresponding map $\varphi_f : I \rightarrow \mathbb{C} \cup \{\infty\}$ given by

$$\varphi_f(x) = \int_0^{\tau(x)} f(\psi_t x) dt.$$

If u is a solution of the cohomological equation $Xu = f$ then

$$(1.2) \quad v(Tx) - v(x) = \varphi_f(x) \text{ on } I,$$

where v is the restriction of u to the curve I . Therefore the existence and a regularity of the solution to the cohomological equation (1.2) is an obvious necessary condition for the existence and the same regularity of the solution to $Xu = f$. As shown in [12] (Theorem 1.2), this is also a sufficient condition under additional assumptions related to the vanishing of certain distributions $\mathfrak{C}_{\sigma,l}^k$ and $\mathfrak{d}_{\sigma,j}^k$ on f . Moreover, the regularity of the solution u depends on the regularity of the solution v and the vanishing of all the mentioned distributions up to some level of their order or hat-order. For this reason, in the present paper, we primarily focus on the cohomological equation $v \circ T - v = \varphi_f$. The regularity of φ_f was completely understood in [12].

It was shown there that $\varphi_f \in C^{n+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$, i.e. is piecewise C^{m+1} and its n -th derivative has polynomial (of degree at most $0 < a < 1$) or logarithmic (if $a = 0$) singularities at discontinuities of the IET T . The degree of smoothness n depends on the maximal order of vanishing for the distributions $\mathfrak{C}_{\sigma,l}^k$, see Theorem 1.1 in [12].

For any $k \in \mathbb{N} \cup \{\infty\}$ denote by $\Phi^k(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ the space of functions φ_f for $f \in C^k(M)$. The main tool used to solve the chomological equation (1.2) is a spectral analysis of the functional version (on $\Phi^k(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$) of the Kontsevich-Zorich cocycle $S(j)$ (see Section 4.2 for the definition). Some kind of spectral analysis (for positive Lyapunov exponents) of the cocycle $S(j)$ was used in [13] and [11] to fully understand the deviation for ergodic integrals of smooth observables for a.a. locally Hamiltonian flows. Our techniques are motivated by correction operators invented by Marmi-Moussa-Yoccoz in [18] (see also [19] and [20]) in its simplest version (without singularities) and later extended in [13] and [11].

To represent formally the main spectral result, let us consider an equivalence relation \sim on the set of triples \mathcal{TC} , introduced in [12]. Two triples $(\sigma, k, l), (\sigma, k, l') \in \mathcal{TC}$ are equivalent with respect to the equivalence relation \sim if the angular sectors $U_{\sigma,l}$ and $U_{\sigma,l'}$ are connected through a chain of saddle loops emanating from the saddle σ . For every equivalence class $[(\sigma, k, l)] \in \mathcal{TC} / \sim$ let

$$\mathfrak{C}_{[(\sigma,k,l)]}(f) := \sum_{(\sigma,k,l') \sim (\sigma,k,l)} \mathfrak{C}_{\sigma,l'}^k(f).$$

For any $k \geq 0$ let $\Gamma_k(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ be the space of functions which are polynomials of degree at most k on any interval I_α , $\alpha \in \mathcal{A}$. This space plays an important role in solving cohomological equations in [10]. We will define two families of functions $\{h_{\bar{t}} : \bar{t} \in \mathcal{TF}^*\}$ and $\{\xi_{[(\sigma,k,l)]} : [(\sigma, k, l)] \in \mathcal{TC} / \sim\}$, which are the keys for understanding the spectral properties of the Kontsevich-Zorich cocycle $S(j)$. First, they meet the following properties:

$$(1.3) \quad \begin{aligned} & h_{\bar{t}} \in \Gamma_k(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \text{ if } \bar{t} = (k, \cdot, \cdot); \\ & \lim_{j \rightarrow \infty} \frac{1}{j} \log \|S(j)h_{\bar{t}}\|_{\text{sup}} = \lim_{j \rightarrow \infty} \frac{1}{j} \log \|S(j)h_{\bar{t}}\|_{L^1} = -\lambda_1 \mathfrak{o}(\bar{t}); \end{aligned}$$

$$(1.4) \quad \begin{aligned} & \xi_{[(\sigma,k,l)]} \in C^{m+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \text{ with } n = \lceil \mathfrak{o}(\sigma, k) \rceil, \quad a = \mathfrak{o}(\sigma, k) - n; \\ & \lim_{j \rightarrow \infty} \frac{1}{j} \log \|S(j)\xi_{[(\sigma,k,l)]}\|_{L^1} = -\lambda_1 \mathfrak{o}(\sigma, k); \end{aligned}$$

$$(1.5) \quad \lim_{j \rightarrow \infty} \frac{1}{j} \log \|S(j)\xi_{[(\sigma,k,l)]}\|_{\text{sup}} = -\lambda_1 \mathfrak{o}(\sigma, k) \text{ if } \mathfrak{o}(\sigma, k) > 0.$$

Then the main spectral result is as follows.

Theorem 1.3 (spectral theorem). *Let $\psi_{\mathbb{R}}$ be a locally Hamiltonian flow such that its restriction to a minimal component M' satisfies FFDC. For every $r > -\frac{m-2}{m}$ and $f \in C^{kr}(M)$ we have*

$$\varphi_f = \sum_{\substack{\bar{t} \in \mathcal{TF}^* \\ \mathfrak{o}(\bar{t}) < r}} \mathfrak{F}_{\bar{t}}(f)h_{\bar{t}} + \sum_{\substack{[(\sigma,k,l)] \in \mathcal{TC} / \sim \\ \mathfrak{o}(\sigma,k) < r}} \mathfrak{C}_{[(\sigma,k,l)]}(f)\xi_{[(\sigma,k,l)]} + \mathfrak{r}_r(f)$$

so that

$$(1.6) \quad \begin{aligned} \limsup_{j \rightarrow \infty} \frac{1}{j} \log \|S(j)\mathfrak{r}_r(f)\|_{\text{sup}} &\leq -\lambda_1 r \text{ if } r > 0 \text{ and} \\ \limsup_{j \rightarrow \infty} \frac{1}{j} \log \|S(j)\mathfrak{r}_r(f)\|_{L^1} &\leq -\lambda_1 r \text{ if } r \leq 0. \end{aligned}$$

This theorem can be seen as a counterpart to spectral results from [3] in general (non-pseudo-Anosov) setting. However, the most important advantage of Theorem 1.3 is that it (more precisely its preceding version, Theorem 5.6) is used to solve (in Section 6.2) the regularity of solutions to the cohomological equation $v \circ T - v = \varphi_f$ (see Theorem 6.8). In Section 6, we modify techniques developed by Marmi-Yoccoz in [20] to study the regularity of solutions in Hölder scale.

1.3. A new family of invariant distributions via extended correction operators. In [12], the authors defined two families of invariant distributions $\mathfrak{C}_{\sigma,l}^k$ and $\mathfrak{D}_{\sigma,j}^k$ inspired by local analysis of higher order derivatives φ_f around ends of intervals exchanged by T . In the current paper, we introduce a new family $\mathfrak{f}_{\bar{t}}$, $\bar{t} \in \mathcal{I}\mathcal{F}^*$ of invariant distributions over IETs and transport them to the level of the surface M by composing with the operator $f \mapsto \varphi_f$. The resulting distributions $\mathfrak{F}_{\bar{t}}$, $\bar{t} \in \mathcal{I}\mathcal{F}^*$ generalize (emulate) the notion of Forni's invariant distributions (associated with Lyapunov exponents of Kontsevitch-Zorich cocycle), but the method of construction is completely different from the original.

The invariant distributions $\mathfrak{f}_{\bar{t}}$ over IETs are defined on the space of $C^{n+\text{PaG}}$ -function (if $\sigma(\bar{t}) < n - a$). In [11], the authors constructed invariant distributions, for $n = 0$, using correction through piecewise constant functions. They constructed so-called correction operators \mathfrak{h}_j , $1 \leq j \leq g$ but the construction was limited to the unstable subspace (corresponding to positive Lyapunov exponents) of the Kontsevitch-Zorich cocycle. The original idea of correcting smooth functions was introduced by Marmi-Moussa-Yoccoz in [18] and then developed in [13] and [11].

In this paper there are three types of new functionals that arise from other parts of the Oseledets splitting (+/−/0 denoting unstable/stable/central resp.) associated to Lyapunov exponents (see Section 5.3). Their construction is based on the use of new correction operators $\mathfrak{h}_{-j,i}$, \mathfrak{h}_j^* , \mathfrak{h}_0 and their higher-order derivatives. The new correction operators allow us to correct φ_f by piecewise constant functions, not only related to unstable vectors as before, but also by central and stable vectors. The construction of these three new types of correction operators is the most important technical novelty of the article, which allows defining the counterparts of Forni's invariant distribution for flows with saddle loops. Together with the previously defined local invariant distributions $\mathfrak{C}_{\sigma,l}^k$ and $\mathfrak{D}_{\sigma,j}^k$ they give a complete and optimal knowledge of the regularity of solutions in Hölder scale. This optimality of regularity seems to be the most important overall novelty of the article.

1.4. Structure of the paper. In § 2, we recall some basic notions related to IETs, Rauzy-Veech induction and accelerations of the Kontsevitch-Zorich cocycle. In § 3, we review Oseledets filtration of accelerated KZ-cocycles and formulate the corresponding Full Filtration Diophantine Condition (FFDC). We set up a new infinite series that are necessary for constructing extended correction operators in the next section. In § 4, extended correction operators $\mathfrak{h}_{-j,i}$, \mathfrak{h}_j^* , \mathfrak{h}_0 are constructed and their basic properties are proven. In § 5, we compute Lyapunov exponents of renormalization cocycle $S(j)$ for piecewise polynomial function $h_{i,l}$, $c_{s,l}$, $h_{-j,l}$. These three

classes of functions are then used to construct the functionals $\mathfrak{f}_{\tilde{t}}$. The culmination of this section is the proof of the spectral result (Theorem 5.6) which is the main component of the proof of the Theorem 1.3. Cohomological equations for IET and the regularity of their solutions are studied in § 6. Finally, in § 7, we conclude the regularity of solutions to cohomological equations for locally Hamiltonian flows. The main results are obtained from main theorems in [12] and results of § 6.

2. INTERVAL EXCHANGE TRANSFORMATIONS (IET)

Let \mathcal{A} be a d -element alphabet and let $\pi = (\pi_0, \pi_1)$ be a pair of bijections $\pi_\varepsilon : \mathcal{A} \rightarrow \{1, \dots, d\}$ for $\varepsilon = 0, 1$. For every $\lambda = (\lambda_\alpha)_{\alpha \in \mathcal{A}} \in \mathbb{R}_{>0}^{\mathcal{A}}$ let $|\lambda| := \sum_{\alpha \in \mathcal{A}} \lambda_\alpha$, $I := [0, |\lambda|)$ and for every $\alpha \in \mathcal{A}$,

$$I_\alpha := [l_\alpha, r_\alpha), \text{ where } l_\alpha = \sum_{\pi_0(\beta) < \pi_0(\alpha)} \lambda_\beta, \quad r_\alpha = \sum_{\pi_0(\beta) \leq \pi_0(\alpha)} \lambda_\beta.$$

Denote by $\mathcal{S}_{\mathcal{A}}^0$ the subset of *irreducible* pairs, i.e. $\pi_1 \circ \pi_0^{-1} \{1, \dots, k\} \neq \{1, \dots, k\}$ for $1 \leq k < d$. We will always assume that $\pi \in \mathcal{S}_{\mathcal{A}}^0$. An *interval exchange transformation* $T = T_{(\pi, \lambda)} : I \rightarrow I$ is a piecewise translation determined by the data (π, λ) , so that $T_{(\pi, \lambda)}$ translates the interval I_α for each $\alpha \in \mathcal{A}$ so that $T(x) = x + w_\alpha$ for $x \in I_\alpha$, where $w = \Omega_\pi \lambda$ and Ω_π is the matrix $[\Omega_{\alpha\beta}]_{\alpha, \beta \in \mathcal{A}}$ given by

$$\Omega_{\alpha\beta} = \begin{cases} +1 & \text{if } \pi_1(\alpha) > \pi_1(\beta) \text{ and } \pi_0(\alpha) < \pi_0(\beta), \\ -1 & \text{if } \pi_1(\alpha) < \pi_1(\beta) \text{ and } \pi_0(\alpha) > \pi_0(\beta), \\ 0 & \text{in all other cases.} \end{cases}$$

An IET $T_{(\pi, \lambda)}$ satisfies the *Keane condition* (see [17]) if $T_{(\pi, \lambda)}^m l_\alpha \neq l_\beta$ for all $m \geq 1$ and for all $\alpha, \beta \in \mathcal{A}$ with $\pi_0(\beta) \neq 1$.

2.1. Rauzy-Veech induction. Rauzy-Veech induction [21] and its accelerations are standard renormalization procedures for IETs. For general background, we refer the readers to the lecture notes by Yoccoz [28, 29] or Viana [26].

Let $T = T_{(\pi, \lambda)}$ be an interval exchange transformation satisfying Keane's condition. Let $\tilde{I} := [0, \max(l_{\pi_0^{-1}(d)}, l_{\pi_1^{-1}(d)}))$ and denote by $\mathcal{R}(T) = \tilde{T} : \tilde{I} \rightarrow \tilde{I}$ the first return map of T to the interval \tilde{I} . Let

$$\epsilon = \epsilon(\pi, \lambda) = \begin{cases} 0 & \text{if } \lambda_{\pi_0^{-1}(d)} > \lambda_{\pi_1^{-1}(d)}, \\ 1 & \text{if } \lambda_{\pi_0^{-1}(d)} < \lambda_{\pi_1^{-1}(d)} \end{cases}$$

and

$$A(T) = A(\pi, \lambda) = Id + E_{\pi_\epsilon^{-1}(d) \pi_{1-\epsilon}^{-1}(d)} \in SL_{\mathcal{A}}(\mathbb{Z}),$$

where Id is the identity matrix and $(E_{ij})_{kl} = \delta_{ik} \delta_{jl}$, using the Kronecker delta notation. Then, by Rauzy (see [21]), \tilde{T} is also an IET on d -intervals satisfying Keane's condition and $\tilde{T} = T_{(\tilde{\pi}, \tilde{\lambda})}$ for some $\tilde{\pi} = (\tilde{\pi}_0, \tilde{\pi}_1) \in \mathcal{S}_{\mathcal{A}}^0$ and $\tilde{\lambda} = A^{-1}(\pi, \lambda) \lambda$. Moreover, the renormalized version of the matrix $\Omega_{\tilde{\pi}}$ is of the form

$$\Omega_{\tilde{\pi}} = A^t(\pi, \lambda) \cdot \Omega_\pi \cdot A(\pi, \lambda).$$

Thus taking $H(\pi) = \Omega_\pi(\mathbb{R}^{\mathcal{A}})$, we have $H(\tilde{\pi}) = A^t(\pi, \lambda)H(\pi)$.

2.2. Kontsevich-Zorich cocycle and its accelerations. Let $T = T_{(\pi, \lambda)}$ be an IET satisfying Keane's condition. For every $n \geq 1$,

$$A^{(n)}(T) = A(T) \cdot A(\mathcal{R}(T)) \cdot \dots \cdot A(\mathcal{R}^{n-1}(T)) \in SL_{\mathcal{A}}(\mathbb{Z}).$$

This defines a multiplicative cocycle A over the transformation \mathcal{R} and it is called the *Kontsevich-Zorich cocycle*. Let $(n_k)_{k \geq 0}$ be an increasing sequence of integers with $n_0 = 0$ called an *accelerating sequence*. For every $k \geq 0$, let $T^{(k)} := \mathcal{R}^{n_k}(T) : I^{(k)} \rightarrow I^{(k)}$. Then $T^{(k)} : I^{(k)} \rightarrow I^{(k)}$ is the first return map of $T : I \rightarrow I$ to the interval $I^{(k)} \subset I$. The sequence of IETs $(T^{(k)})_{k \geq 0}$ gives an *acceleration* of the Rauzy-Veech renormalization procedure associated with the accelerating sequence $(n_k)_{k \geq 0}$.

Let $(\pi^{(k)}, \lambda^{(k)})$ be the pair defining $T^{(k)}$ and let $I_{\alpha}^{(k)}$, $\alpha \in \mathcal{A}$ be intervals exchanged by $T^{(k)}$. Then $\lambda^{(k)} = (\lambda_{\alpha}^{(k)})_{\alpha \in \mathcal{A}}$, where $\lambda_{\alpha}^{(k)} = |I_{\alpha}^{(k)}|$ for $\alpha \in \mathcal{A}$.

For every $k \geq 0$ let $Z(k+1) := A^{(n_{k+1}-n_k)}(\mathcal{R}^{n_k}(T))^t$. We then have

$$\lambda^{(k)} = Z(k+1)^t \lambda^{(k+1)}, \quad k \geq 0.$$

By following notations from [18], for each $0 \leq k < l$ let

$$Q(k, l) = Z(l) \cdot Z(l-1) \cdot \dots \cdot Z(k+2) \cdot Z(k+1) = A^{(n_l-n_k)}(\mathcal{R}^{n_k}(T))^t.$$

Then, $Q(k, l) \in SL_{\mathcal{A}}(\mathbb{Z})$ and $\lambda^{(k)} = Q(k, l)^t \lambda^{(l)}$. We write $Q(k) = Q(0, k)$.

2.3. Rokhlin towers related to accelerations. Note that $Q_{\alpha\beta}(k)$ is the time spent by any point of $I_{\alpha}^{(k)}$ in I_{β} until it returns to $I^{(k)}$. Then $Q_{\alpha}(k) = \sum_{\beta \in \mathcal{A}} Q_{\alpha\beta}(k)$ is the first return time of points of $I_{\alpha}^{(k)}$ to $I^{(k)}$. Then the IET $T : I \rightarrow I$ splits into a set of d *Rokhlin tower* of the form

$$\{T^i(I_{\alpha}^{(k)}), 0 \leq i < Q_{\alpha}(k)\}, \quad \alpha \in \mathcal{A}$$

so that $Q_{\alpha}(k)$ floors of the α -th tower are pairwise disjoint intervals.

3. DIOPHANTINE CONDITIONS FOR IETS

In this section we introduce a new Diophantine condition for IETs which is a full measure condition on the set of IETs. The Diophantine condition is a modified version of the previously introduced one in [11] (see also [14]), so called Filtration Diophantine condition (FDC). It is improved by extending the Oseledets filtration to stable and central subspaces. Based on this condition, we show that certain series involving matrices of the accelerated cocycle grows in a controlled way.

3.1. Oseledets filtration. Fix $\pi \in \mathcal{S}_{\mathcal{A}}^0$. Suppose that there exist $\lambda_1 > \dots > \lambda_g > \lambda_{g+1} = 0$ such that for a.e. IET (π, λ) there exists a filtration of linear subspaces (Oseledets filtration)

$$(3.1) \quad \begin{aligned} \{0\} &= E_0(\pi, \lambda) \subset E_{-1}(\pi, \lambda) \subset \dots \subset E_{-g}(\pi, \lambda) \subset E_{cs}(\pi, \lambda) \\ &= E_{g+1}(\pi, \lambda) \subset E_g(\pi, \lambda) \subset \dots \subset E_1(\pi, \lambda) = \Gamma := \mathbb{R}^{\mathcal{A}} \end{aligned}$$

such that for every $1 \leq i \leq g$ we have

$$(3.2) \quad \begin{aligned} \lim_{n \rightarrow +\infty} \frac{\log \|Q(n)h\|}{n} &= \lambda_{-i} := -\lambda_i \text{ for all } h \in E_{-i}(\pi, \lambda) \setminus E_{-i+1}(\pi, \lambda) \\ \lim_{n \rightarrow +\infty} \frac{\log \|Q(n)h\|}{n} &= 0 \text{ for all } h \in E_{cs}(\pi, \lambda) \setminus E_{-g}(\pi, \lambda) \\ \lim_{n \rightarrow +\infty} \frac{\log \|Q(n)h\|}{n} &= \lambda_i \text{ for all } h \in E_i(\pi, \lambda) \setminus E_{i+1}(\pi, \lambda) \\ \dim E_{-i}(\pi, \lambda) - \dim E_{-i+1}(\pi, \lambda) &= \dim E_i(\pi, \lambda) - \dim E_{i+1}(\pi, \lambda) = 1. \end{aligned}$$

Suppose that there exists a filtration of linear subspaces which is complementary to the Oseledets filtration (3.1):

$$(3.3) \quad \begin{aligned} \{0\} &= U_1 \subset U_2 \subset \dots \subset U_g \subset U_{g+1} \subset U_{-g} \subset \dots \subset U_{-1} \subset U_0 = \Gamma \\ \text{such that } U_{g+1} &\subset H(\pi) \text{ and } E_j(\pi, \lambda) \oplus U_j = \Gamma \text{ for } -g \leq j \leq g+1. \end{aligned}$$

As $E_{-g} \oplus U_{g+1} = H(\pi)$, $U_{j+1} = U_j \oplus (U_{j+1} \cap E_j)$ and $\dim(U_{j+1} \cap E_j) = 1$, for every $j \in \pm\{1, \dots, g\}$ there exists $h_j \in U_{j+1} \cap E_j$ such that

$$h_j \in H(\pi), \quad U_{j+1} = U_j \oplus \mathbb{R}h_j \text{ and } \lim_{n \rightarrow +\infty} \frac{\log \|Q(n)h_j\|}{n} = \lambda_j.$$

Let $c_1, \dots, c_{\gamma-1}$ be a basis of $U_{-g} \cap E_{g+1}$. Then for every $2 \leq j \leq g+1$ the linear subspace $U_j \subset \Gamma$ is generated by h_1, \dots, h_{j-1} and for every $0 \leq j \leq g$ the linear subspace $U_{-j} \subset \Gamma$ is generated by $h_1, \dots, h_g, c_1, \dots, c_{\gamma-1}$ and h_{-g}, \dots, h_{-j-1} . Moreover,

$$(3.4) \quad \text{if } 0 \neq h \in U_j \text{ then } \lim_{n \rightarrow +\infty} \frac{\log \|Q(n)h\|}{n} \geq \lambda_{j-1},$$

where $\lambda_{-g-1} = -\lambda_{g+1} = 0$.

For every $k \geq 0$ and $-g \leq j \leq g+1$ let $E_j^{(k)} := Q(k)E_j$ and $U_j^{(k)} := Q(k)U_j$.

3.2. Rokhlin Tower Condition and Filtration Diophantine Condition. The following Rokhlin Towers Condition (RTC) was introduced in [14].

Definition 1 (RTC). An IET $T_{(\pi, \lambda)}$ together with an acceleration satisfies RTC if there exists a constant $0 < \delta < 1$ such that

$$(RT) \quad \begin{aligned} &\text{for any } k \geq 1 \text{ there exists number } 0 < p_k \leq \min_{\alpha \in \mathcal{A}} Q_\alpha(k) \text{ such that} \\ &\{T^i I^{(k)} : 0 \leq i < p_k\} \text{ is a Rokhlin of intervals with measure } \geq \delta|I|. \end{aligned}$$

For any sequence $(r_n)_{n \geq 0}$ of real numbers and for all $0 \leq k \leq l$, we will use the notation $r(k, l) := \sum_{k \leq j < l} r_j$.

Definition 2 (FFDC). An IET $T : I \rightarrow I$ satisfying Keane's condition and Oseledets generic (i.e. there is a filtration of linear subspaces (3.1) satisfying (3.2)), satisfies the *Full Filtration Diophantine Condition (FFDC)* if for every $\tau > 0$ there exist constants $C, \kappa \geq 1$, an accelerating sequence $(n_k)_{k \geq 0}$, a sequence of natural numbers $(r_n)_{n \geq 0}$ with $r_0 = 0$ and a complementary filtration $(U_j)_{-g \leq j \leq g+1}$ (satisfying (3.3))

such that (RT) holds and

$$(3.5) \quad \lim_{n \rightarrow +\infty} \frac{r(0, n)}{n} \in (1, 1 + \tau)$$

$$(3.6) \quad \left\| Q|_{E_j^{(k)}}(k, l) \right\| \leq C e^{(\lambda_j + \tau)r(k, l)} \text{ for all } 0 \leq k < l \text{ and } 1 \leq j \leq g + 1$$

$$(3.7) \quad \left\| Q|_{E_{-j}^{(k)}}(k, l) \right\| \leq C e^{(-\lambda_j + \tau)r(k, l)} \text{ for all } 0 \leq k < l \text{ and } 1 \leq j \leq g$$

$$(3.8) \quad \left\| Q|_{U_j^{(k)}}(k, l)^{-1} \right\| \leq C e^{(-\lambda_{j-1} + \tau)r(k, l)} \text{ for all } 0 \leq k < l \text{ and } 2 \leq j \leq g + 1$$

$$(3.9) \quad \left\| Q|_{U_{-j}^{(k)}}(k, l)^{-1} \right\| \leq C e^{(\lambda_{j+1} + \tau)r(k, l)} \text{ for all } 0 \leq k < l \text{ and } 0 \leq j \leq g$$

$$(3.10) \quad \|Z(k + 1)\| \leq C e^{\tau k} \text{ for all } k \geq 0$$

$$(3.11) \quad C^{-1} e^{\lambda_1 k} \leq \|Q(k)\| \leq C e^{\lambda_1(1+\tau)k} \text{ for all } k \geq 0$$

$$(3.12) \quad \max_{\alpha \in \mathcal{A}} \frac{|I^{(k)}|}{|I_\alpha^{(k)}|} \leq \kappa \text{ for all } k \geq 0$$

$$(3.13) \quad |\sin \angle(E_j^{(k)}, U_j^{(k)})| \geq c \|Q(k)\|^{-\tau} \text{ for all } k \geq 0 \text{ and } -g \leq j \leq g + 1.$$

Definition 3. A locally Hamiltonian flow $\psi_{\mathbb{R}}$ on M with isolated fixed points and restricted to its minimal component $M' \subset M$ satisfies the *Full Filtration Diophantine Condition (FFDC)* if there exists a transversal $I \subset M'$ such that the corresponding IET $T : I \rightarrow I$ satisfies the FFDC.

Theorem 3.1. *Almost every IET satisfies FFDC.*

Proof. Most of the proof of Theorem follows similarly from the proof of Theorem 3.2 in [11]. In addition to the proof of FDC condition in [11], it suffices to slightly modify the construction of the full measure set Ξ (coming from [11]) to show that every $(\pi, \lambda) \in \Xi$ satisfies (3.7), (3.9) and (3.13) not only on the non-negative part of the filtration (as shown in [11]) but also on its negative part, i.e. on the $E_{-j}^{(k)}$ and $U_{-j}^{(k)}$ for $1 \leq j \leq g$. Since this modification is straightforward, we omit the details. \square

Remark 3.2. In view of Theorem 3.1, almost every (with respect to the Katok fundamental class) locally Hamiltonian flow $\psi_{\mathbb{R}}$ on M with isolated fixed points and restricted to its minimal component $M' \subset M$ satisfies the FFDC.

Remark 3.3. As $1 = |I| \leq |I^{(n)}| \|Q(n)\| \leq |I|/\kappa = \kappa^{-1}$, by (3.11), we have

$$(3.14) \quad |I^{(n)}|^{-1} \leq \|Q(n)\| \leq C e^{(\lambda_1 + \tau)n} \text{ and } |I^{(n)}| \leq \kappa^{-1} \|Q(n)\|^{-1} \leq \kappa^{-1} C e^{-\lambda_1 n}.$$

As $\lim_{n \rightarrow +\infty} n/r(0, n) > 1/(1 + \tau) > 1 - \tau$, there exists $c > 0$ such that

$$(3.15) \quad (1 - \tau)r(0, n) - c \leq n \leq r(0, n) \text{ for all } n \geq 0.$$

Remark 3.4. Let us consider the map $\bar{\xi} : I \rightarrow \mathbb{R}$ given by $\bar{\xi}(x) = x$ and the corresponding coboundary $\bar{\xi} \circ T - \bar{\xi}$. Then $\bar{\xi} \circ T - \bar{\xi} \in \Gamma$ and for every $k \geq 0$,

$$Q(k)(\bar{\xi} \circ T - \bar{\xi})_\alpha = \bar{\xi}(T^{Q_\alpha(k)} x) - \bar{\xi}(x) = T^{Q_\alpha(k)} x - x \text{ for any } x \in I_\alpha^{(k)}.$$

Therefore $\|Q(k)(\bar{\xi} \circ T - \bar{\xi})\| \leq |I^{(k)}| \leq \kappa^{-1} C e^{-\lambda_1 k}$. By (3.2), $\bar{\xi} \circ T - \bar{\xi} \in E_{-1}(\pi, \lambda)$. Since the space $E_{-1}(\pi, \lambda)$ is one-dimensional, we have $h_{-1} = c(\bar{\xi} \circ T - \bar{\xi})$ for some $c \neq 0$.

For any $k \geq 0$ and $-g \leq j \leq g+1$ denote by $P_{E_j^{(k)}} : \mathbb{R}^A \rightarrow E_j^{(k)}$ and $P_{U_j^{(k)}} : \mathbb{R}^A \rightarrow U_j^{(k)}$ the corresponding projections, i.e. $P_{E_j^{(k)}} + P_{U_j^{(k)}} = Id_{\mathbb{R}^A}$. In view of (3.13), using the arguments of the proof of Lemma 3.5 in [11], for any $\tau > 0$ there exists $C > 0$ such that for all $k \geq 0$ and $-g \leq j \leq g+1$,

$$(3.16) \quad \|P_{E_j^{(k)}}\| \leq C \|Q(k)\|^\tau, \quad \|P_{U_j^{(k)}}\| \leq C \|Q(k)\|^\tau.$$

Moreover, by definition, for any pair $0 \leq k < l$ and any $-g \leq j \leq g+1$ we have

$$Q(k, l) \circ P_{E_j^{(k)}} = P_{E_j^{(l)}} \circ Q(k, l) \text{ and } Q(k, l) \circ P_{U_j^{(k)}} = P_{U_j^{(l)}} \circ Q(k, l).$$

3.3. Diophantine series. For every $a \geq 0$ and $s \geq 1$, let $\langle s \rangle^a = s^a$ if $a > 0$ and $\langle s \rangle^a = 1 + \log s$ if $a = 0$.

Definition 4. For every IET $T : I \rightarrow I$ satisfying Keane's condition, any $0 \leq a < 1$, any $2 \leq i \leq g+1$, any $\tau > 0$ and any accelerating sequence we define sequences $(K_k^{a,i,\tau}(T))_{k \geq 0}$, $(C_k^{a,i,\tau}(T))_{k \geq 0}$ so that

$$K_k^{a,i,\tau}(T) := \sum_{l \geq k} \|Q|_{U_i^{(k)}}(k, l+1)^{-1}\| \|Z(l+1)\| \langle \|Q(l)\| \rangle^a \|Q(l+1)\|^\tau,$$

$$C_k^{a,i,\tau}(T) := \sum_{0 \leq l < k} \|Q|_{E_i^{(l+1)}}(l+1, k)\| \|Z(l+1)\| \langle \|Q(l)\| \rangle^a \|Q(l+1)\|^\tau.$$

Proposition 3.5. [11, Proposition 3.6] *Let $T : I \rightarrow I$ be an IET satisfying FFDC and let $0 \leq a < 1$. Suppose that $2 \leq i \leq g+1$ is chosen such that $a\lambda_1 < \lambda_{i-1}$. Then for every $0 < \tau < \frac{\lambda_{i-1} - \lambda_1 a}{3(1+\lambda_1)}$ the sequences $(K_k^{a,i,\tau})_{k \geq 0}$, $(C_k^{a,i,\tau})_{k \geq 0}$ are well defined and*

$$(3.17) \quad K_k^{a,i,\tau}(T) \leq C_\tau e^{(\lambda_1 a + 5\tau(1+\lambda_1))r(0,k)},$$

$$C_k^{a,i,\tau}(T) \leq C_\tau e^{(\max\{\lambda_i, \lambda_1 a\} + 3\tau(1+\lambda_1))r(0,k)}.$$

The series was originally designed to construct some correction operators (see [11, §6]) on C^{0+P_a} . We now present a new type of series for the similar purpose on C^{m+P_a} .

Definition 5. For every IET $T : I \rightarrow I$ satisfying Keane's condition, any $2 \leq j \leq g+1$, any non-negative sequence $\bar{s} = (s_k)_{k \geq 0}$, any $\tau > 0$ and any accelerating sequence, we define sequences $(V_k^{j,\tau}(T, \bar{s}))_{k \geq 0}$, $(W_k^{j,\tau}(T, \bar{s}))_{k \geq 0}$ so that

$$V_k^{j,\tau}(T, \bar{s}) := \sum_{l \geq k} \|Q|_{U_{-j}^{(k)}}(k, l+1)^{-1}\| \|Q(l+1)\|^\tau \|Z(l+1)\| s_l,$$

$$W_k^{j,\tau}(T, \bar{s}) := \sum_{0 \leq l < k} \|Q|_{E_{-j}^{(l+1)}}(l+1, k)\| \|Q(l+1)\|^\tau \|Z(l+1)\| s_l.$$

Proposition 3.6. *Let $T : I \rightarrow I$ be an IET satisfying FFDC. Fix $0 \leq j \leq g$, $\lambda_{j+1} < \rho$ and $0 < \tau < \frac{\rho - \lambda_{j+1}}{\lambda_1 + 3}$. Then there exists $C_\tau > 0$ such that for any non-negative sequence $\bar{s} = (s_k)_{k \geq 0}$ with $s_k \leq D e^{-\rho r(0,k+1)}$ for all $k \geq 0$ we have*

$$(3.18) \quad V_k^{j,\tau}(T, \bar{s}) \leq C_\tau D e^{(-\rho + (\lambda_1 + 2)\tau)r(0,k)},$$

$$(3.19) \quad W_k^{j,\tau}(T, \bar{s}) \leq C_\tau D e^{(\max\{-\rho, -\lambda_j\} + (\lambda_1 + 3)\tau)r(0,k)}.$$

Proof. By Definition 2,

$$\begin{aligned}
V_k^{j,\tau} &\leq \sum_{l \geq k} C^3 D e^{(\lambda_{j+1} + \tau)r(k, l+1)} e^{(\lambda_1 + \tau)\tau(l+1)} e^{\tau(l+1)} e^{-\rho r(0, l+1)} \\
&\leq \sum_{l \geq k} C^3 D e^{(\lambda_{j+1} + \tau)r(k, l+1)} e^{(-\rho + \tau(\lambda_1 + 2))r(0, l+1)} \\
&= D e^{(-\rho + \tau(\lambda_1 + 2))r(0, k)} \sum_{l \geq k} C^3 e^{(-\rho + \lambda_{j+1} + \tau(\lambda_1 + 3))r(k, l+1)} \\
&\leq D e^{(-\rho + \tau(\lambda_1 + 2))r(0, k)} \sum_{l \geq k} C^3 e^{(-\rho + \lambda_{j+1} + \tau(\lambda_1 + 3))(l+1-k)} \\
&= D e^{(-\rho + \tau(\lambda_1 + 2))r(0, k)} \sum_{l \geq 1} C^3 e^{(-\rho + \lambda_{j+1} + \tau(\lambda_1 + 3))l}.
\end{aligned}$$

As $-\rho + \lambda_{j+1} + \tau(\lambda_1 + 3) < 0$, the above series is convergent, so we get (3.18). Moreover, again by Definition 2,

$$\begin{aligned}
W_k^{j,\tau} &\leq \sum_{0 \leq l < k} C^3 D e^{(-\lambda_j + \tau)r(l+1, k)} e^{(\lambda_1 + \tau)\tau(l+1)} e^{\tau(l+1)} e^{-\rho r(0, l+1)} \\
&\leq \sum_{1 \leq l \leq k} C^3 D e^{(-\lambda_j + \tau)r(l, k)} e^{(-\rho + \tau(\lambda_1 + 2))r(0, l)} \leq C^3 D k e^{(\max\{-\lambda_j, -\rho\} + \tau(\lambda_1 + 2))r(0, k)} \\
&\leq C^3 D C' e^{(\max\{-\lambda_j, -\rho\} + \tau(\lambda_1 + 3))r(0, k)},
\end{aligned}$$

which gives (3.19). \square

4. EXTENDED CORRECTION OPERATORS

In this section we define three types of new correction operators \mathfrak{h}_j^* , $\mathfrak{h}_{-j, i}$ and \mathfrak{h}_0 for $2 \leq i \leq g+1$ and $0 \leq j \leq g$. These operators are motivated by the correction operator \mathfrak{h}_i previously defined on C^{0+P_a} , the space of functions with polynomial singularities. In [11, §6], the maps from C^{0+P_a} were corrected by piecewise constant functions coming from the unstable subspace. Our new operators are constructed to correct piecewise smooth functions whose higher order derivatives have polynomial singularities (elements of $C^{n+P_a^G}$) by piecewise polynomial functions.

4.1. $C^{n+P_a^G}$ space. Fix $0 \leq a < 1$ and an integer $n \geq 0$. Following [12, §2], $C^{n+P_a}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ is the space of C^{n+1} -functions on $\bigcup_{\alpha \in \mathcal{A}} \text{Int } I_\alpha$ such that

$$p_a(D^n \varphi) := \max_{\alpha \in \mathcal{A}} \sup_{x \in (l_\alpha, r_\alpha)} \max\{|D^{n+1} \varphi(x)(x - l_\alpha)^{1+a}|, |D^{n+1} \varphi(x)(r_\alpha - x)^{1+a}|\}$$

is finite and

$$\begin{aligned}
C_{\alpha, n}^{a,+}(\varphi) &= (-1)^n C_\alpha^+(D^n \varphi) := (-1)^{n+1} \lim_{x \searrow l_\alpha} D^{n+1} \varphi(x)(x - l_\alpha)^{1+a}, \\
C_{\alpha, n}^{a,-}(\varphi) &= C_\alpha^-(D^n \varphi) := \lim_{x \nearrow r_\alpha} D^{n+1} \varphi(x)(r_\alpha - x)^{1+a}
\end{aligned}$$

exist. The space $C^{n+P_a}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ is a Banach space equipped with the norm

$$\|\varphi\|_{C^{n+P_a}} := \sum_{k=0}^n \|D^k \varphi\|_{L^1(I)} + p_a(D^n \varphi).$$

We denote by $C^{n+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \subset C^{n+\text{Pa}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ the space of functions with *geometric type*, i.e. such that

$$C_{\pi_0^{-1}(d),n}^{a,-} \cdot C_{\pi_1^{-1}(d),n}^{a,-} = 0 \quad \text{and} \quad C_{\pi_0^{-1}(1),n}^{a,+} \cdot C_{\pi_1^{-1}(1),n}^{a,+} = 0.$$

4.2. Special Birkhoff sums. Assume that an IET $T : I \rightarrow I$ satisfies Keane's condition. For any $0 \leq k < l$ and any measurable map $\varphi : I^{(k)} \rightarrow \mathbb{R}$ over the IET $T^{(k)} : I^{(k)} \rightarrow I^{(k)}$, denote by $S(k, l)\varphi : I^{(l)} \rightarrow \mathbb{R}$ the renormalized map over $T^{(l)}$ given by

$$S(k, l)\varphi(x) = \sum_{0 \leq i < Q_\beta(k, l)} \varphi((T^{(k)})^i x) \text{ for } x \in I_\beta^{(l)}.$$

Sums of this form are called *special Birkhoff sums*. In convention, we write $S(k)\varphi$ for $S(0, k)\varphi$ and $S(k, k)\varphi = \varphi$. If φ is integrable then

$$(4.1) \quad \|S(k, l)\varphi\|_{L^1(I^{(l)})} \leq \|\varphi\|_{L^1(I^{(k)})} \quad \text{and} \quad \int_{I^{(l)}} S(k, l)\varphi(x) dx = \int_{I^{(k)}} \varphi(x) dx.$$

If additionally $\varphi \in \text{BV}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha^{(k)})$ (is of bounded variation), then

$$(4.2) \quad \text{Var } S(k, l)\varphi \leq \text{Var } \varphi \quad \text{and} \quad \|S(k, l)\varphi\|_{\text{sup}} \leq \|Q(k, l)\| \|\varphi\|_{\text{sup}},$$

where $\text{Var } \varphi$ is the sum of variations of φ restricted to $\text{Int } I_\alpha$ for $\alpha \in \mathcal{A}$.

Denote by $\Gamma^{(k)}$ the set of functions on $I^{(k)}$ which are constant on all $I_\alpha^{(k)}$, $\alpha \in \mathcal{A}$. Clearly, $S(k, l)\Gamma^{(k)} = \Gamma^{(l)}$ and $S(k, l)$ is the linear automorphism of $\mathbb{R}^{\mathcal{A}}$ whose matrix in the canonical basis is $Q(k, l)$.

Remark 4.1. In view of §5 in [11], $S(k, l) : C^{n+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha^{(k)}) \rightarrow C^{n+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha^{(l)})$. Moreover, for every IET T satisfying FFDC, there exists $C \geq 1$ such that for all $0 \leq k \leq l$ and for every function $\varphi \in C^{0+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha^{(k)})$

$$(4.3) \quad \begin{aligned} p_a(S(k, l)\varphi) &\leq C p_a(\varphi) \text{ if } 0 < a < 1, \\ p_a(S(k, l)\varphi) &\leq C(1 + \log \|Q(k, l)\|) p_a(\varphi) \text{ if } a = 0. \end{aligned}$$

4.3. Correction operator on $C^{0+\text{PaG}}$. For any integrable map $f : I \rightarrow \mathbb{R}$ and any subinterval $J \subset I$, let $m(f, J)$ stand for the mean value of f on J , that is

$$m(f, J) = \frac{1}{|J|} \int_J f(x) dx.$$

For the IET $T^{(k)}$ let $\mathcal{M}^{(k)} : L^1(I^{(k)}) \rightarrow \Gamma^{(k)}$ be the corresponding mean value *projection operator* given by

$$\mathcal{M}^{(k)}(f) = \sum_{\alpha \in \mathcal{A}} m(f, I_\alpha^{(k)}) \chi_{I_\alpha^{(k)}}.$$

This operator projects any map onto a piecewise constant function, whose values are equal to the mean value of f on the exchanged intervals $I_\alpha^{(k)}$, $\alpha \in \mathcal{A}$.

Theorem 4.2. [11, Theorem 6.1] *Assume that T satisfies FFDC. For any $0 \leq a < 1$, take $2 \leq j \leq g + 1$ so that $\lambda_1 a < \lambda_{j-1}$. There exists a bounded linear operator $\mathfrak{h}_j : C^{0+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow U_j$ such that for any $\tau > 0$ there exists a constant $C = C_\tau \geq 1$ such that for every $\varphi \in C^{0+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ with $\mathfrak{h}_j(\varphi) = 0$ we have*

$$(4.4) \quad \|\mathcal{M}^{(k)}(S(k)\varphi)\| \leq C \left((K_k^{a,j,\tau} + C_k^{a,j,\tau}) p_a(\varphi) + \|Q_{E_j}(k)\| \frac{\|\varphi\|_{L^1(I^{(0)})}}{|I^{(0)}|} \right).$$

The operator $\mathfrak{h}_j : C^{0+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow U_j \subset H(\pi)$ called the *correction operator* is given by

$$(4.5) \quad \begin{aligned} \mathfrak{h}_j(\varphi) &= \lim_{k \rightarrow \infty} Q(0, k)^{-1} \circ P_{U_j^{(k)}} \circ \mathcal{M}^{(k)} \circ S(k)(\varphi) \\ &= \sum_{l \geq 0} Q(0, l)^{-1} \circ P_{U_j^{(l)}} \circ (\mathcal{M}^{(l)} \circ S(l) - Z(l) \circ \mathcal{M}^{(l-1)} \circ S(l-1))(\varphi) \end{aligned}$$

where $\mathcal{M}^{(-1)} = 0$.

Remark 4.3. Note that for all $2 \leq j' \leq j \leq g+1$ we have $P_{U_{j'}^{(0)}} \circ P_{U_j^{(0)}} = P_{U_{j'}^{(0)}}$. It follows that $P_{U_{j'}^{(0)}} \circ \mathfrak{h}_j = \mathfrak{h}_{j'}$, hence $\mathfrak{h}_j(\varphi) = 0$ implies $\mathfrak{h}_{j'}(\varphi) = 0$. Moreover, by definition, $\mathfrak{h}_j(h) = 0$ for every $h \in E_j$ and $\mathfrak{h}_j(h) = h$ for every $h \in U_j$, in particular $\mathfrak{h}_j \circ \mathfrak{h}_j = \mathfrak{h}_j$.

4.4. First step: correction operator \mathfrak{h}_j^* on BV. As a first step, we construct an initial extended correction operator \mathfrak{h}_j^* on the space of bounded variation functions taking value in the space U_{-j} from the complimentary filtration.

By definition, for every $\varphi \in \text{BV}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha^{(k)})$ we have

$$(4.6) \quad \|\mathcal{M}^{(k)}(\varphi)\| \leq \|\varphi\|_{\text{sup}} \quad \text{and} \quad \|\varphi - \mathcal{M}^{(k)}(\varphi)\|_{\text{sup}} \leq \text{Var } \varphi.$$

Let $P_0^{(k)} : L^1(\sqcup_{\alpha \in \mathcal{A}} I_\alpha^{(k)}) \rightarrow L^1(\sqcup_{\alpha \in \mathcal{A}} I_\alpha^{(k)})$ be a linear operator given by

$$P_0^{(k)}(\varphi) = \varphi - \mathcal{M}^{(k)}(\varphi).$$

If $\varphi \in \text{BV}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$, then

$$(4.7) \quad \|P_0^{(k)}(S(k)\varphi)\|_{\text{sup}} \leq \text{Var}(S(k)\varphi).$$

By §6.1 in [11], for every $0 \leq a < 1$ and $\varphi \in C^{0+\text{Pa}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha^{(k)})$,

$$(4.8) \quad \|\mathcal{M}^{(k)}(\varphi)\|_{L^1(I^{(k)})} \leq 2 \|\varphi\|_{L^1(I^{(k)})}$$

$$(4.9) \quad \|\varphi - \mathcal{M}^{(k)}(\varphi)\|_{L^1(I^{(k)})} \leq \frac{2^{2+a}d}{1-a} p_a(\varphi) |I^{(k)}|^{1-a}.$$

Therefore for $\varphi \in C^{0+\text{Pa}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ we obtain

$$(4.10) \quad \frac{\|S(k)\varphi\|_{L^1(I^{(k)})}}{|I^{(k)}|} \leq \|\mathcal{M}^{(k)}(S(k)\varphi)\| + p_a(S(k)\varphi) \frac{2^{2+a}}{(1-a)|I^{(k)}|^a}.$$

As

$$(4.11) \quad \frac{|I^{(k)}| \|h\|}{\kappa} \leq \min_{\beta \in \mathcal{A}} |I_\beta^{(k)}| \|h\| \leq \|h\|_{L^1(I^{(k)})} \leq |I^{(k)}| \|h\| \quad \text{for every } h \in \Gamma^{(k)},$$

by (4.8), for every $\varphi \in C^{0+\text{Pa}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$,

$$(4.12) \quad \|\mathcal{M}^{(k)}(\varphi)\| \leq \frac{2\kappa}{|I^{(k)}|} \|\varphi\|_{L^1(I^{(k)})}.$$

Lemma 4.4. *Let $0 \leq j \leq g$ and $\varphi \in \text{BV}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ be such that*

$$(4.13) \quad \sum_{l \geq 1} \|Q|_{U_{-j}^{(0)}}(l)^{-1}\| \|Q(l)\|^\tau \|Z(l)\| \text{Var}(S(l-1)\varphi) < +\infty.$$

Then the limit

$$(4.14) \quad \mathfrak{h}_j^*(\varphi) = \lim_{l \rightarrow \infty} Q(0, l)^{-1} \circ P_{U_{-j}^{(l)}} \circ \mathcal{M}^{(l)} \circ S(l)(\varphi) \in U_{-j}$$

exists and there exists a universal constant $C > 0$ such that

$$(4.15) \quad \|\mathfrak{h}_j^*(\varphi)\| \leq C \left(\|\varphi\|_{\sup} + \sum_{l \geq 1} \|Q|_{U_{-j}^{(0)}}(l)^{-1}\| \|Q(l)\|^\tau \|Z(l)\| \text{Var}(S(l-1)\varphi) \right).$$

Moreover, for every $k \geq 1$ we have

$$(4.16) \quad \begin{aligned} & \|\mathcal{M}^{(k)}(S(k)(\varphi - \mathfrak{h}_j^*(\varphi)))\| \\ & \leq C \left(\sum_{l > k} \|Q|_{U_{-j}^{(k)}}(k, l)^{-1}\| \|Q(l)\|^\tau \|Z(l)\| \text{Var}(S(l-1)\varphi) \right. \\ & \quad \left. + \sum_{1 \leq l \leq k} \|Q|_{E_{-j}^{(l)}}(l, k)\| \|Q(l)\|^\tau \|Z(l)\| \text{Var}(S(l-1)\varphi) + \|Q|_{E_{-j}^{(0)}}(k)\| \|\varphi\|_{\sup} \right). \end{aligned}$$

Proof. Let $v_k := \mathcal{M}^{(k)} \circ S(k)(\varphi)$. Direct calculation shows that

$$(4.17) \quad \begin{aligned} & (S(k, k+1) \circ P_0^{(k)} \circ S(k)(\varphi) - P_0^{(k+1)} \circ S(k, k+1) \circ S(k)(\varphi)) \\ & = -S(k, k+1) \circ \mathcal{M}^{(k)} \circ S(k)(\varphi) + \mathcal{M}^{(k+1)} \circ S(k+1)(\varphi) \\ & = -Z(k+1)v_k + v_{k+1}. \end{aligned}$$

Then, by (4.7) and (4.2),

$$\begin{aligned} \|S(k, k+1) \circ P_0^{(k)} \circ S(k)(\varphi)\|_{\sup} & \leq \|Z(k+1)\| \|P_0^{(k)} \circ S(k)(\varphi)\|_{\sup} \\ & \leq \|Z(k+1)\| \text{Var}(S(k)\varphi) \end{aligned}$$

and

$$\|P_0^{(k+1)} \circ S(k+1)(\varphi)\|_{\sup} \leq \text{Var}(S(k+1)\varphi) \leq \text{Var}(S(k)\varphi).$$

This gives

$$(4.18) \quad \|Z(k+1)v_k - v_{k+1}\| \leq 2\|Z(k+1)\| \text{Var}(S(k)\varphi).$$

For any sequence $(x_k)_{k \geq 0}$ in \mathbb{R}^A , let $\Delta x_{k+1} = x_{k+1} - Z(k+1)x_k$ for $k \geq 0$ and $\Delta x_0 = x_0$. Then, by telescoping,

$$(4.19) \quad x_k = \sum_{j=0}^k Q(j, k) \Delta x_j.$$

By (4.6) and (4.18),

$$(4.20) \quad \|\Delta v_0\| \leq \|\varphi\|_{\sup} \text{ and } \|\Delta v_{k+1}\| \leq 2\|Z(k+1)\| \text{Var}(S(k)\varphi).$$

For every $k \geq 0$ let $e_k = P_{E_{-j}^{(k)}} v_k \in E_{-j}^{(k)}$ and $u_k = P_{U_{-j}^{(k)}} v_k \in U_{-j}^{(k)}$. Then $v_k = u_k + e_k$. Since $Z(k+1)(E_{-j}^{(k)}) = E_{-j}^{(k+1)}$ and $Z(k+1)(U_{-j}^{(k)}) = U_{-j}^{(k+1)}$ we have

$$(4.21) \quad \begin{aligned} \Delta u_{k+1} & = u_{k+1} - Z(k+1)u_k = P_{U_{-j}^{(k+1)}} \Delta v_{k+1}, \\ \Delta e_{k+1} & = e_{k+1} - Z(k+1)e_k = P_{E_{-j}^{(k+1)}} \Delta v_{k+1}, \\ \Delta u_0 & = u_0 = P_{U_{-j}^{(0)}} \Delta v_0, \quad \Delta e_0 = e_0 = P_{E_{-j}^{(0)}} \Delta v_0. \end{aligned}$$

In view of (3.16) and (4.20), we have

$$(4.22) \quad \|\Delta u_0\| \leq C \|\Delta v_0\| \leq C \|\varphi\|_{\sup}, \quad \|\Delta e_0\| \leq C \|\Delta v_0\| \leq C \|\varphi\|_{\sup}$$

and for every $k \geq 1$ we have

$$(4.23) \quad \begin{aligned} \|\Delta u_k\| &\leq 2C\|Q(k)\|^\tau \|Z(k)\| \operatorname{Var}(S(k-1)\varphi), \\ \|\Delta e_k\| &\leq 2C\|Q(k)\|^\tau \|Z(k)\| \operatorname{Var}(S(k-1)\varphi). \end{aligned}$$

Let us consider the infinite series $v := \sum_{l \geq 0} Q(l)^{-1} \Delta u_l$. Since

$$(4.24) \quad \begin{aligned} &\sum_{l \geq 0} \|Q|_{U_{-j}^{(0)}}(l)^{-1}\| \|\Delta u_l\| \\ &\leq C(\|\varphi\|_{\sup} + 2 \sum_{l \geq 1} \|Q|_{U_{-j}^{(0)}}(l)^{-1}\| \|Q(l)\|^\tau \|Z(l)\| \operatorname{Var}(S(l-1)\varphi)) \end{aligned}$$

is finite, $v \in U_{-j}$ is well defined. In view of (4.17) and (4.21),

$$\begin{aligned} Q(l)^{-1} \Delta u_l &= Q(l)^{-1} \circ P_{U_{-j}^{(l)}}(\mathcal{M}^{(l)} \circ S(l)(\varphi) - S(l-1, l) \circ \mathcal{M}^{(l-1)} \circ S(l-1)(\varphi)) \\ &= Q(l)^{-1} \circ P_{U_{-j}^{(l)}} \circ \mathcal{M}^{(l)} \circ S(l)(\varphi) - Q(l-1)^{-1} \circ P_{U_{-j}^{(l-1)}} \circ \mathcal{M}^{(l-1)} \circ S(l-1)(\varphi). \end{aligned}$$

It follows that $\mathfrak{h}_j^*(\varphi)$ is well defined and $\mathfrak{h}_j^*(\varphi) = v$, so by (4.24), we obtain (4.15).

By the definition of v , (4.19) and (4.23), for every $k \geq 0$ we have

$$(4.25) \quad \begin{aligned} \|Q(k)v - u_k\| &= \left\| \sum_{l > k} Q|_{U_{-j}^{(k)}}(k, l)^{-1} \Delta u_l \right\| \leq \sum_{l > k} \|Q|_{U_{-j}^{(k)}}(k, l)^{-1}\| \|\Delta u_l\| \\ &\leq 2C \sum_{l > k} \|Q|_{U_{-j}^{(k)}}(k, l)^{-1}\| \|Q(l)\|^\tau \|Z(l)\| \operatorname{Var}(S(l-1)\varphi). \end{aligned}$$

To obtain the bound of norm of $e_k \in E_{-j}^{(k)}$, we apply (4.19), (4.23) and (4.22),

$$\begin{aligned} \|e_k\| &\leq \sum_{0 \leq l \leq k} \|Q(l, k) \Delta e_l\| \leq \sum_{0 \leq l \leq k} \|Q|_{E_{-j}^{(l)}}(l, k)\| \|\Delta e_l\| \\ &\leq C(\|Q|_{E_{-j}^{(0)}}(k)\| \|\varphi\|_{\sup} + 2 \sum_{1 \leq l \leq k} \|Q|_{E_{-j}^{(l)}}(l, k)\| \|Q(l)\|^\tau \|Z(l)\| \operatorname{Var}(S(l-1)\varphi)). \end{aligned}$$

Combining with (4.25), we conclude

$$\begin{aligned} \|Q(k)v - v_k\| &\leq 2C \left(\sum_{l > k} \|Q|_{U_{-j}^{(k)}}(k, l)^{-1}\| \|Q(l)\|^\tau \|Z(l)\| \operatorname{Var}(S(l-1)\varphi) \right. \\ &\quad \left. + \sum_{1 \leq l \leq k} \|Q|_{E_{-j}^{(l)}}(l, k)\| \|Q(l)\|^\tau \|Z(l)\| \operatorname{Var}(S(l-1)\varphi) + \|Q|_{E_{-j}^{(0)}}(k)\| \|\varphi\|_{\sup} \right). \end{aligned}$$

Since $\mathcal{M}^{(k)}(S(k)(\mathfrak{h}_j^*(\varphi))) = Q(k)v$, this gives (4.16). \square

Remark 4.5. Suppose that $0 \leq j \leq j' \leq g$. Then the operator $\mathfrak{h}_{j'}^*$ is well defined and $P_{U_{-j'}^{(0)}} \circ \mathfrak{h}_j^* = \mathfrak{h}_{j'}^*$. Hence $\mathfrak{h}_j^*(\varphi) = 0$ implies $\mathfrak{h}_{j'}^*(\varphi) = 0$. In view of (4.5) and (4.14), the same arguments show that for every $2 \leq l \leq g+1$ we have $P_{U_l^{(0)}} \circ \mathfrak{h}_j^* = \mathfrak{h}_l$. Hence $\mathfrak{h}_j^*(\varphi) = 0$ implies $\mathfrak{h}_l(\varphi) = 0$. Moreover, by definition, $\mathfrak{h}_j^*(h) = 0$ for every $h \in E_{-j}$ and $\mathfrak{h}_j^*(h) = h$ for every $h \in U_{-j}$.

4.5. Second step: correction operator $\mathfrak{h}_{-j,i}$. Now we introduce second type correction operators $\mathfrak{h}_{-j,i} : C^{1+\text{P}_a\text{G}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow U_{-j}$ for $2 \leq i \leq g+1$ and $1 \leq j \leq g$. They extend previous (standard) correction operators \mathfrak{h}_i to the complement of the stable part of the Oseledets filtration. For this purpose, we use a certain modification of the operator \mathfrak{h}_j^* , which we link with the derivative of \mathfrak{h}_i .

For all $0 \leq a < 1$ and $2 \leq i \leq g+1$ let

$$C_i^{1+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) = \{\varphi \in C^{1+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) : \mathfrak{h}_i(D\varphi) = 0\}.$$

Let us consider the sequence $\bar{s} = (s_k)_{k \geq 0}$ given by $s_k := |I^{(k)}|(K_k^{a,i,\tau} + C_k^{a,i,\tau})$.

Theorem 4.6. *Assume that T satisfies FFDC. Let $0 \leq a < 1$, $2 \leq i \leq g+1$ and $1 \leq j \leq g$ so that $a\lambda_1 < \lambda_{i-1}$ and $\max\{a\lambda_1, \lambda_i\} < \lambda_1 - \lambda_{j+1}$. Then the linear operator $\mathfrak{h}_j^* : C_i^{1+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow U_{-j}$ is well defined and bounded. Moreover, for any $0 < \tau < \frac{\max\{a\lambda_1, \lambda_i\} - \lambda_1 + \lambda_{j+1}}{9(1+\lambda_1)}$ there exists a constant $C = C_\tau \geq 1$ such that for any $\varphi \in C_i^{1+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ with $\mathfrak{h}_j^*(\varphi) = 0$ we have*

$$(4.26) \quad \text{Var}(S(k)\varphi) \leq C s_k \|D\varphi\|_{C^{0+\text{Pa}}}$$

$$(4.27) \quad \|\mathcal{M}^{(k)}(S(k)\varphi)\| \leq C((W_k^{j,\tau}(\bar{s}) + V_k^{j,\tau}(\bar{s})) \|D\varphi\|_{C^{0+\text{Pa}}} + \|Q_{E_{-j}^{(0)}}(k)\| \|\varphi\|_{\text{sup}})$$

with

$$\begin{aligned} s_k &= O(e^{(\max\{\lambda_i, \lambda_1 a\} - \lambda_1 + 6\tau(1+\lambda_1))r(0,k+1)}) \\ V_k^{j,\tau}(T, \bar{s}) &= O(e^{(\max\{\lambda_i, \lambda_1 a\} - \lambda_1 + 8\tau(1+\lambda_1))r(0,k)}) \\ W_k^{j,\tau}(T, \bar{s}) &= O(e^{(\max\{\lambda_i - \lambda_1, \lambda_1 a - \lambda_1, -\lambda_j\} + 9\tau(1+\lambda_1))r(0,k)}). \end{aligned}$$

Proof. As $\varphi \in C_i^{1+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$, we have $D\varphi \in C^{0+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ and $\mathfrak{h}_i(D\varphi) = 0$. By (4.10), (4.3), (3.14) and Theorem 4.2,

$$\begin{aligned} \text{Var}(S(k)\varphi) &= \|S(k)(D\varphi)\|_{L^1(I^{(k)})} \\ &\leq |I^{(k)}| \left(\|\mathcal{M}^{(k)}(S(k)D\varphi)\| + p_a(S(k)D\varphi) \frac{2^{a+2}}{(1-a)|I^{(k)}|^a} \right) \\ &\leq C_a |I^{(k)}| \left((K_k^{a,i,\tau} + C_k^{a,i,\tau}) p_a(D\varphi) + \|Q_{E_i}(k)\| \frac{\|D\varphi\|_{L^1(I^{(0)})}}{|I^{(0)}|} \right) \\ &\leq C_a |I^{(k)}| (K_k^{a,i,\tau} + C_k^{a,i,\tau}) \|D\varphi\|_{C^{0+\text{Pa}}} \leq C_\tau s_k \|D\varphi\|_{C^{0+\text{Pa}}}, \end{aligned}$$

which gives (4.26). It follows that

$$\begin{aligned} &\|Q|_{U_{-j}^{(k)}}(k, l)^{-1}\| \|Q(l)\|^\tau \|Z(l)\| \text{Var}(S(l-1)\varphi) \\ &= O(\|Q|_{U_{-j}^{(k)}}(k, l)^{-1}\| \|Q(l)\|^\tau \|Z(l)\|_{s_{l-1}} \|D\varphi\|_{C^{0+\text{Pa}}}), \\ &\|Q|_{E_{-j}^{(l)}}(l, k)\| \|Q(l)\|^\tau \|Z(l)\| \text{Var}(S(l-1)\varphi) \\ &= O(\|Q|_{E_{-j}^{(l)}}(l, k)\| \|Q(l)\|^\tau \|Z(l)\|_{s_{l-1}} \|D\varphi\|_{C^{0+\text{Pa}}}). \end{aligned}$$

In view of (3.17), (3.14) and (3.15), we have

$$\begin{aligned} s_k &= O(e^{-\lambda_1 k} e^{(\max\{\lambda_i, \lambda_1 a\} + 5\tau(1+\lambda_1))r(0,k)}) \\ &= O(e^{-\lambda_1(1-\tau)r(0,k+1)} e^{(\max\{\lambda_i, \lambda_1 a\} + 5\tau(1+\lambda_1))r(0,k)}) \\ &= O(e^{(\max\{\lambda_i, \lambda_1 a\} - \lambda_1 + 6\tau(1+\lambda_1))r(0,k+1)}). \end{aligned}$$

As $\max\{\lambda_i, \lambda_1 a\} + \lambda_{j+1} - \lambda_1 + 6\tau(1+\lambda_1) + \tau(3+\lambda_1) < 0$, by Proposition 3.6,

$$\begin{aligned} V_k^{j,\tau}(T, \bar{s}) &= O(e^{(\max\{\lambda_i, \lambda_1 a\} - \lambda_1 + 8\tau(1+\lambda_1))r(0,k)}) \\ W_k^{j,\tau}(T, \bar{s}) &= O(e^{(\max\{\lambda_i - \lambda_1, \lambda_1 a - \lambda_1, -\lambda_j\} + 9\tau(1+\lambda_1))r(0,k)}). \end{aligned}$$

As $V_0^{j,\tau}(T, \bar{s})$ is finite, the series (4.13) is convergent. By Lemma 4.4 (see (4.15)), the operator $\mathfrak{h}_j^* : C_i^{1+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow U_{-j}$ is well defined and bounded. Moreover, in view of (4.16), this also gives (4.27). \square

For every $\varphi \in L^1(I)$ denote by $\tilde{\varphi} \in AC(I)$ its primitive integral $\tilde{\varphi}(x) = \int_0^x \varphi(y) dy$.

Corollary 4.7. *Assume that T satisfies FFDC. Let $0 \leq a < 1$, $2 \leq i \leq g+1$ and $1 \leq j \leq g$ so that $a\lambda_1 < \lambda_{i-1}$ and $\max\{a\lambda_1, \lambda_i\} < \lambda_1 - \lambda_{j+1}$. There exists a bounded operator $\mathfrak{h}_{-j,i} : C^{1+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow U_{-j}$ such that for every $\varphi \in C^{1+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ with $\mathfrak{h}_{-j,i}(\varphi) = 0$ and $\mathfrak{h}_i(D\varphi) = 0$ we have*

$$(4.28) \quad \|S(k)\varphi\|_{\text{sup}} \leq O(e^{(\max\{\lambda_i - \lambda_1, \lambda_1 a - \lambda_1, -\lambda_j\} + \tau)r(0,k)}) \|\varphi\|_{C^{1+\text{Pa}}} \text{ for every } \tau > 0.$$

Proof. Let $K_i : C^{1+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow C_i^{1+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ be the bounded operator defined by $K_i(\varphi) = \varphi - \mathfrak{h}_i(\widetilde{D\varphi})$. Since $\mathfrak{h}_i(DK_i(\varphi)) = \mathfrak{h}_i(D\varphi) - \mathfrak{h}_i(\mathfrak{h}_i(D\varphi)) = 0$, we really have $K_i(\varphi) \in C_i^{1+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$. We can use Theorem 4.6 to define $\mathfrak{h}_{-j,i} : C^{1+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow U_{-j}$ as $\mathfrak{h}_{-j,i} := \mathfrak{h}_j^* \circ K_i$.

Suppose that $\varphi \in C^{1+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ is such that $\mathfrak{h}_{-j,i}(\varphi) = 0$ and $\mathfrak{h}_i(D\varphi) = 0$. Then $\mathfrak{h}_j^*(K_i(\varphi)) = \mathfrak{h}_{-j,i}(\varphi) = 0$ and $\varphi = K_i(\varphi) + \mathfrak{h}_i(\widetilde{D\varphi}) = K_i(\varphi)$, so $\mathfrak{h}_j^*(\varphi) = 0$. In view of (4.7) and Theorem 4.6,

$$\|S(k)(\varphi)\|_{\text{sup}} \leq \|\mathcal{M}^{(k)}(S(k)\varphi)\| + \text{Var}(S(k)\varphi) = O(e^{(\max\{\lambda_i - \lambda_1, \lambda_1 a - \lambda_1, -\lambda_j\} + \tau)r(0,k)}).$$

\square

Remark 4.8. Suppose that $1 \leq j \leq j' \leq g$ and $2 \leq i' \leq i \leq g+1$. Then the operator $\mathfrak{h}_{-j',i'}$ is well defined. By Remarks 4.3 and 4.5, $\mathfrak{h}_{-j,i}(\varphi) = 0$ and $\mathfrak{h}_i(D\varphi) = 0$ imply $\mathfrak{h}_{-j',i'}(\varphi) = 0$, $\mathfrak{h}_{i'}(D\varphi) = 0$ and $\mathfrak{h}_l(\varphi) = 0$ for every $2 \leq l \leq g+1$. Moreover, by the same remarks, we also have $\mathfrak{h}_{-j,i}(h) = 0$ for every $h \in E_{-j}$ and $\mathfrak{h}_{-j,i}(h) = h$ for every $h \in U_{-j}$, in particular $\mathfrak{h}_{-j,i} \circ \mathfrak{h}_{-j,i} = \mathfrak{h}_{-j,i}$.

4.6. Third step: correction operator \mathfrak{h}_0 . The last correction operator $\mathfrak{h}_0 : C^{2+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow U_0 = \Gamma$ plays the same roles as $\mathfrak{h}_{-j,i}$ but for the parameter $j = 0$. As in the construction of $\mathfrak{h}_{-j,i}$, we also use the operator \mathfrak{h}_j^* (for $j = 0$), but we need to link it with the derivative of $\mathfrak{h}_{-g,2}$ and the second derivative of \mathfrak{h}_2 .

Theorem 4.9. *Assume that T satisfies FFDC. Let $0 \leq a < 1$. There exists a bounded operator $\mathfrak{h}_0 : C^{2+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow U_0$ such that if $\varphi \in C^{2+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ with*

$$\mathfrak{h}_0(\varphi) = 0, \quad \mathfrak{h}_{-g,2}(D\varphi) = 0, \quad \mathfrak{h}_2(D^2\varphi) = 0 \text{ and}$$

$$\|S(k)D\varphi\|_{\text{sup}} = O(e^{-\rho r(0,k)})c(D\varphi) \text{ for some } \rho > 0,$$

then for every $0 < \tau < \min\{\lambda_1 - \lambda_2, \lambda_1(1-a), \lambda_g, \rho\}/3(1 + \max\{\lambda_1, \rho\})$, we have

$$(4.29) \quad \|S(k)\varphi\|_{\text{sup}} = O(e^{(-\rho - \lambda_1 + 2\tau(\lambda_1 + \rho + 1))r(0,k)})c(D\varphi).$$

Proof. Let us consider

$$C_{-g,2}^{2+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) = \{\varphi \in C^{2+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) : \mathfrak{h}_{-g,2}(D\varphi) = 0, \mathfrak{h}_2(D^2\varphi) = 0\}.$$

By Corollary 4.7, for every $\varphi \in C_{-g,2}^{2+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ we have

$$\begin{aligned} \|S(k)(D\varphi)\|_{\text{sup}} &= O(e^{(\max\{\lambda_2 - \lambda_1, \lambda_1 a - \lambda_1, -\lambda_g\} + \tau)r(0,k)}) \|D\varphi\|_{C^{1+\text{Pa}}} \\ &= O(e^{-\rho_0 r(0,k)}) \|D\varphi\|_{C^{1+\text{Pa}}} \end{aligned}$$

with $\rho_0 := \min\{\lambda_1 - \lambda_2, \lambda_1(1 - a), \lambda_g\} - \tau > 0$. As

$$\text{Var}(S(k)\varphi) = \|S(k)D\varphi\|_{L^1(I^{(k)})} \leq |I^{(k)}| \|S(k)D\varphi\|_{\text{sup}},$$

it follows that for $l > k$ we have

$$\begin{aligned} & \|Q|_{U_0^{(k)}}(k, l)^{-1} \| \|Q(l)\|^\tau \|Z(l)\| \text{Var}(S(l-1)\varphi) \\ & \leq \|Q(k, l)^{-1} \| \|Q(l)\|^\tau \|Z(l)\| |I^{(l-1)}| \|S(l-1)D\varphi\|_{\text{sup}} \\ & = O(e^{(\lambda_1+\tau)r(k,l)} e^{\tau(\lambda_1+\tau)l} e^{\tau l} e^{-\lambda_1(l-1)} e^{-\rho_0 r(0, l-1)}) \|D\varphi\|_{C^{1+\text{Pa}}}. \end{aligned}$$

By (3.15), it follows that

$$\begin{aligned} & \|Q|_{U_0^{(k)}}(k, l)^{-1} \| \|Q(l)\|^\tau \|Z(l)\| \text{Var}(S(l-1)\varphi) \\ (4.30) \quad & = O(e^{(\lambda_1+\tau)r(k,l)} e^{\tau(\lambda_1+2)r(0,l)} e^{-(1-\tau)\lambda_1 r(0,l)} e^{-(1-\tau)\rho_0 r(0,l)}) \|D\varphi\|_{C^{1+\text{Pa}}} \\ & = O(e^{(-\lambda_1-\rho_0+\tau(3\lambda_1+2))r(0,k)} e^{(-\rho_0+\tau(3\lambda_1+3))r(k,l)}) \|D\varphi\|_{C^{1+\text{Pa}}}. \end{aligned}$$

The same arguments show that if additionally $\|S(k)D\varphi\|_{\text{sup}} = O(e^{-\rho r(0,k)})c(D\varphi)$ then

$$\begin{aligned} & \|Q|_{U_0^{(k)}}(k, l)^{-1} \| \|Q(l)\|^\tau \|Z(l)\| \text{Var}(S(l-1)\varphi) \\ (4.31) \quad & = O(e^{(-\lambda_1-\rho+\tau(2\lambda_1+\rho+2))r(0,k)} e^{(-\rho+\tau(2\lambda_1+\rho+3))r(k,l)}) c(D\varphi). \end{aligned}$$

As $-\rho_0 + 3\tau(\lambda_1 + 1) < 0$, by (4.30), the series (4.13) is convergent for $j = 0$. By Lemma 4.4, the operator $\mathfrak{h}_0^* : C_{-g,2}^{2+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow U_0 = \Gamma$ is well defined and if $\mathfrak{h}_0^*(\varphi) = 0$ then

$$\|\mathcal{M}^{(k)}(S(k)\varphi)\| \leq C \sum_{l>k} \|Q|_{U_0^{(k)}}(k, l)^{-1} \| \|Q(l)\|^\tau \|Z(l)\| \text{Var}(S(l-1)\varphi).$$

Therefore

$$\begin{aligned} & \|S(k)\varphi\|_{\text{sup}} \leq \|\mathcal{M}^{(k)}(S(k)\varphi)\| + \text{Var}(S(k)\varphi) \\ (4.32) \quad & \leq 2C \sum_{l>k} \|Q|_{U_0^{(k)}}(k, l)^{-1} \| \|Q(l)\|^\tau \|Z(l)\| \text{Var}(S(l-1)\varphi). \end{aligned}$$

Let $K : C^{2+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow C_{-g,2}^{2+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ be the bounded operator defined by

$$K(\varphi) := \varphi - \widetilde{\mathfrak{h}_2(D^2\varphi)} - \widetilde{\mathfrak{h}_{-g,2}(D\varphi)} + \widetilde{\mathfrak{h}_{-g,2}(\mathfrak{h}_2(D^2\varphi))}.$$

Then

$$DK(\varphi) = D\varphi - \widetilde{\mathfrak{h}_2(D^2\varphi)} - \widetilde{\mathfrak{h}_{-g,2}(D\varphi - \mathfrak{h}_2(D^2\varphi))}, \quad D^2K(\varphi) = D^2\varphi - \mathfrak{h}_2(D^2\varphi).$$

Since $\mathfrak{h}_2(D^2K(\varphi)) = \mathfrak{h}_2(D^2\varphi) - \mathfrak{h}_2(\mathfrak{h}_2(D^2\varphi)) = 0$ and

$$\mathfrak{h}_{-g,2}(DK(\varphi)) = \mathfrak{h}_{-g,2}(D\varphi - \widetilde{\mathfrak{h}_2(D^2\varphi)}) - \mathfrak{h}_{-g,2}(\mathfrak{h}_{-g,2}(D\varphi - \widetilde{\mathfrak{h}_2(D^2\varphi)})) = 0,$$

we really have $K(\varphi) \in C_{-g,2}^{2+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$. Finally we define $\mathfrak{h}_0 : C^{2+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow U_0$ as $\mathfrak{h}_0 = \mathfrak{h}_0^* \circ K$.

Suppose that $\varphi \in C^{2+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ is such that $\mathfrak{h}_0(\varphi) = 0$, $\mathfrak{h}_{-g,2}(D\varphi) = 0$, $\mathfrak{h}_2(D^2\varphi) = 0$ and $\|S(k)D\varphi\|_{\text{sup}} = O(e^{-\rho r(0,k)})c(D\varphi)$. Then $\mathfrak{h}_0^*(K(\varphi)) = \mathfrak{h}_0(\varphi) = 0$ with $K(\varphi) = \varphi$, so $\mathfrak{h}_0^*(\varphi) = 0$. In view of (4.32) and (4.31), this gives

$$\|S(k)(\varphi)\|_{\text{sup}} \leq e^{(-\lambda_1-\rho+\tau(2\lambda_1+\rho+2))r(0,k)} O\left(\sum_{l>k} e^{(-\rho+\tau(2\lambda_1+\rho+3))r(k,l)}\right) c(D\varphi).$$

As

$$\sum_{l>k} e^{(-\rho+\tau(2\lambda_1+\rho+3))r(k,l)} \leq \sum_{l\geq 1} e^{(-\rho+\tau(2\lambda_1+\rho+3))l} < +\infty,$$

this gives (4.29). \square

Remark 4.10. Using Remark 4.5 and 4.8, we obtain that $\mathfrak{h}_0(\varphi) = 0$, $\mathfrak{h}_{-g,2}(D\varphi) = 0$ and $\mathfrak{h}_2(D^2\varphi) = 0$ imply $\mathfrak{h}_{-j,i}(\varphi) = 0$ for any pair i, j and $\mathfrak{h}_l(\varphi) = 0$ for any $2 \leq l \leq g+1$. By Remark 4.5, we also have $\mathfrak{h}_0(h) = h$ for every $h \in \Gamma$, in particular $\mathfrak{h}_0 \circ \mathfrak{h}_0 = \mathfrak{h}_0$.

Finally, we prove a fast decay of special Birkhoff sums for $\varphi \in C^{n+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ under some vanishing conditions for derivatives of previously defined correction operators. This result is a key step in the construction of invariant distributions $\mathfrak{f}_{\bar{t}}$ and the proof of the spectral theorem.

Theorem 4.11. *Assume that T satisfies FFDC. Let $0 \leq a < 1$, $2 \leq i \leq g+1$ and $1 \leq j \leq g$ with $a\lambda_1 < \lambda_{i-1}$ and $\max\{a\lambda_1, \lambda_i\} < \lambda_1 - \lambda_{j+1}$. Let $n \geq 1$. Suppose that $\varphi \in C^{n+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ is such that $\mathfrak{h}_{-j,i}(D^{n-1}\varphi) = 0$, $\mathfrak{h}_i(D^n\varphi) = 0$ and $\mathfrak{h}_0(D^l\varphi) = 0$ for all $0 \leq l < n-1$. Then for every small enough $\tau > 0$, we have*

$$(4.33) \quad \|S(k)\varphi\|_{\text{sup}} \leq O(e^{(-n\lambda_1 + \max\{\lambda_i, \lambda_1 a, \lambda_1 - \lambda_j\} + \tau)r(0,k)}) \|\varphi\|_{C^{n+\text{Pa}}}$$

Proof. First we show that $\mathfrak{h}_{-g,2}(D^{l+1}\varphi) = 0$, $\mathfrak{h}_2(D^{l+2}\varphi) = 0$ for all $0 \leq l < n-1$. As $\mathfrak{h}_{-j,i}(D^{n-1}\varphi) = 0$ and $\mathfrak{h}_i(D^n\varphi) = 0$, by Remark 4.8 applied to $D^{n-1}\varphi$, we have $\mathfrak{h}_{-g,2}(D^{n-1}\varphi) = 0$, $\mathfrak{h}_2(D^n\varphi) = 0$ and $\mathfrak{h}_2(D^{n-1}\varphi) = 0$. This gives our claim for $l = n-2$. As $\mathfrak{h}_0(D^{n-2}\varphi) = 0$, by Remark 4.10 applied to $D^{n-2}\varphi$, we obtain $\mathfrak{h}_{-g,2}(D^{n-2}\varphi) = 0$. Together with $\mathfrak{h}_2(D^{n-1}\varphi) = 0$ this gives our claim for $l = n-3$. Repeating the same arguments for lower-order derivatives and using induction, we get our claim for every $0 \leq l < n-1$.

The proof of (4.33) is also done by induction on n . The base case $n = 1$ follows directly from Corollary 4.7. Assume that the induction hypothesis (4.33) holds for a particular $n \geq 1$. Suppose that $\varphi \in C^{n+1+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ is such that $\mathfrak{h}_{-j,i}(D^n\varphi) = 0$, $\mathfrak{h}_i(D^{n+1}\varphi) = 0$ and $\mathfrak{h}_0(D^l\varphi) = 0$ for all $0 \leq l < n$. By the induction hypothesis, applied to $D\varphi$, for every small enough $\tau > 0$, we have

$$\|S(k)D\varphi\|_{\text{sup}} \leq O(e^{(-n\lambda_1 + \max\{\lambda_i, \lambda_1 a, \lambda_1 - \lambda_j\} + \tau)r(0,k)}) \|D\varphi\|_{C^{n+\text{Pa}}}.$$

By assumption and the first part of the proof, $\mathfrak{h}_0(\varphi) = 0$, $\mathfrak{h}_{-g,2}(D\varphi) = 0$, $\mathfrak{h}_2(D^2\varphi) = 0$. In view of Theorem 4.9 applied to $\rho = n\lambda_1 - \max\{\lambda_i, \lambda_1 a, \lambda_1 - \lambda_j\} - \tau$, we get

$$\|S(k)\varphi\|_{\text{sup}} \leq O(e^{(-(n+1)\lambda_1 + \max\{\lambda_i, \lambda_1 a, \lambda_1 - \lambda_j\} + 2(n+1)(\lambda_1+1)\tau)r(0,k)}) \|\varphi\|_{C^{n+1+\text{Pa}}}.$$

\square

5. SPECTRUM OF THE FUNCTIONAL KZ-COCYCLES

Special Birkhoff sums cocycle $S(k)$ is an infinite dimensional extension of the KZ-cocycle. In this section we compute Lyapunov exponents of the cocycle $S(k)$ on $C^{n+\text{Pa}}$. We construct a finite set of piecewise polynomial functions that form the basis for the spectral Theorem 5.6. These piecewise polynomials are obtained by applying correction operators constructed in the previous section and their Lyapunov exponents correspond to Lyapunov exponents of standard KZ-cocycle.

5.1. Lyapunov exponents for piecewise polynomials. For every $l \geq 0$ denote by $\mathbb{R}_l[x]$ the linear space of polynomials of degree not greater than l . Since every linear operator defined on a finite dimensional linear space is bounded, for every $l \geq 0$ there exists a constant $c_l > 0$ such that for every $f \in \mathbb{R}_l[x]$ we have $c_l \|D^l f\|_{C^0([0,1])} \leq \|f\|_{L^1([0,1])}$. Therefore, for every interval $I = [a, b] \subset \mathbb{R}$ we obtain

$$\frac{\|f\|_{L^1(I)}}{|I|} = \|f(a + |I|(\cdot))\|_{L^1([0,1])} \geq c_l \left\| \frac{d^l}{dx^l} f(a + |I|x) \right\|_{C^0([0,1])} = c_l |I|^l \|D^l f\|_{C^0(I)}.$$

For every $l \geq 0$ denote by $\Gamma_l(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ the space of maps $f : I \rightarrow \mathbb{R}$ such that for every $\alpha \in \mathcal{A}$ the restriction of f to I_α belongs to $\mathbb{R}_l[x]$. Then $\Gamma_0 = \Gamma$ and for every $f \in \Gamma_l(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ we have $D^l f \in \Gamma$ and

$$(5.1) \quad \frac{1}{|I|} \|f\|_{L^1(I)} \geq c_l \left(\min_{\alpha \in \mathcal{A}} |I_\alpha| \right)^l \|D^l f\|.$$

Let $h_1, \dots, h_g, c_1, \dots, c_{\gamma-1}, h_{-g}, \dots, h_{-1}$ be a basis of Γ described in Section 3.1. Then

$$(5.2) \quad \lim_{k \rightarrow \infty} \frac{\log \|Q(k)h_i\|}{k} = \lambda_i \text{ for } 1 \leq |i| \leq g, \quad \lim_{k \rightarrow \infty} \frac{\log \|Q(k)c_s\|}{k} = 0 \text{ for } 1 \leq s < \gamma.$$

For every $2 \leq i \leq g+1$ choose $1 \leq j_i \leq g$ such that $\lambda_1 - \lambda_{j_i} \leq \lambda_i < \lambda_1 - \lambda_{j_i+1}$ and for every $1 \leq j \leq g$ choose $2 \leq i_j \leq g+1$ such that $\lambda_{i_j} \leq \lambda_1 - \lambda_j < \lambda_{i_j-1}$.

Definition 6. For every $l \geq 0$ let $h_{i,l}$ for $1 \leq i \leq g$, $c_{s,l}$ for $1 \leq s < \gamma$, and $h_{-j,l}$ for $1 \leq j \leq g$ be elements of $\Gamma_l(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ defined inductively as follows:

$$\begin{aligned} h_{i,0} &= h_i, \quad h_{i,1} = \widetilde{h}_i - \mathfrak{h}_{-j_i,i}(\widetilde{h}_i), \quad h_{i,l+1} = \widetilde{h}_{i,l} - \mathfrak{h}_0(\widetilde{h}_{i,l}) \text{ for } l \geq 1 \text{ if } 2 \leq i \leq g, \\ h_{1,0} &= h_1, \quad h_{1,1} = \widetilde{h}_1 - \mathfrak{h}_{g+1}(\widetilde{h}_1), \quad h_{1,2} = \widetilde{h}_{1,1} - \mathfrak{h}_{-1,g+1}(\widetilde{h}_{1,1}), \\ h_{1,l+1} &= \widetilde{h}_{1,l} - \mathfrak{h}_0(\widetilde{h}_{1,l}) \text{ for } l \geq 2, \\ c_{s,0} &= c_s, \quad c_{s,1} = \widetilde{c}_s - \mathfrak{h}_{-1,g+1}(\widetilde{c}_s), \quad c_{s,l+1} = \widetilde{c}_{s,l} - \mathfrak{h}_0(\widetilde{c}_{s,l}) \text{ for } l \geq 1, \\ h_{-j,0} &= h_{-j}, \quad h_{-j,l+1} = \widetilde{h}_{-j,l} - \mathfrak{h}_0(\widetilde{h}_{-j,l}) \text{ for } l \geq 0. \end{aligned}$$

Since $\mathfrak{h}_0 \circ \mathfrak{h}_0 = \mathfrak{h}_0$, $\mathfrak{h}_{g+1} \circ \mathfrak{h}_{g+1} = \mathfrak{h}_{g+1}$ and $\mathfrak{h}_{-1,g+1} \circ \mathfrak{h}_{-1,g+1} = \mathfrak{h}_{-1,g+1}$, we obtain

$$(5.3) \quad D^n h_{i,l} = h_{i,l-n}, \quad D^n c_{s,l} = c_{s,l-n}, \quad D^n h_{-j,l} = h_{-j,l-n} \text{ if } 0 \leq n \leq l,$$

$$(5.4) \quad \mathfrak{h}_{-j_i,i}(h_{i,1}) = 0, \quad \mathfrak{h}_0(h_{i,l}) = 0 \text{ for } l \geq 2 \text{ if } 2 \leq i \leq g,$$

$$(5.5) \quad \mathfrak{h}_{g+1}(h_{1,1}) = 0, \quad \mathfrak{h}_{-1,g+1}(h_{1,2}) = 0, \quad \mathfrak{h}_0(h_{1,l}) = 0 \text{ for } l \geq 3,$$

$$(5.6) \quad \mathfrak{h}_{-1,g+1}(c_{s,1}) = 0, \quad \mathfrak{h}_0(c_{s,l}) = 0 \text{ for } l \geq 2,$$

$$(5.7) \quad \mathfrak{h}_0(h_{-j,l}) = 0 \text{ for } l \geq 1.$$

In view of (5.3), $h_{i,l}$ for $1 \leq |i| \leq g$, $0 \leq l \leq n$ together with $c_{s,l}$ for $1 \leq s < \gamma$, $0 \leq l \leq n$ is a basis of the space $\Gamma_n(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$. Hence every $h \in \Gamma_n(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ has a unique decomposition

$$h = \sum_{0 \leq l \leq n} \left(\sum_{1 \leq |i| \leq g} d(h, h_{i,l}) h_{i,l} + \sum_{1 \leq s < \gamma} d(h, c_{s,l}) c_{s,l} \right).$$

Lyapunov exponents of $S(k)$ for $h_{i,l}$, $c_{s,l}$ are computed by adapting inductive definitions and using Theorem 4.11. Their lower bounds are obtained by FFDC properties of T .

Proposition 5.1. *Assume that T satisfies FFDC. Then for every $l \geq 0$,*

$$(5.8) \quad \lim_{k \rightarrow \infty} \frac{\log \|S(k)h_{i,l}\|_{\text{sup}}}{k} = \lim_{k \rightarrow \infty} \frac{\log(\|S(k)h_{i,l}\|_{L^1(I^{(k)})} / |I^{(k)}|)}{k} = \lambda_i - l\lambda_1$$

$$\lim_{k \rightarrow \infty} \frac{\log \|S(k)c_{s,l}\|_{\text{sup}}}{k} = \lim_{k \rightarrow \infty} \frac{\log(\|S(k)c_{s,l}\|_{L^1(I^{(k)})} / |I^{(k)}|)}{k} = -l\lambda_1$$

for $i \in \pm\{1, \dots, g\}$ and for $1 \leq s < \gamma$. Moreover, for every $h \in \Gamma_n(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$,

$$(5.9) \quad \lim_{k \rightarrow \infty} \frac{\log \|S(k)h\|_{\text{sup}}}{k} = \max(\{\lambda_i - l\lambda_1 : 0 \leq l \leq n, 1 \leq |i| \leq g, d(h, h_{i,l}) \neq 0\} \\ \cup \{-l\lambda_1 : 0 \leq l \leq n, 1 \leq s < \gamma, d(h, c_{s,l}) \neq 0\}).$$

Proof. If $l = 0$ then (5.8) follows directly from (5.2).

Suppose that $\varphi = h_{-j,l}$ for some $l \geq 1$. Then $\varphi \in C^{l+1+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ with $a = 0$, $\mathfrak{h}_{-j,i_j}(D^l\varphi) = 0$, $\mathfrak{h}_{i_j}(D^{l+1}\varphi) = 0$ and $\mathfrak{h}_0(D^p\varphi) = 0$ for all $0 \leq p < l$. Indeed, as $D^l\varphi = h_{-j} \in E_{-j}$, by Remark 4.8, we have $\mathfrak{h}_{-j}(D^l\varphi) = \mathfrak{h}_{-j}(h_{-j}) = 0$ and $D^{l+1}\varphi = Dh_{-j} = 0$. In view of Theorem 4.11, this gives

$$\limsup_{k \rightarrow \infty} \frac{\log \|S(k)\varphi\|_{\text{sup}}}{k} \leq -(l+1)\lambda_1 + \max\{\lambda_{i_j}, \lambda_1 a, \lambda_1 - \lambda_j\} = -l\lambda_1 - \lambda_j.$$

Suppose that $\varphi = h_{i,l}$ for some $2 \leq i \leq g$ and $l \geq 1$. Then $\varphi \in C^{l+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ with $a = 0$, $\mathfrak{h}_{-j,i,i}(D^{l-1}\varphi) = 0$, $\mathfrak{h}_i(D^l\varphi) = 0$ and $\mathfrak{h}_0(D^p\varphi) = 0$ for all $0 \leq p < l-1$. Indeed, as $D^l\varphi = h_i \in E_i$, by Remark 4.3, we have $\mathfrak{h}_i(D^l\varphi) = 0$. Moreover, by definition, $\mathfrak{h}_{-j,i,i}(D^{l-1}h_{i,l}) = 0$. In view of Theorem 4.11, this gives

$$\limsup_{k \rightarrow \infty} \frac{\log \|S(k)\varphi\|_{\text{sup}}}{k} \leq -l\lambda_1 + \max\{\lambda_i, \lambda_1 a, \lambda_1 - \lambda_j\} = -l\lambda_1 + \lambda_i.$$

Suppose that $\varphi = h_{1,1}$. Then $\varphi \in C^{0+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ with $a = 0$ and $\mathfrak{h}_{g+1}(\varphi) = \mathfrak{h}_{g+1}(h_{1,1}) = 0$. In view of Theorem 4.2 and Proposition 3.5, for every $\tau > 0$ small enough, $\|\mathcal{M}^{(k)}(S(k)\varphi)\| = O(e^{\tau k})$. As $\varphi = h_{1,1}$ is of bounded variation, we also have $\text{Var}(S(k)\varphi) \leq \text{Var}(\varphi)$. Since $\|S(k)\varphi\|_{\text{sup}} \leq \|\mathcal{M}^{(k)}(S(k)\varphi)\| + \text{Var}(S(k)\varphi)$, this gives

$$\limsup_{k \rightarrow \infty} \frac{\log \|S(k)\varphi\|_{\text{sup}}}{k} \leq 0 = -\lambda_1 + \lambda_1.$$

Suppose that $\varphi = h_{1,l}$ for some $l \geq 2$. Then $\varphi \in C^{l-1+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ with $a = 0$, $\mathfrak{h}_{-1,g+1}(D^{l-2}\varphi) = 0$, $\mathfrak{h}_{g+1}(D^{l-1}\varphi) = 0$ and $\mathfrak{h}_0(D^p\varphi) = 0$ for all $0 \leq p < l-2$. In view of Theorem 4.11, this gives

$$\limsup_{k \rightarrow \infty} \frac{\log \|S(k)\varphi\|_{\text{sup}}}{k} \leq -(l-1)\lambda_1 + \max\{\lambda_{g+1}, \lambda_1 a, \lambda_1 - \lambda_1\} = -l\lambda_1 + \lambda_1.$$

Suppose that $\varphi = c_{s,l}$ for some $l \geq 1$. Then $\varphi \in C^{l+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ with $a = 0$, $\mathfrak{h}_{-1,g+1}(D^{l-1}\varphi) = 0$, $\mathfrak{h}_{g+1}(D^l\varphi) = 0$ and $\mathfrak{h}_0(D^p\varphi) = 0$ for all $0 \leq p < l-1$. Indeed, as $D^l\varphi = c_s \in E_{g+1}$, by Remark 4.3, we have $\mathfrak{h}_{g+1}(D^l\varphi) = 0$. In view of Theorem 4.11, this gives

$$\limsup_{k \rightarrow \infty} \frac{\log \|S(k)\varphi\|_{\text{sup}}}{k} \leq -l\lambda_1 + \max\{\lambda_{g+1}, \lambda_1 a, \lambda_1 - \lambda_1\} = -l\lambda_1.$$

In summary, for every $\varphi \in \Gamma_l(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ of the form $h_{i,l}$, $c_{s,l}$ or $h_{-j,l}$ we have $D^l \varphi \in \Gamma$ and

$$(5.10) \quad \limsup_{k \rightarrow \infty} \frac{\log \|S(k)\varphi\|_{\text{sup}}}{k} \leq -l\lambda_1 + \lambda(D^l \varphi) \text{ for } \lambda(D^l \varphi) = \lim_{k \rightarrow \infty} \frac{\log \|Q(k)D^l \varphi\|}{k}.$$

It follows that (5.10) holds also for any $\varphi \in \Gamma_l(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$. On the other hand, if additionally $D^l \varphi \neq 0$ then, by (5.1), (3.12) and (3.14),

$$\frac{1}{|I^{(k)}|} \|S(k)\varphi\|_{L^1(I^{(k)})} \geq c_l \kappa^l |I^{(k)}|^l \|S(k)D^l \varphi\| \geq c_l \kappa^l C^{-l} e^{-(\lambda_1 + \tau)lk} \|Q(k)D^l \varphi\|.$$

It follows that

$$\liminf_{k \rightarrow \infty} \frac{\log(\|S(k)\varphi\|_{L^1(I^{(k)})} / |I^{(k)}|)}{k} \geq -l\lambda_1 + \lambda(D^l \varphi),$$

so

$$\lim_{k \rightarrow \infty} \frac{\log \|S(k)\varphi\|_{\text{sup}}}{k} = \lim_{k \rightarrow \infty} \frac{\log(\|S(k)\varphi\|_{L^1(I^{(k)})} / |I^{(k)}|)}{k} = -l\lambda_1 + \lambda(D^l \varphi).$$

This completes the proof. □

5.2. New functionals arising from correcting operators. In this section, we develop the idea of constructing invariant distributions by decomposing correction operators with respect to the base elements, introduced in [14] and [11, §9.1]. The original idea is to decompose the operator $\mathfrak{h}_i : C^{0+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow U_i$ relative to its base elements h_1, \dots, h_{i-1} of U_i . We extend this idea by taking the decomposition of correction operators $\mathfrak{h}_{-j,i}$ and \mathfrak{h}_0 . Using an inductive procedure, we get a new family of functionals defined on $C^{n+\text{Pa}}$, which in Section 5.3 are adjusted to define invariant distributions \mathfrak{f}_t .

For every $0 \leq a < 1$ let $2 \leq i_a \leq g+1$ and $1 \leq j_a \leq g$ such that $\lambda_{i_a} \leq \lambda_1 a < \lambda_{i_a-1}$ and $\lambda_1 - \lambda_{j_a} \leq \lambda_1 a < \lambda_1 - \lambda_{j_a+1}$. Let us consider the bounded operators $d_{i,0}^+ : C^{0+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow \mathbb{R}$ for $1 \leq i < i_a$ such that for every $\varphi \in C^{0+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$,

$$(5.11) \quad \mathfrak{h}_{i_a}(\varphi) = \sum_{1 \leq i < i_a} d_{i,0}^+(\varphi) h_i.$$

Since $\mathfrak{h}_{i_a} : C^{0+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow U_{i_a}$ is bounded and h_1, \dots, h_{i_a-1} is a basis of U_{i_a} , they are well defined and bounded.

Next let us consider the bounded operators $d_{i,1}^+ : C^{1+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow \mathbb{R}$ for $1 \leq i \leq g$, $d_{s,1}^0 : C^{1+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow \mathbb{R}$ for $1 \leq s < \gamma$, $d_{-j,1}^- : C^{1+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow \mathbb{R}$ for $j_a < j \leq g$, such that for every $\varphi \in C^{1+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$,

$$(5.12) \quad \begin{aligned} & \mathfrak{h}_{-j_a, i_a} \left(\varphi - \sum_{1 \leq i < i_a} d_{i,0}^+(D\varphi) h_{i,1} \right) \\ &= \sum_{1 \leq i \leq g} d_{i,1}^+(\varphi) h_i + \sum_{1 \leq s < \gamma} d_{s,1}^0(\varphi) c_s + \sum_{j_a < j \leq g} d_{-j,1}^-(\varphi) h_{-j}. \end{aligned}$$

Since $\mathfrak{h}_{-j_a, i_a} : C^{1+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow U_{-j_a}$ is bounded and $h_1, \dots, h_g, c_1, \dots, c_s, h_{-g}, \dots, h_{-j_a+1}$ is a basis of U_{-j_a} , they are well defined and bounded.

Next let us consider the bounded operators $d_{i,2}^+ : C^{2+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow \mathbb{R}$ for $1 \leq i \leq g$, $d_{s,2}^0 : C^{2+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow \mathbb{R}$ for $1 \leq s < \gamma$, $d_{-j,2}^- : C^{2+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow \mathbb{R}$ for

$1 \leq j \leq g$, such that for every $\varphi \in C^{2+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$,

$$(5.13) \quad \begin{aligned} & \mathfrak{h}_0 \left(\varphi - \sum_{1 \leq i < i_\alpha} d_{i,0}^+(D^2\varphi)h_{i,2} - \sum_{1 \leq i \leq g} d_{i,1}^+(D\varphi)h_{i,1} - \sum_{1 \leq s < \gamma} d_{s,1}^0(D\varphi)c_{s,1} \right) \\ &= \sum_{1 \leq i \leq g} d_{i,2}^+(\varphi)h_i + \sum_{1 \leq s < \gamma} d_{s,2}^0(\varphi)c_s + \sum_{1 \leq j \leq g} d_{-j,2}^-(\varphi)h_{-j}. \end{aligned}$$

For any $l \geq 3$ let us consider the bounded operators $d_{i,l}^+ : C^{l+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow \mathbb{R}$ for $1 \leq i \leq g$, $d_{s,l}^0 : C^{l+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow \mathbb{R}$ for $1 \leq s < \gamma$, $d_{-j,l}^- : C^{l+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow \mathbb{R}$ for $1 \leq j \leq g$ such that for every $\varphi \in C^{l+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$,

$$(5.14) \quad \begin{aligned} & \mathfrak{h}_0 \left(\varphi - \sum_{1 \leq i \leq g} d_{i,l-2}^+(D^2\varphi)h_{i,2} - \sum_{1 \leq i \leq g} d_{i,l-1}^+(D\varphi)h_{i,1} - \sum_{1 \leq s < \gamma} d_{s,l-1}^0(D\varphi)c_{s,1} \right) \\ &= \sum_{1 \leq i \leq g} d_{i,l}^+(\varphi)h_i + \sum_{1 \leq s < \gamma} d_{s,l}^0(\varphi)c_s + \sum_{1 \leq j \leq g} d_{-j,l}^-(\varphi)h_{-j}. \end{aligned}$$

The following lemma is necessary for proving lower bounds for the growth of the cocycle $S(k)$ in the sense of L^1 -norm.

Lemma 5.2. *Assume that T satisfies FFDC. Let $0 \leq a < 1$ and $n \geq 0$. Then for every $\varphi \in C^{n+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ with $\sum_{\alpha \in \mathcal{A}} (|C_{\alpha,n}^{a,+}(\varphi)| + |C_{\alpha,n}^{a,-}(\varphi)|) > 0$, we have*

$$(5.15) \quad \liminf_{k \rightarrow \infty} \frac{\log(\|S(k)(\varphi)\|_{L^1(I^{(k)})} / |I^{(k)}|)}{k} \geq (a - n)\lambda_1.$$

Proof. By the proof of Theorem 1.1 (see Part V) in [11], if $C_\alpha^\pm(D^n\varphi) \neq 0$ then there exists $\varepsilon > 0$ and a sequence of intervals $\widehat{J}^{(k)} \subset I_\alpha^{(k)}$, $k \geq 1$ such that

$$(5.16) \quad |\widehat{J}^{(k)}| \geq \frac{\varepsilon |I_\alpha^{(k)}|}{4} \quad \text{and} \quad |(S(k)D^n\varphi)(x)| \geq \frac{|C_\alpha^\pm|}{|I_\alpha^{(k)}|^a} \quad \text{for all } x \in \widehat{J}^{(k)} \text{ and } k \geq 1.$$

An elementary argument shows that if $f : I \rightarrow \mathbb{R}$ is a C^1 function such that $|Df(x)| \geq a > 0$ for all $x \in I$, then there exists a subinterval $J \subset I$ such that $|J| \geq |I|/4$ and $|f(x)| \geq a|I|/4$ (see [11, Lemma 4.7]). It follows that for every $n \geq 1$ if $f : I \rightarrow \mathbb{R}$ is a C^n function such that $|D^n f(x)| \geq a > 0$ for all $x \in I$, then there exists a subinterval $J \subset I$ such that $|J| \geq |I|/4^n$ and $|f(x)| \geq a|I|^n/4^{n(n+1)/2}$.

In view of (5.16), it follows that there exists a sequence of intervals $J^{(k)} \subset \widehat{J}^{(k)} \subset I_\alpha^{(k)}$, $k \geq 1$ such that

$$|J^{(k)}| \geq \frac{\varepsilon |I_\alpha^{(k)}|}{4^{n+1}} \quad \text{and} \quad |(S(k)\varphi)(x)| \geq \varepsilon^n \frac{|C_\alpha^\pm|}{|I_\alpha^{(k)}|^a} \frac{|I_\alpha^{(k)}|^n}{4^{n(n+3)/2}} \quad \text{for all } x \in J^{(k)} \text{ and } k \geq 1.$$

Therefore,

$$\frac{1}{|I^{(k)}|} \|S(k)(\varphi)\|_{L^1(I^{(k)})} \geq \varepsilon^{n+1} \frac{1}{|I^{(k)}|} \frac{|C_\alpha^\pm|}{|I_\alpha^{(k)}|^a} \frac{|I_\alpha^{(k)}|^{n+1}}{4^{(n+1)^2}}.$$

By (3.12) and (3.14),

$$\frac{|I_\alpha^{(k)}|^{n+1-a}}{|I^{(k)}|} \geq \kappa^{n+1-a} |I^{(k)}|^{n-a} \geq \kappa^{n+1-a} C^{a-n} e^{-(\lambda_1 + \tau)(n-a)k}.$$

This gives (5.15). □

In the following theorem, we prove the first version of the spectral result for the cocycle $S(k)$ on $C^{n+\text{PaG}}$. Any map $\varphi \in C^{n+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ is decomposed with respect to the base elements $h_{i,l}, c_{s,l}, h_{-j,l}$ with weights determined by the derivatives of the functionals defined at the beginning of the subsection. The main tool of the proof is again Theorem 4.11.

Theorem 5.3. *Assume that T satisfies FFDC. For any $0 \leq a < 1$ and $n \geq 1$ there exists a bounded operator $\mathbf{r}_{a,n} : C^{n+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow C^{n+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ such that for every $\varphi \in C^{n+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$,*

$$\begin{aligned} \varphi &= \mathbf{r}_{a,n}(\varphi) + \sum_{1 \leq i < i_a} d_{i,0}^+(D^n \varphi) h_{i,n} \\ &+ \sum_{1 \leq i \leq g} d_{i,1}^+(D^{n-1} \varphi) h_{i,n-1} + \sum_{1 \leq s < \gamma} d_{s,1}^0(D^{n-1} \varphi) c_{s,n-1} + \sum_{j_a < j \leq g} d_{-j,1}^-(D^{n-1} \varphi) h_{-j,n-1} \\ &+ \sum_{2 \leq l \leq n} \left(\sum_{1 \leq i \leq g} d_{i,l}^+(D^{n-l} \varphi) h_{i,n-l} + \sum_{1 \leq s < \gamma} d_{s,l}^0(D^{n-l} \varphi) c_{s,n-l} + \sum_{1 \leq j \leq g} d_{-j,l}^-(D^{n-l} \varphi) h_{-j,n-l} \right) \end{aligned}$$

and for any $\tau > 0$,

$$(5.17) \quad \|S(k) \mathbf{r}_{a,n}(\varphi)\|_{\text{sup}} \leq O(e^{\lambda_1(a-n+\tau)k}) \|\mathbf{r}_{a,n}(\varphi)\|_{C^{n+\text{Pa}}}.$$

If additionally $\sum_{\alpha \in \mathcal{A}} (|C_{\alpha,n}^{a,+}(\varphi)| + |C_{\alpha,n}^{a,-}(\varphi)|) > 0$ then

$$(5.18) \quad \lim_{k \rightarrow \infty} \frac{\log \|S(k) \mathbf{r}_{a,n}(\varphi)\|_{\text{sup}}}{k} = \lim_{k \rightarrow \infty} \frac{\log \frac{\|S(k) \mathbf{r}_{a,n}(\varphi)\|_{L^1(I(k))}}{|I(k)|}}{k} = (a-n)\lambda_1.$$

Proof. In view of (5.3), for every $0 \leq m \leq n-1$,

$$\begin{aligned} D^m \mathbf{r}_{a,n}(\varphi) &= D^m \varphi - \sum_{1 \leq i < i_a} d_{i,0}^+(D^n \varphi) h_{i,n-m} \\ &- \sum_{1 \leq i \leq g} d_{i,1}^+(D^{n-1} \varphi) h_{i,n-1-m} - \sum_{1 \leq s < \gamma} d_{s,1}^0(D^{n-1} \varphi) c_{s,n-1-m} - \sum_{j_a < j \leq g} d_{-j,1}^-(D^{n-1} \varphi) h_{-j,n-1-m} \\ &- \sum_{2 \leq l \leq n-m} \left(\sum_{1 \leq i \leq g} d_{i,l}^+(D^{n-l} \varphi) h_{i,n-l-m} - \sum_{1 \leq s < \gamma} d_{s,l}^0(D^{n-l} \varphi) c_{s,n-l-m} - \sum_{1 \leq j \leq g} d_{-j,l}^-(D^{n-l} \varphi) h_{-j,n-l-m} \right). \end{aligned}$$

Suppose that $0 \leq m \leq n-3$. Since $\mathfrak{h}_0(h_{i,l}) = 0$ for $l \geq 3$ (see (5.5)), $\mathfrak{h}_0(c_{s,l}) = 0$ for $l \geq 2$ (see (5.6)), $\mathfrak{h}_0(h_{-j,l}) = 0$ for $l \geq 1$ (see (5.7)) and $\mathfrak{h}_0(h) = h$ for $h \in \Gamma$ (see Remark 4.10), it follows that

$$\begin{aligned} \mathfrak{h}_0(D^m \mathbf{r}_{a,n}(\varphi)) &= \mathfrak{h}_0 \left(D^m \varphi - \sum_{1 \leq i \leq g} d_{i,n-m-2}^+(D^{m+2} \varphi) h_{i,2} \right. \\ &- \sum_{1 \leq i \leq g} d_{i,n-m-1}^+(D^{m+1} \varphi) h_{i,1} - \sum_{1 \leq s < \gamma} d_{s,n-m-1}^0(D^{m+1} \varphi) c_{s,1} \\ &- \left. \sum_{1 \leq i \leq g} d_{i,n-m}^+(D^m \varphi) h_i - \sum_{1 \leq s < \gamma} d_{s,n-m}^0(D^m \varphi) c_s - \sum_{1 \leq j \leq g} d_{-j,n-m}^-(D^m \varphi) h_{-j} \right). \end{aligned}$$

In view of (5.14), this gives $\mathfrak{h}_0(D^m \mathbf{r}_{a,n}(\varphi)) = 0$. The same arguments show that

$$\begin{aligned} \mathfrak{h}_0(D^{n-2} \mathbf{r}_{a,n}(\varphi)) &= \mathfrak{h}_0\left(D^{n-2} \varphi - \sum_{1 \leq i < i_a} d_{i,0}^+(D^n \varphi) h_{i,2} \right. \\ &\quad \left. - \sum_{1 \leq i \leq g} d_{i,1}^+(D^{n-1} \varphi) h_{i,1} - \sum_{1 \leq s < \gamma} d_{s,1}^0(D^{n-1} \varphi) c_{s,1} \right) \\ &\quad - \sum_{1 \leq i \leq g} d_{i,2}^+(D^{n-2} \varphi) h_i - \sum_{1 \leq s < \gamma} d_{s,2}^0(D^{n-2} \varphi) c_s - \sum_{1 \leq j \leq g} d_{-j,2}^-(D^{n-2} \varphi) h_{-j}. \end{aligned}$$

In view of (5.13), this gives $\mathfrak{h}_0(D^{n-2} \mathbf{r}_{a,n}(\varphi)) = 0$.

Next we pass to the $n-1$ -th derivative,

$$\begin{aligned} D^{n-1} \mathbf{r}_{a,n}(\varphi) &= D^{n-1} \varphi - \sum_{1 \leq i < i_a} d_{i,0}^+(D^n \varphi) h_{i,1} \\ &\quad - \sum_{1 \leq i \leq g} d_{i,1}^+(D^{n-1} \varphi) h_i - \sum_{1 \leq s < \gamma} d_{s,1}^0(D^{n-1} \varphi) c_s - \sum_{j_a < j \leq g} d_{-j,1}^-(D^{n-1} \varphi) h_{-j}. \end{aligned}$$

In view of (5.12), this gives $\mathfrak{h}_{-j_a, i_a}(D^{n-1} \mathbf{r}_{a,n}(\varphi)) = 0$.

Finally we pass to the n -th derivative,

$$D^n \mathbf{r}_{a,n}(\varphi) = D^n \varphi - \sum_{1 \leq i < i_a} d_{i,0}^+(D^n \varphi) h_i.$$

In view of (5.11), this gives $\mathfrak{h}_{i_a}(D^n \mathbf{r}_{a,n}(\varphi)) = 0$.

Since $\max\{\lambda_{i_a}, a\lambda_1, \lambda_1 - \lambda_{j_a}\} = a\lambda_1$, by Theorem 4.11, for any $\tau > 0$,

$$\|S(k) \mathbf{r}_{a,n}(\varphi)\|_{\text{sup}} = O(e^{\lambda_1(a-n+\tau)k}) \|\mathbf{r}_{a,n}(\varphi)\|_{C^{n+P_a}}.$$

The final lower bound in (5.18) follows directly from Lemma 5.2. \square

Remark 5.4. Theorem 5.3 remains true also in the case when $n = 0$, except that in formulas (5.17) and (5.18) we must replace the sup norm by the L^1 norm. Here, $\mathbf{r}_{a,0} : C^{0+P_a\text{G}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow C^{0+P_a\text{G}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ is given by

$$\mathbf{r}_{a,0}(\varphi) = \varphi - \sum_{1 \leq i < i_a} d_{i,0}^+(\varphi) h_i,$$

so $\mathfrak{h}_{i_a}(\mathbf{r}_{a,0}(\varphi)) = 0$ for every $\varphi \in C^{0+P_a\text{G}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$. By Theorem 4.2 and Proposition 3.5, for any $\tau > 0$,

$$\|\mathcal{M}^{(k)}(S(k)(\mathbf{r}_{a,0}(\varphi)))\| = O(e^{(a\lambda_1+\tau)k}) \|\mathbf{r}_{a,0}(\varphi)\|_{C^{0+P_a}}.$$

In view of (4.10) and (4.3), it follows that

$$\|S(k)(\mathbf{r}_{a,0}(\varphi))\|_{L^1(I^{(k)})} / |I^{(k)}| = O(e^{(a\lambda_1+\tau)k}) \|\mathbf{r}_{a,0}(\varphi)\|_{C^{0+P_a}}.$$

The lower bound follows again directly from Lemma 5.2.

5.3. Invariant distributions on $C^{n+P_a\text{G}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$. For every $0 \leq a < 1$ and $n \geq 0$ denote by $\mathcal{T}_{a,n}^*$ ($\mathcal{T}_{a,n}$ resp.) the subset of triples $\bar{t} \in \mathcal{T}\mathcal{F}^*$ ($\mathcal{T}\mathcal{F}$ resp.) of the form $(l, +, i)$, $(l, 0, s)$ or $(l, -, j)$ such that $0 \leq l \leq n$ with the additional restriction that

- if $l = n$ then we deal only with $(n, +, i)$ for $1 \leq i < i_a$;
- if $l = n-1$ then we deal only with $(n-1, +, i)$ for all $1 \leq i \leq g$, $(n-1, 0, s)$ for all $1 \leq s < \gamma$ and $(n-1, -, j)$ for $j_a < j \leq g$.

Recall that $\mathcal{T}\mathcal{F}$ is the subset of triples in $\mathcal{T}\mathcal{F}^*$ after removing all triples of the form $(l, -, 1)$.

Remark 5.5. By definition,

$$\bar{t} \in \mathcal{T}_{a,n}^* \iff \mathfrak{o}(\bar{t}) \leq (n - \frac{\lambda_{i_a-1}}{\lambda_1}) \vee (n - 1 + \frac{\lambda_{j_a+1}}{\lambda_1}).$$

As $\lambda_{i_a} \leq \lambda_1 a < \lambda_{i_a-1}$ and $\lambda_{j_a+1} < \lambda_1(1-a) \leq \lambda_{j_a}$, it follows that

$$(5.19) \quad \bar{t} \in \mathcal{T}_{a,n}^* \iff \mathfrak{o}(\bar{t}) < n - a.$$

Definition 7. For every $\bar{t} \in \mathcal{T}_{a,n}^*$ let $\mathfrak{f}_{\bar{t}} : C^{n+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow \mathbb{C}$ and $h_{\bar{t}} \in \Gamma_n(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ be defined as follows:

- $\mathfrak{f}_{\bar{t}} = d_{i,n-l}^+ \circ D^l$ and $h_{\bar{t}} := h_{i,l}$ if $\bar{t} = (l, +, i)$;
- $\mathfrak{f}_{\bar{t}} = d_{s,n-l}^0 \circ D^l$ and $h_{\bar{t}} := c_{s,l}$ if $\bar{t} = (l, 0, s)$;
- $\mathfrak{f}_{\bar{t}} = d_{-j,n-l}^- \circ D^l$ and $h_{\bar{t}} := h_{-j,l}$ if $\bar{t} = (l, -, j)$.

Theorem 5.6. *Assume that T satisfies FFDC. Then given $0 \leq a < 1$ and $n \geq 0$, every $\varphi \in C^{n+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ is decomposed as follows:*

$$(5.20) \quad \varphi = \sum_{\bar{t} \in \mathcal{T}_{a,n}^*} \mathfrak{f}_{\bar{t}}(\varphi) h_{\bar{t}} + \mathfrak{r}_{a,n}(\varphi),$$

so that for any $\tau > 0$ and for all $0 \leq l < n$,

$$(5.21) \quad \|S(k)(D^l \mathfrak{r}_{a,n}(\varphi))\|_{\text{sup}} = O(e^{(-\lambda_1(n-l-a)+\tau)k}) \|D^l \mathfrak{r}_{a,n}(\varphi)\|_{C^{n-l+\text{Pa}}},$$

$$(5.22) \quad \|S(k)(D^n \mathfrak{r}_{a,n}(\varphi))\|_{L^1(I^{(k)})} / |I^{(k)}| = O(e^{(\lambda_1 a + \tau)k}) \|D^n \mathfrak{r}_{a,n}(\varphi)\|_{C^{0+\text{Pa}}} \text{ and}$$

$$(5.23) \quad \lim_{k \rightarrow \infty} \frac{1}{k} \log \left\| S(k) \sum_{\bar{t} \in \mathcal{T}_{a,n}^*} a_{\bar{t}} h_{\bar{t}} \right\|_{\text{sup}} = -\lambda_1 \min\{\mathfrak{o}(\bar{t}) : \bar{t} \in \mathcal{T}_{a,n}^*, a_{\bar{t}} \neq 0\}.$$

If additionally $\sum_{\alpha \in \mathcal{A}} (|C_{\alpha,n}^{a,+}(\varphi)| + |C_{\alpha,n}^{a,-}(\varphi)|) > 0$ then

$$(5.24) \quad \lim_{k \rightarrow \infty} \frac{1}{k} \log \|S(k)(D^l \mathfrak{r}_{a,n}(\varphi))\|_{\text{sup}} = -\lambda_1(n-l-a) \text{ for } 0 \leq l < n \text{ and}$$

$$(5.25) \quad \lim_{k \rightarrow \infty} \frac{1}{k} \log (\|S(k)(D^l \mathfrak{r}_{a,n}(\varphi))\|_{L^1(I^{(k)})} / |I^{(k)}|) = -\lambda_1(n-l-a) \text{ for } 0 \leq l \leq n.$$

Moreover, for each $\bar{t} \in \mathcal{T}_{a,n}^*$ the functional $\mathfrak{f}_{\bar{t}} : C^{n+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow \mathbb{C}$ is invariant, i.e. for every $\varphi \in C^{n+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ such that $\varphi = v \circ T - v$ for some $v \in C^r(I)$ with $\mathfrak{o}(\bar{t}) < r \leq n-a$, we have $\mathfrak{f}_{\bar{t}}(\varphi) = 0$. Also, the functionals $C_{\alpha,n}^{a,\pm} : C^{n+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow \mathbb{C}$ are invariant, i.e. if $\varphi = v \circ T - v$ for some $v \in C^r(I)$ with $n-a < r$, then $C_{\alpha,n}^{a,\pm}(\varphi) = 0$ for every $\alpha \in \mathcal{A}$.

Proof. All claims of the theorem, in addition to invariance, are derived directly from Proposition 5.1, Theorem 5.3 and Remark 5.4, so we focus only on invariance.

Suppose that $\varphi = v \circ T - v$ for some $v \in C^r(I)$ with $r \leq n-a$. Let $r = m + b$ with an integer $0 \leq m < n$ and $0 < b \leq 1$. By (5.20), for every $0 \leq j \leq n$,

$$(5.26) \quad D^j(\varphi - \mathfrak{r}_{a,n}(\varphi)) = \sum_{\bar{t} \in \mathcal{T}_{a,n}^*} \mathfrak{f}_{\bar{t}}(\varphi) D^j h_{\bar{t}}.$$

Then for every $0 \leq j \leq m$ we have $D^j v \in C^{m-j+b}(I)$ and for every $x \in I_\alpha^{(k)}$,

$$|S(k) D^j \varphi(x)| = |D^j v(T^{Q_\alpha^{(k)}}(x)) - D^j v(x)| \leq \begin{cases} \|D^j v\|_{C^1} |I^{(k)}| & \text{if } 0 \leq j < m \\ \|D^m v\|_{C^b} |I^{(k)}|^b & \text{if } j = m. \end{cases}$$

This also gives

$$\left| \int_{I_\alpha^{(k)}} S(k) D^{m+1} \varphi(x) dx \right| = |S(k) D^m \varphi(r_\alpha^{(k)}) - S(k) D^m \varphi(l_\alpha^{(k)})| \leq 2 \|v\|_{C^{m+b}} |I^{(k)}|^b.$$

As $|I^{(k)}| = O(e^{-\lambda_1 k})$, $|I_\alpha^{(k)}|^{-1} = O(|I^{(k)}|^{-1})$ and $|I^{(k)}|^{-1} = O(e^{(\lambda_1 + \tau)k})$ for every $\tau > 0$, we obtain

$$\begin{aligned} \limsup_{k \rightarrow \infty} \frac{1}{k} \log \|S(k) D^j \varphi\|_{\text{sup}} &\leq -\lambda_1 \text{ if } j < m; \\ \limsup_{k \rightarrow \infty} \frac{1}{k} \log \|S(k) D^m \varphi\|_{\text{sup}} &\leq -b\lambda_1; \\ \limsup_{k \rightarrow \infty} \frac{1}{k} \log \|\mathcal{M}^{(k)}(S(k) D^{m+1} \varphi)\| &\leq (1-b)\lambda_1. \end{aligned}$$

As $m < n$, in view of (5.21) and (5.22), it follows that

$$(5.27) \quad \limsup_{k \rightarrow \infty} \frac{1}{k} \log \|S(k)(D^j(\varphi - \mathbf{r}_{a,n}(\varphi)))\|_{\text{sup}} \leq -\lambda_1 \text{ if } 0 \leq j < m;$$

$$(5.28) \quad \limsup_{k \rightarrow \infty} \frac{1}{k} \log \|S(k)(D^m(\varphi - \mathbf{r}_{a,n}(\varphi)))\|_{\text{sup}} \leq -b\lambda_1;$$

$$(5.29) \quad \limsup_{k \rightarrow \infty} \frac{1}{k} \log \|\mathcal{M}^{(k)}(S(k)(D^{m+1}(\varphi - \mathbf{r}_{a,n}(\varphi))))\| \leq (1-b)\lambda_1.$$

In view of (5.26), $\tilde{\varphi} = D^{m+1}(\varphi - \mathbf{r}_{a,n}(\varphi)) \in \Gamma_{n-m-1}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$. Therefore, by (4.6) and (4.2),

$$\|S(k)\tilde{\varphi}\|_{\text{sup}} \leq \|\mathcal{M}^{(k)}(S(k)\tilde{\varphi})\| + \text{Var}(S(k)\tilde{\varphi}) \leq \|\mathcal{M}^{(k)}(S(k)\tilde{\varphi})\| + \text{Var} \tilde{\varphi}.$$

In view of (5.29), this gives

$$(5.30) \quad \limsup_{k \rightarrow \infty} \frac{1}{k} \log \|S(k)(D^{m+1}(\varphi - \mathbf{r}_{a,n}(\varphi)))\|_{\text{sup}} \leq (1-b)\lambda_1.$$

On the other hand, by (5.26) and (5.9),

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{1}{k} \log \|S(k)(D^j(\varphi - \mathbf{r}_{a,n}(\varphi)))\|_{\text{sup}} &= \lim_{k \rightarrow \infty} \frac{1}{k} \log \|S(k) \sum_{\bar{t} \in \mathcal{T}_{a,n}^*} \mathbf{f}_{\bar{t}}(\varphi) D^j h_{\bar{t}}\|_{\text{sup}} \\ &= \lambda_1 \max \left\{ -\mathbf{o}(\bar{t}) + j : \bar{t} \in \mathcal{T}_{a,n}^*, \mathbf{f}_{\bar{t}}(\varphi) \neq 0, D^j h_{\bar{t}} \neq 0 \right\}. \end{aligned}$$

In view of (5.27), (5.28), (5.30), this yields

$$(5.31) \quad \min \left\{ \mathbf{o}(\bar{t}) : \bar{t} \in \mathcal{T}_{a,n}^*, \mathbf{f}_{\bar{t}}(\varphi) \neq 0, D^l h_{\bar{t}} \neq 0 \right\} \geq l + 1 \text{ if } 0 \leq l < m,$$

$$(5.32) \quad \min \left\{ \mathbf{o}(\bar{t}) : \bar{t} \in \mathcal{T}_{a,n}^*, \mathbf{f}_{\bar{t}}(\varphi) \neq 0, D^m h_{\bar{t}} \neq 0 \right\} \geq m + b,$$

$$(5.33) \quad \min \left\{ \mathbf{o}(\bar{t}) : \bar{t} \in \mathcal{T}_{a,n}^*, \mathbf{f}_{\bar{t}}(\varphi) \neq 0, D^{m+1} h_{\bar{t}} \neq 0 \right\} \geq m + b.$$

Let $\bar{t} \in \mathcal{T}_{a,n}$ be any triple such that $\mathbf{o}(\bar{t}) < r = m + b$. By definition, $\mathbf{f}_{\bar{t}}$, $h_{\bar{t}}$ and $\mathbf{o}(\bar{t})$ are of the form:

$$\begin{aligned} \mathbf{f}_{\bar{t}} &= d_{i,n-l}^+ \circ D^l, \quad h_{\bar{t}} = h_{i,l} \text{ and } \mathbf{o}(\bar{t}) = l - \frac{\lambda_i}{\lambda_1} \text{ or} \\ \mathbf{f}_{\bar{t}} &= d_{s,n-l}^0 \circ D^l, \quad h_{\bar{t}} = c_{s,l} \text{ and } \mathbf{o}(\bar{t}) = l \text{ or} \\ \mathbf{f}_{\bar{t}} &= d_{-j,n-l}^- \circ D^l, \quad h_{\bar{t}} = h_{-j,l} \text{ and } \mathbf{o}(\bar{t}) = l + \frac{\lambda_j}{\lambda_1} \text{ with } j \neq 1 \end{aligned}$$

for $0 \leq l \leq m+1$. If $0 \leq l < m$, then $D^l h_{\bar{t}} \neq 0$ and $\mathfrak{o}(\bar{t}) \leq l + \lambda_2/\lambda_1 < l+1$. Then, by (5.31), $\mathfrak{f}_{\bar{t}}(\varphi) = 0$. If $l = m$ or $m+1$, then $D^l h_{\bar{t}} \neq 0$ and $\mathfrak{o}(\bar{t}) < m+b$. Then, by (5.32) and (5.33), $\mathfrak{f}_{\bar{t}}(\varphi) = 0$ as well. This completes the proof of invariance for the functionals $\mathfrak{f}_{\bar{t}}$, $\bar{t} \in \mathcal{T}_{a,n}$.

Suppose that $\varphi = v \circ T - v$ for some $v \in C^r(I)$ with $r > n-a$.

Assume that $0 < a < 1$. Then $D^{n-1}\varphi = D^{n-1}v \circ T - D^{n-1}v$ with $D^{n-1}v \in C^{1-a+\tau}(I)$, where $0 < \tau < (r-n+a) \wedge a$. Therefore, $D^{n-1}\varphi$ is $(1-a+\tau)$ -Hölder on any interval I_α , $\alpha \in \mathcal{A}$. Suppose, contrary to our claim, that $C_\alpha^+(D^n\varphi) = C_{\alpha,n}^{a,+}(\varphi) \neq 0$. Then there exists $\varepsilon > 0$ such that

$$0 < c := |C_\alpha^+(D^n\varphi)|/2 \leq |D^{n+1}\varphi(x)||x - l_\alpha|^{1+a} \text{ for } x \in (l_\alpha, l_\alpha + \varepsilon].$$

Hence, for every $x \in (l_\alpha, l_\alpha + \varepsilon]$,

$$\begin{aligned} \left| \frac{c}{a(x-l_\alpha)^a} - \frac{c}{a\varepsilon^a} \right| &= \int_x^{l_\alpha+\varepsilon} \frac{c}{(s-l_\alpha)^{1+a}} ds \leq \left| \int_x^{l_\alpha+\varepsilon} D^{n+1}\varphi(s) ds \right| \\ &\leq |D^n\varphi(x) - D^n\varphi(l_\alpha + \varepsilon)|. \end{aligned}$$

It follows that there exists $0 < \delta < \varepsilon$ such that

$$\frac{c}{2a(x-l_\alpha)^a} \leq |D^n\varphi(x)| \text{ for } x \in (l_\alpha, l_\alpha + \delta].$$

Hence, for every $x, y \in (l_\alpha, l_\alpha + \delta]$,

$$\begin{aligned} \frac{c}{2a(1-a)} |(y-l_\alpha)^{1-a} - (x-l_\alpha)^{1-a}| &= \int_x^y \frac{c}{2a(s-l_\alpha)^a} ds \leq \left| \int_x^y D^n\varphi(s) ds \right| \\ &\leq |D^{n-1}\varphi(x) - D^{n-1}\varphi(y)| \leq \|D^{n-1}\varphi\|_{C^{1-a+\tau}} |(y-l_\alpha) - (x-l_\alpha)|^{1-a+\tau}. \end{aligned}$$

It follows that $c \leq 2a(1-a)\|D^{n-1}\varphi\|_{C^{1-a+\tau}} s^\tau$ for every $s \in (0, \delta]$, contrary to $|C_\alpha^+(D^n\varphi)| = 2c > 0$. This gives $C_{\alpha,n}^{a,+}(\varphi) = C_\alpha^+(D^n\varphi) = 0$ and the same arguments also show that $C_{\alpha,n}^{a,-}(\varphi) = C_\alpha^-(D^n\varphi) = 0$.

If $a = 0$ then the proof runs in the same way. In this case $D^n\varphi = D^n v \circ T - D^n v$ with $D^n v \in C^\tau(I)$, where $0 < \tau < (r-n) \wedge 1$. Therefore, $D^n\varphi$ is τ -Hölder on any I_α , $\alpha \in \mathcal{A}$. Suppose that $C_{\alpha,n}^{a,+}(\varphi) \neq 0$. As in the previous case, there exists $\varepsilon > 0$ such that

$$0 < c := |C_\alpha^+(D^n\varphi)|/2 \leq |D^{n+1}\varphi(x)||x - l_\alpha| \text{ for } x \in (l_\alpha, l_\alpha + \varepsilon].$$

Hence, for every $x, y \in (l_\alpha, l_\alpha + \varepsilon]$,

$$\begin{aligned} c |\log(y-l_\alpha) - \log(x-l_\alpha)| &= \int_x^y \frac{c}{s-l_\alpha} ds \leq \left| \int_x^y D^{n+1}\varphi(s) ds \right| \\ &\leq |D^n\varphi(x) - D^n\varphi(y)| \leq \|D^n\varphi\|_{C^\tau} |(y-l_\alpha) - (x-l_\alpha)|^\tau. \end{aligned}$$

It follows that $c \log 2 \leq \|D^n\varphi\|_{C^\tau} s^\tau$ for every $s \in (0, \varepsilon/2]$, contrary to $|C_\alpha^+(D^n\varphi)| = 2c > 0$. This completes the proof. \square

Lemma 5.7. *The decomposition (5.20) is unique, i.e. if*

$$\varphi = \sum_{\bar{t} \in \mathcal{T}_{a,n}^*} a_{\bar{t}} h_{\bar{t}} + \tilde{\varphi} \quad \text{with} \quad \limsup_{k \rightarrow \infty} \frac{1}{k} \log \|S(k)\tilde{\varphi}\|_{\text{sup}} \leq -\lambda_1(n-a),$$

then $a_{\bar{t}} = \mathfrak{f}_{\bar{t}}(\varphi)$ for every $\bar{t} \in \mathcal{T}_{a,n}^*$. In particular, $\mathfrak{f}_{\bar{t}}(\mathfrak{r}_{a,n}(\varphi)) = 0$ for every $\bar{t} \in \mathcal{T}_{a,n}^*$.

Proof. By assumption, $\tilde{\varphi} - \mathbf{r}_{a,n}(\varphi) = \sum_{\bar{t} \in \mathcal{I}_{a,n}^*} (\mathbf{f}_{\bar{t}}(\varphi) - a_{\bar{t}})h_{\bar{t}}$ and

$$\limsup_{k \rightarrow \infty} \frac{1}{k} \log \|S(k)(\tilde{\varphi} - \mathbf{r}_{a,n}(\varphi))\|_{\text{sup}} \leq -\lambda_1(n - a).$$

On the other hand, by (5.23),

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log \left\| S(k) \sum_{\bar{t} \in \mathcal{I}_{a,n}^*} (\mathbf{f}_{\bar{t}}(\varphi) - a_{\bar{t}})h_{\bar{t}} \right\|_{\text{sup}} = -\lambda_1 \min\{\mathbf{o}(\bar{t}) : \bar{t} \in \mathcal{I}_{a,n}^*, \mathbf{f}_{\bar{t}}(\varphi) \neq a_{\bar{t}}\}.$$

In view of (5.19), both give $a_{\bar{t}} = \mathbf{f}_{\bar{t}}(\varphi)$ for every $\bar{t} \in \mathcal{I}_{a,n}^*$. \square

Remark 5.8. Let us consider two pairs $(n_1, a_1), (n_2, a_2)$ such that $n_1 - a_1 < n_2 - a_2$. Then $C^{n_2+P_{a_2}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \subset C^{n_1+P_{a_1}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ and $\mathcal{I}_{a_1, n_1}^* \subset \mathcal{I}_{a_2, n_2}^*$. Suppose that $\bar{t}_1 \in \mathcal{I}_{a_1, n_1}^*, \bar{t}_2 \in \mathcal{I}_{a_2, n_2}^*$ are such that $\bar{t}_1 = \bar{t}_2$. By Lemma 5.7, $\mathbf{f}_{\bar{t}_1} : C^{n_1+P_{a_1}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow \mathbb{C}$ is an extension of $\mathbf{f}_{\bar{t}_2} : C^{n_2+P_{a_2}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow \mathbb{C}$.

6. SOLVING COHOMOLOGICAL EQUATIONS ON IET

Given $\varphi \in C^{n+P_a G}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$, we provide a smooth solution v (whose some derivative is Hölder) of the cohomological equation $v \circ T - v = \varphi$ for the IET T provided that the sequence $S(k)\varphi$ decays fast enough. Combining this with the spectral result (Theorem 5.6), we get a regularity of the solutions depending on the vanishing of the invariant distributions $\mathbf{f}_{\bar{t}}$. Main estimates for regularity are carried out by decompositions of orbits and space decompositions invented by Marmi-Moussa-Yoccoz [18, §2.2.3] and [20, §3.7-8].

Time decomposition.

- Let T an IET satisfying Keane's condition, $x \in I$ and $N \geq 1$. Let y be the point of the orbit $(T^j x)_{0 \leq j < N}$ which is closest to 0.
- We split the orbit into positive/negative parts $(T^j y)_{0 \leq j < N^+}$ and $(T^j y)_{N^- \leq j < 0}$, where $N = N^+ - N^-$.
- Let $k \geq 0$ be the largest number such that at least one element of $(T^j y)_{0 < j < N^+}$ belongs to $I^{(k)}$.
- Let $y, T^{(k)}y, \dots, (T^{(k)})^{q(k)}y$ be all points of $(T^j y)_{0 \leq j < N^+}$ that belong to $I^{(k)}$ for some $q(k) > 0$. Let $y(k) := y$.
- We define $y(l), q(l)$ inductively backward for $0 \leq l < k$. Let $y(k-1) = (T^{(k)})^{q(k)}(y)$ and let $y(l) = T^{N^{(l)}}(y)$ be the last point of the orbit $(T^j y)_{0 \leq j < N}$ which belongs to $I^{(l+1)}$. Let $y(l), T^{(l)}(y(l)), \dots, (T^{(l)})^{q(l)}(y(l)) := y(l-1)$ be all points of $(T^j y)_{N^{(l)} \leq j < N^+}$ that belong to $I^{(l)}$ for some $q(l) \geq 0$.

Then,

$$(6.1) \quad \sum_{0 \leq j < N^+} \varphi(T^j y) = \sum_{l=0}^k \sum_{0 \leq j < q(l)} S(l)\varphi((T^{(l)})^j(y(l))) \text{ with } q(l) \leq \|Z(l+1)\|.$$

The negative part of the orbit is divided in a similar way.

Space decomposition. Recall the partition into Rokhlin towers in § 2.3

$$I = \bigcup_{\alpha \in \mathcal{A}} \bigcup_{i=0}^{Q_\alpha(k)-1} T^i(I_\alpha^{(k)}).$$

- For any pair $x_- < x_+$ of points in I , let $k \geq 0$ be the smallest integer such that (x_-, x_+) contains at least of one interval of the k -th partition.
- Let $J^{(k)}(1), \dots, J^{(k)}(q(k))$ be all intervals of the k -th partition contained in (x_-, x_+) . Then $0 < q(k) \leq \|Z(k)\|$.
- For every $l \geq k$, let $x_+(l) < x_+$ be the largest end point of an interval of the l -th partition. Then $x_+(l) \geq x_+(l-1)$ for any $l > k$.
- For any $l > k$ the interval $(x_+(l-1), x_+(l))$ is the union of intervals $J_+^{(l)}(1), \dots, J_+^{(l)}(q_+(l))$ of the l -th partition for some $0 \leq q_+(l) \leq \|Z(l)\|$.
- The point $x_-(l)$, $0 \leq q_-(l) \leq \|Z(l)\|$ and intervals $J_-^{(l)}(1), \dots, J_-^{(l)}(q_-(l))$ of the l -th partition are defined in the similar way.

This yields the following decomposition of (x_-, x_+) :

$$(6.2) \quad (x_-, x_+) = \bigcup_{1 \leq q \leq q(k)} J^{(k)}(q) \cup \bigcup_{l > k} \bigcup_{\epsilon = \pm} \bigcup_{1 \leq q \leq q_\epsilon(l)} J_\epsilon^{(l)}(q).$$

6.1. Hölder solutions. In this section solutions of the cohomological equation $v \circ T - v = \varphi$ are obtained by applying standard Gottschalk-Hedlund arguments for $\varphi \in C^{1+\text{PaG}}$. A Hölder regularity of solutions follows from exponential decay of $S(k)\varphi$ and some bounds on the growth of $S(k)D\varphi$.

Lemma 6.1. *Suppose that $0 \leq a < 1$ and $\varphi \in C^{1+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ is such that for any $\tau > 0$ we have $\|S(k)\varphi\|_{\text{sup}} = O(e^{(-\lambda_1(1-a)+\tau)k})c_1(\varphi)$. Then there exists a continuous solution $v \in C^0(I)$ of the cohomological equation $\varphi = v \circ T - v$ such that $v(0) = 0$ and*

$$(6.3) \quad \sup\{|v(x) - v(y)| : x, y \in I\} \leq 2 \sum_{l=0}^{\infty} \|Z(l+1)\| \|S(l)\varphi\|_{\text{sup}}.$$

Proof. In view of (6.1) for any $n \in \mathbb{N}$,

$$\|\varphi^{(n)}\|_{\text{sup}} \leq 2 \sum_{l=0}^{\infty} \|Z(l+1)\| \|S(l)\varphi\|_{\text{sup}}.$$

As $\|Z(l+1)\| = O(e^{\tau l})$ and $\|S(l)\varphi\|_{\text{sup}} = O(e^{(-\lambda_1(1-a)+\tau)l})c_1(\varphi)$, the series on the right side of the inequality converges and the n -th Birkhoff sums of φ are uniformly bounded. By classical Gottschalk-Hedlund type arguments (see [19, Theorem 3.4]), the cohomological equation has a continuous solution v . Moreover, for any $x \in I$ and $n \geq 1$,

$$|v(T^n x) - v(x)| = |\varphi^{(n)}(x)| \leq 2 \sum_{l=0}^{\infty} \|Z(l+1)\| \|S(l)\varphi\|_{\text{sup}}.$$

As the orbit $\{T^n x\}_{n \geq 0}$ is dense and v is continuous, this gives (6.3).

Since the function v is unique up to an additive constant, it can be always chosen so that $v(0) = 0$. In what follows, we will always deal with solutions satisfying $v(0) = 0$. \square

For any interval $J \subset I$, let $\text{osc}(v, J) := \sup\{|v(x) - v(y)| : x, y \in J\}$.

Corollary 6.2. *Let $\varphi \in C^{1+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ be such that for any $\tau > 0$ we have $\|S(k)\varphi\|_{\text{sup}} = O(e^{(-\lambda_1(1-a)+\tau)k})c_1(\varphi)$. Then for every $\tau > 0$,*

$$(6.4) \quad \text{osc}(v, I^{(k)}) = O(e^{(-\lambda_1(1-a)+\tau)k})c_1(\varphi).$$

Proof. As $\varphi = v \circ T - v$, for every $k \geq 0$ we have $S(k)\varphi = v \circ T^{(k)} - v$ on $I^{(k)}$. Then, by (6.3) applied to $T^{(k)} : I^{(k)} \rightarrow I^{(k)}$, we have

$$\text{osc}(v, I^{(k)}) = \sup\{|v(x) - v(y)| : x, y \in I^{(k)}\} \leq 2 \sum_{l \geq k}^{\infty} \|Z(l+1)\| \|S(l)\varphi\|_{\text{sup}}.$$

As $\|Z(l+1)\| = O(e^{\tau l})$ and $\|S(l)\varphi\|_{\text{sup}} = O(e^{(-\lambda_1(1-a)+\tau)l})c_1(\varphi)$, this gives (6.4). \square

The following elementary calculations will be used in estimating $\text{osc}(v, T^i(I_\alpha^{(k)}))$ for $1 \leq i < Q_\alpha(k)$ in Lemma 6.4.

Lemma 6.3. *Let $\varphi \in C^{0+\text{P}_a\text{G}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$. Then for every $\alpha \in \mathcal{A}$ and any Borel set $J \subset I_\alpha$,*

$$(6.5) \quad \int_J |\varphi(x)| dx \leq \begin{cases} \frac{\|\varphi\|_{L^1(I)}|J|}{|I|} + \frac{2^{a+3}p_a(\varphi)|J|^{1-a}}{a(1-a)} & \text{if } 0 < a < 1, \\ \frac{\|\varphi\|_{L^1(I)}|J|}{|I|} + 4p_a(\varphi)|J|(1 + \log \frac{|J|}{|I|}) & \text{if } a = 0. \end{cases}$$

Proof. By Remark 2.1 in [12], for any $x \in \text{Int } I_\alpha$,

$$|\varphi(x)| \leq \frac{\|\varphi\|_{L^1}}{|I|} + p_a(\varphi) \left(\frac{1}{a \min\{x - l_\alpha, r_\alpha - x\}^a} + \frac{2^{a+2}}{a(1-a)|I_\alpha|^a} \right) \text{ if } 0 < a < 1,$$

$$|\varphi(x)| \leq \frac{\|\varphi\|_{L^1}}{|I|} + p_a(\varphi) \left(\log \frac{|I_\alpha|}{2 \min\{x - l_\alpha, r_\alpha - x\}} + 2 \right) \text{ if } a = 0.$$

It follows that if $0 < a < 1$ then

$$\int_J |\varphi(x)| dx \leq \frac{\|\varphi\|_{L^1(I)}|J|}{|I|} + \frac{2^{a+2}p_a(\varphi)|J|}{a(1-a)|I|^a} + \frac{2p_a(\varphi)}{a} \int_0^{|J|} x^{-a} dx$$

and if $a = 0$ then

$$\int_J |\varphi(x)| dx \leq \frac{\|\varphi\|_{L^1(I)}|J|}{|I|} + 2p_a(\varphi)|J| - 2p_a(\varphi) \int_0^{|J|} \log(x/|I|) dx.$$

This gives (6.5). \square

Lemma 6.4. *Suppose that $\varphi \in C^{1+\text{P}_a\text{G}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ is such that for any $\tau > 0$ we have $\|S(k)\varphi\|_{\text{sup}} = O(e^{(-\lambda_1(1-a)+\tau)k})c_1(\varphi)$ and $\frac{\|S^{(k)}D\varphi\|_{L^1(I^{(k)})}}{|I^{(k)}|} = O(e^{(\lambda_1 a + \tau)k})c_0(D\varphi)$. Then for any $k \geq 0$, $\alpha \in \mathcal{A}$ and $0 \leq N < Q_\alpha(k)$,*

$$(6.6) \quad \text{osc}(v, T^N(I_\alpha^{(k)})) = \text{osc}(v, I_\alpha^{(k)}) + O(e^{(-\lambda_1(1-a)+\tau)k})(c_0(D\varphi) + p_a(D\varphi)).$$

Proof. Since $\varphi = v \circ T - v$, by telescoping, for any $x_1, x_2 \in I_\alpha^{(k)}$

$$v(T^N x_2) - v(T^N x_1) - (v(x_2) - v(x_1)) = \varphi^{(N)}(x_2) - \varphi^{(N)}(x_1) = \int_{x_1}^{x_2} \sum_{i=0}^{N-1} D\varphi(T^i x) dx.$$

Hence

$$(6.7) \quad \text{osc}(v, T^N(I_\alpha^{(k)})) \leq \text{osc}(v, I_\alpha^{(k)}) + \int_{I_\alpha^{(k)}} \left| \sum_{i=0}^{N-1} D\varphi(T^i x) \right| dx.$$

In view of (6.1), for every $x \in I_\alpha^{(k)}$ we have

$$(6.8) \quad \sum_{i=0}^{N-1} D\varphi(T^i x) = \sum_{l=0}^k \sum_{0 \leq i < q(l)} S(l)D\varphi((T^{(l)})^i x(l))$$

with $0 \leq q(l) \leq \|Z(l+1)\|$ and $I_\alpha^{(k)} \ni x \mapsto x(l) \in J_l \subset I^{(l)}$ is a translation and J_l is the image of $I_\alpha^{(k)}$ by this translation. It follows that

$$(6.9) \quad \int_{I_\alpha^{(k)}} \left| \sum_{i=0}^{N-1} D\varphi(T^i x) \right| dx \leq \sum_{l=0}^k \sum_{0 \leq i < q(l)} \int_{(T^{(l)})^i J_l} |S(l)D\varphi(x)| dx.$$

Assume that $0 < a < 1$. As $|(T^{(l)})^i J_l| = |J_l| = |I_\alpha^{(k)}|$, in view of (6.5),

$$\int_{(T^{(l)})^i J_l} |S(l)D\varphi(x)| dx \leq \frac{\|S(l)D\varphi\|_{L^1(I^{(l)})} |I_\alpha^{(k)}|}{|I^{(l)}|} + \frac{2^{a+3} p_a(S(l)D\varphi) |I_\alpha^{(k)}|^{1-a}}{a(1-a)}.$$

By (4.3), there exists $C > 0$ such that

$$(6.10) \quad \begin{aligned} p_a(S(l)D\varphi) &\leq C p_a(D\varphi) \text{ if } 0 < a < 1, \\ p_a(S(l)D\varphi) &\leq C(1 + \log \|Q(l)\|) p_a(D\varphi) \text{ if } a = 0. \end{aligned}$$

As $\frac{\|S(l)D\varphi\|_{L^1(I^{(l)})}}{|I^{(l)}|} = O(e^{(\lambda_1 a + \tau)l}) c_0(D\varphi)$ and $|I^{(k)}| = O(e^{-\lambda_1 k})$, it follows that

$$(6.11) \quad \int_{(T^{(l)})^i J_l} |S(l)D\varphi(x)| dx = O(e^{(-\lambda_1(1-a) + \tau)k}) (c_0 + p_a)(D\varphi).$$

If $a = 0$ then, by (6.5),

$$\int_{(T^{(l)})^i J_l} |S(l)D\varphi(x)| dx \leq \frac{\|S(l)D\varphi\|_{L^1(I^{(l)})} |I_\alpha^{(k)}|}{|I^{(l)}|} + 4p_a(S(l)D\varphi) |I_\alpha^{(k)}| (1 + \log \frac{|I^{(l)}|}{|I_\alpha^{(k)}|}).$$

In view of (3.14), $\log |I^{(l)}| / |I_\alpha^{(k)}| \leq \log |I| / |I_\alpha^{(k)}| = \log O(e^{(\lambda_1 + \tau)k}) = O(e^{\tau k})$ and $\log \|Q(l)\| = O(e^{\tau k})$ for $l \leq k$, and by (6.10) we also get (6.11) when $a = 0$. By (6.9), this gives

$$\begin{aligned} \int_{I_\alpha^{(k)}} \left| \sum_{i=0}^{N-1} D\varphi(T^i x) \right| dx &= k \|Z(k+1)\| O(e^{(-\lambda_1(1-a) + \tau)k}) (c_0 + p_a)(D\varphi) \\ &= O(e^{(-\lambda_1(1-a) + 3\tau)k}) (c_0 + p_a)(D\varphi). \end{aligned}$$

In view of (6.7), this gives (6.6). □

By combining previous lemmas, under a decaying condition on $S(k)\varphi$ and some bound on the growth of $S(k)D\varphi$, a Hölder solution of the cohomological equation is obtained.

Theorem 6.5. *Suppose that $\varphi \in C^{1+P_a G}(\sqcup_{\alpha \in A} I_\alpha)$ is such that for any $\tau > 0$ we have $\|S(k)\varphi\|_{\text{sup}} = O(e^{(-\lambda_1(1-a) + \tau)k}) c_1(\varphi)$ and $\frac{\|S(k)D\varphi\|_{L^1(I^{(k)})}}{|I^{(k)}|} = O(e^{(\lambda_1 a + \tau)k}) c_0(D\varphi)$. There exists a continuous solution $v : I \rightarrow \mathbb{R}$ of the cohomological equation $\varphi = v \circ T - v$ such that $v(0) = 0$ and for any $0 < \tau < 1 - a$ we have $v \in C^{(1-a) - \tau}(I)$. Moreover, there exists $C_\tau > 0$ such that $\|v\|_{C^{(1-a) - \tau}} \leq C_\tau (c_1(\varphi) + c_0(D\varphi) + p_a(D\varphi))$.*

Proof. For any pair $x < y$ of points in I we use the space decomposition of the interval (x, y) introduced in the beginning of the section. Then

$$|v(y) - v(x)| \leq \sum_{q=1}^{q(k)} \text{osc}(v, J^{(k)}(q)) + \sum_{l>k} \sum_{\epsilon=\pm} \sum_{q=1}^{q_\epsilon(l)} \text{osc}(v, J_\epsilon^{(l)}(q))$$

with $q(k) \leq \|Z(k)\|$ and $q_{\pm}(l) \leq \|Z(l)\|$. As each $J^{(k)}(q)$ is of the form $T^n I_{\alpha}^{(k)}$ for some $0 \leq n < Q_{\alpha}(k)$ and each $J_{\pm}^{(l)}(q)$ is of the form $T^n I_{\alpha}^{(l)}$ for some $0 \leq n < Q_{\alpha}(l)$, in view of Corollary 6.2 and Lemma 6.4, for any $\tau > 0$,

$$\begin{aligned} \text{osc}(v, J^{(k)}(q)) &\leq O(e^{(-\lambda_1(1-a)+\tau)k})(c_1(\varphi) + c_0(D\varphi) + p_a(D\varphi)), \\ \text{osc}(v, J_{\pm}^{(l)}(q)) &\leq O(e^{(-\lambda_1(1-a)+\tau)l})(c_1(\varphi) + c_0(D\varphi) + p_a(D\varphi)). \end{aligned}$$

It follows that

$$|v(y) - v(x)| \leq O\left(\sum_{l \geq k} \|Z(l)\| e^{(-\lambda_1(1-a)+\tau)l}\right)(c_1(\varphi) + c_0(D\varphi) + p_a(D\varphi)).$$

As $\|Z(l)\| = O(e^{\tau l})$, we obtain

$$|v(y) - v(x)| \leq O(e^{(-\lambda_1(1-a)+2\tau)k})(c_1(\varphi) + c_0(D\varphi) + p_a(D\varphi)).$$

By the choice of k , $|y - x| \geq \min_{\alpha \in \mathcal{A}} |I_{\alpha}^{(k)}| \geq c_{\tau} e^{-(\lambda_1+\tau)k}$ for some $c_{\tau} > 0$. It follows that

$$|v(y) - v(x)| \leq O(1)(c_1(\varphi) + c_0(D\varphi) + p_a(D\varphi))|y - x|^{\frac{\lambda_1(1-a)-2\tau}{\lambda_1+\tau}}.$$

As $v(0) = 0$, this completes the proof. \square

6.2. Higher regularity. Higher regularity of solutions is obtained by applying Theorem 6.5 as the initial step of induction.

Theorem 6.6. *Let $n \geq 1$ and $0 \leq a < 1$. Assume that T satisfies the FFDC. Let $\varphi \in C^{n+P_a G}(\sqcup_{\alpha \in \mathcal{A}} I_{\alpha})$ be a map such that for any $\tau > 0$ we have*

$$(6.12) \quad \|S(k)D^l \varphi\|_{\text{sup}} = O(e^{(-\lambda_1(n-l-a)+\tau)k}) \|D^l \varphi\|_{C^{n-l+P_a}} \text{ for } 0 \leq l < n$$

and

$$(6.13) \quad \frac{1}{|I^{(k)}|} \|S(k)D^n \varphi\|_{L^1(I^{(k)})} = O(e^{(\lambda_1 a + \tau)k}) \|D^n \varphi\|_{C^{0+P_a}}.$$

Then there exists a C^{n-1} -solution $v : I \rightarrow \mathbb{R}$ of the cohomological equation $\varphi = v \circ T - v$ such that $v(0) = 0$ and for any $0 < \tau < 1 - a$ we have $v \in C^{n-a-\tau}(I)$. Moreover, there exists $C_{\tau, n} > 0$ such that $\|v\|_{C^{n-a-\tau}} \leq C_{\tau, n} \|\varphi\|_{C^{n+P_a}}$.

Proof. The proof is by induction on n . For $n = 1$, our claim follows from Theorem 6.5 applied to $c_1(\varphi) = \|\varphi\|_{C^{1+P_a}}$ and $c_0(D\varphi) = \|D\varphi\|_{C^{0+P_a}}$.

Suppose that for some $n \geq 1$ if $\varphi \in C^{n+P_a G}(\sqcup_{\alpha \in \mathcal{A}} I_{\alpha})$ satisfies (6.12) and (6.13) then there exists a C^{n-1} -solution v of the cohomological equation such that for any $\tau > 0$ we have $v \in C^{n-a-\tau}(I)$ and $\|v\|_{C^{n-a-\tau}} \leq C_{\tau, n} \|\varphi\|_{C^{n+P_a}}$.

Let $\varphi \in C^{n+1+P_a G}(\sqcup_{\alpha \in \mathcal{A}} I_{\alpha})$ be such that

$$\begin{aligned} \|S(k)D^l \varphi\|_{\text{sup}} &= O(e^{(-\lambda_1(n+1-l-a)+\tau)k}) \|D^l \varphi\|_{C^{n+1-l+P_a}} \text{ for } 0 \leq l \leq n \text{ and} \\ \frac{1}{|I^{(k)}|} \|S(k)D^{n+1} \varphi\|_{L^1(I^{(k)})} &= O(e^{(\lambda_1 a + \tau)k}) \|D^{n+1} \varphi\|_{C^{0+P_a}}. \end{aligned}$$

It follows that $D\varphi \in C^{n+P_a G}(\sqcup_{\alpha \in \mathcal{A}} I_{\alpha})$ satisfies (6.12) and (6.13). By induction hypothesis, there exists $v_0 \in C^{n-1}(I)$ such that $D\varphi = v_0 \circ T - v_0$, $v_0(0) = 0$ and for any $\tau > 0$ we have $v_0 \in C^{n-a-\tau}(I)$ with $\|v_0\|_{C^{n-a-\tau}} \leq C_{\tau, n} \|D\varphi\|_{C^{n+P_a}}$. By integrating, there exists $\chi \in \Gamma$ that satisfies $\varphi = \tilde{v}_0 \circ T - \tilde{v}_0 + \chi$ (recall that $\tilde{v}_0(x) = \int_0^x v_0(s) ds$). Note that for any $k \geq 1$,

$$S(k)\varphi = S(k)(\tilde{v}_0 \circ T - \tilde{v}_0) + Q(k)\chi.$$

By assumption,

$$(6.14) \quad \begin{aligned} \|S(k)\varphi\|_{\text{sup}} &= O(e^{(-\lambda_1(n+1-a)+\tau)k})\|\varphi\|_{C^{n+1+P_a}} \\ &\leq O(e^{-\lambda_1 k} e^{(-\lambda_1(1-a)+\tau)k})\|\varphi\|_{C^{n+1+P_a}} = O(e^{-\lambda_1 k})\|\varphi\|_{C^{n+1+P_a}}. \end{aligned}$$

On the other hand, for any $x \in I_\alpha^{(k)}$,

$$(6.15) \quad |S(k)(\tilde{v}_0 \circ T - \tilde{v}_0)(x)| = |\tilde{v}_0(T^{Q_\alpha(k)}x) - \tilde{v}_0(x)| \leq \|v_0\|_{\text{sup}}|x - T^{Q_\alpha(k)}x|.$$

It follows that

$$\|S(k)(\tilde{v}_0 \circ T - \tilde{v}_0)\|_{\text{sup}} \leq \|v_0\|_{\text{sup}}|I^{(k)}| = O(e^{-\lambda_1 k})\|D\varphi\|_{C^{n+P_a}}.$$

Therefore, $\|Q(k)\chi\| = O(e^{-\lambda_1 k})\|\varphi\|_{C^{n+1+P_a}}$. In view of (3.2), $\chi \in E_{-1}(\pi, \lambda)$. As $E_{-1}(\pi, \lambda)$ is one-dimensional, by Remark 3.4, $\chi = c(\bar{\xi} - \bar{\xi} \circ T)$ for some $c = c(\varphi) \in \mathbb{R}$ (recall that $\bar{\xi}(x) = x$). Note that $|c(\varphi)| \leq \|v_0\|_{\text{sup}}$. Indeed, by (6.15),

$$\|S(k)(\tilde{v}_0 \circ T - \tilde{v}_0)\|_{\text{sup}} \leq \|v_0\|_{\text{sup}} \|S(k)(\bar{\xi} - \bar{\xi} \circ T)\|_{\text{sup}}.$$

As $\frac{1}{k} \log \|S(k)(\bar{\xi} - \bar{\xi} \circ T)\|_{\text{sup}} \rightarrow -\lambda_1$, in view of (6.14), we obtain $\|S(k)\varphi\|_{\text{sup}} = o(\|S(k)(\bar{\xi} - \bar{\xi} \circ T)\|_{\text{sup}})$. It follows that

$$\begin{aligned} |c(\varphi)| \|S(k)(\bar{\xi} - \bar{\xi} \circ T)\|_{\text{sup}} &= \|S(k)\chi\|_{\text{sup}} \leq \|S(k)\varphi\|_{\text{sup}} + \|S(k)(\tilde{v}_0 \circ T - \tilde{v}_0)\|_{\text{sup}} \\ &\leq (\|v_0\|_{\text{sup}} + o(1)) \|S(k)(\bar{\xi} - \bar{\xi} \circ T)\|_{\text{sup}}. \end{aligned}$$

Hence $|c(\varphi)| \leq \|v_0\|_{\text{sup}}$.

Let $v : I \rightarrow \mathbb{R}$, $v = \tilde{v}_0 - c(\varphi)\bar{\xi}$. Then $\varphi = v \circ T - v$ and $v \in C^{n+1-a-\tau}(I)$ with

$$\|Dv\|_{C^{n-a-\tau}} = \|v_0\|_{C^{n-a-\tau}} + |c(\varphi)| \leq \|v_0\|_{C^{n-a-\tau}} + \|v_0\|_{\text{sup}} \leq 2C_{\tau,n}\|D\varphi\|_{C^{n+P_a}}.$$

As $v(0) = 0$, this gives

$$\begin{aligned} \|v\|_{C^{n+1-a-\tau}} &= \|v\|_{\text{sup}} + \|Dv\|_{C^{n-a-\tau}} \leq |I|\|Dv\|_{\text{sup}} + \|Dv\|_{C^{n-a-\tau}} \\ &\leq (|I| + 1)\|Dv\|_{C^{n-a-\tau}} \leq 2(|I| + 1)C_{\tau,n}\|D\varphi\|_{C^{n+P_a}}. \end{aligned}$$

This completes the proof. □

Corollary 6.7. *For every $n \geq 0$ there exists a polynomial $v_n \in \mathbb{R}_{n+1}[x]$ such that $h_{-1,n} = v_n \circ T - v_n$ and $v_n(0) = 0$.*

For every $\bar{t} \in \mathcal{T}\mathcal{F}$ if $\mathfrak{o}(\bar{t}) > r > 0$, then there exists $v_{\bar{t}} \in C^r(I)$ such that $h_{\bar{t}}(0) = 0$ and $h_{\bar{t}} = v_{\bar{t}} \circ T - v_{\bar{t}}$.

Proof. In view of (5.3) and (5.8), for every $0 \leq l \leq n$ we have $D^l h_{-1,n} = h_{-1,n-l}$, and

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log \|S(k)D^l h_{-1,n}\|_{\text{sup}} = -\lambda_1(n-l+1) \text{ and } D^{n+1}h_{-1,n} = 0.$$

Therefore $h_{-1,n} \in C^{n+1+P_a G}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ satisfies (6.12) and (6.13) for $a = 0$. Then, by Theorem 6.6, there exists $v_n \in C^n(I)$ such that $h_{-1,n} = v_n \circ T - v_n$ and $v_n(0) = 0$. As $h_{-1} = D^n h_{-1,n} = D^n v_n \circ T - D^n v_n$, by Remark 3.4 and the ergodicity of T , we have $D^n v_n(x) = \bar{\xi}(x) + c = x + c$. It follows that $v_n \in \mathbb{R}_{n+1}[x]$.

Suppose that $\bar{t} \in \mathcal{T}\mathcal{F}$ and $\mathfrak{o}(\bar{t}) > r > 0$. Let $n := \lceil \mathfrak{o}(\bar{t}) \rceil - 1$, $a := \mathfrak{o}(\bar{t}) - n$ and choose $\tau > 0$ so that $r < n - a - \tau < n - a = \mathfrak{o}(\bar{t})$. In view of (5.3) and (5.8), for every $0 \leq l \leq n$,

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log \|S(k)D^l h_{\bar{t}}\|_{\text{sup}} = -\lambda_1(\mathfrak{o}(\bar{t}) - l) = -\lambda_1(n - l - a).$$

As $h_{\bar{t}} \in C^{n+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$, by Theorem 6.6, there exists $v_{\bar{t}} \in C^{n-a-\tau}(I)$ such that $h_{\bar{t}}(0) = 0$ and $h_{\bar{t}} = v_{\bar{t}} \circ T - v_{\bar{t}}$. As $r < n - a - \tau$, this gives our claim. \square

We finish the section by summarizing the complete conditions for having smooth solutions of the cohomological equations for a.e IETs.

Theorem 6.8. *Let $n \geq 1$, $0 \leq a < 1$ and $0 < r < n - a$ such that $r \notin \{\mathfrak{o}(\bar{t}) : \bar{t} \in \mathcal{T}_{a,n}\}$. Assume that T satisfies the FFDC. Let $\varphi \in C^{n+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ be a map such that $\mathfrak{f}_{\bar{t}}(\varphi) = 0$ for all $\bar{t} \in \mathcal{T}_{a,n}$ with $\mathfrak{o}(\bar{t}) < r$. Then there exists a solution $v \in C^r(I)$ of the cohomological equation $\varphi = v \circ T - v$ such that $v(0) = 0$. The operator*

$$(6.16) \quad \bigcap_{\bar{t} \in \mathcal{T}_{a,n}, \mathfrak{o}(\bar{t}) < r} \ker(\mathfrak{f}_{\bar{t}}) \ni \varphi \mapsto v \in C^r(I)$$

is linear and bounded.

Moreover, there exist bounded operators $\Gamma_n : C^{n+\text{Pa}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow \Gamma_n(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ and $V_n : C^{n+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow C^{n-1}(I)$ such that

$$\varphi = V_n(\varphi) \circ T - V_n(\varphi) + \Gamma_n(\varphi).$$

More precisely, for every $0 < \tau < 1 - a$ the operator V_n takes value in $C^{n-a-\tau}(I)$ and $V_n : C^{n+\text{PaG}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow C^{n-a-\tau}(I)$ is also bounded.

Proof. Assume that $\mathfrak{f}_{\bar{t}}(\varphi) = 0$ for every $\bar{t} \in \mathcal{T}_{a,n}$ with $\mathfrak{o}(\bar{t}) < r$. Then

$$\varphi = \mathfrak{r}_{a,n}(\varphi) + \sum_{\bar{t} \in \mathcal{T}_{a,n}, \mathfrak{o}(\bar{t}) > r} \mathfrak{f}_{\bar{t}}(\varphi) h_{\bar{t}} + \sum_{\bar{t} \in \mathcal{T}_{a,n}^* \setminus \mathcal{T}_{a,n}} \mathfrak{f}_{\bar{t}}(\varphi) h_{\bar{t}}.$$

Choose $\tau > 0$ such that $r < n - a - \tau$. In view of Theorem 5.6 and 6.6, there exists $\bar{v} \in C^{n-a-\tau}(I)$ such that $\mathfrak{r}_{a,n}(\varphi) = \bar{v} \circ T - \bar{v}$ and $\bar{v}(0) = 0$. There exists also $C_{\tau,n} > 0$ such that $\|\bar{v}\|_{C^{n-a-\tau}} \leq C_{\tau,n} \|\mathfrak{r}_{a,n}(\varphi)\|_{C^{n+\text{Pa}}}$. By Corollary 6.7, for every $\bar{t} \in \mathcal{T}_{a,n}^* \setminus \mathcal{T}_{a,n}$ there exists a polynomial $v_{\bar{t}}$ such that $h_{\bar{t}} = v_{\bar{t}} \circ T - v_{\bar{t}}$ and $v_{\bar{t}}(0) = 0$. Moreover, if $\bar{t} \in \mathcal{T}_{a,n}$ and $\mathfrak{o}(\bar{t}) > r > 0$ then, again by Corollary 6.7, there exists $v_{\bar{t}} \in C^r(I)$ such that $h_{\bar{t}} = v_{\bar{t}} \circ T - v_{\bar{t}}$ and $v_{\bar{t}}(0) = 0$. It follows that

$$\varphi = \bar{v} \circ T - \bar{v} + \sum_{\bar{t} \in \mathcal{T}_{a,n}, \mathfrak{o}(\bar{t}) > r} \mathfrak{f}_{\bar{t}}(\varphi) (v_{\bar{t}} \circ T - v_{\bar{t}}) + \sum_{\bar{t} \in \mathcal{T}_{a,n}^* \setminus \mathcal{T}_{a,n}} \mathfrak{f}_{\bar{t}}(\varphi) (v_{\bar{t}} \circ T - v_{\bar{t}})$$

and

$$v = \bar{v} + \sum_{\bar{t} \in \mathcal{T}_{a,n}, \mathfrak{o}(\bar{t}) > r} \mathfrak{f}_{\bar{t}}(\varphi) v_{\bar{t}} + \sum_{\bar{t} \in \mathcal{T}_{a,n}^* \setminus \mathcal{T}_{a,n}} \mathfrak{f}_{\bar{t}}(\varphi) v_{\bar{t}} \in C^r(I)$$

satisfies $\varphi = v \circ T - v$ and $v(0) = 0$. Moreover,

$$\begin{aligned} \|v\|_{C^r} &\leq C_{\tau,n} \|\mathfrak{r}_{a,n}(\varphi)\|_{C^{n+\text{Pa}}} + \sum_{\bar{t} \in \mathcal{T}_{a,n}^*} |\mathfrak{f}_{\bar{t}}(\varphi)| \|v_{\bar{t}}\|_{C^r} \\ &\leq C_{\tau,n} \|\varphi\|_{C^{n+\text{Pa}}} + \sum_{\bar{t} \in \mathcal{T}_{a,n}^*} (C_{\tau,n} \|h_{\bar{t}}\|_{C^{n+\text{Pa}}} + \|v_{\bar{t}}\|_{C^r}) |\mathfrak{f}_{\bar{t}}(\varphi)|. \end{aligned}$$

As all functionals $\mathfrak{f}_{\bar{t}} : C^{n+\text{Pa}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha) \rightarrow \mathbb{C}$ are bounded, the operator (6.16) is bounded as well.

The second part of the theorem follows directly from Theorem 5.6 and 6.6 with $\Gamma_n(\varphi) = \sum_{\bar{t} \in \mathcal{T}_{a,n}^*} \mathfrak{f}_{\bar{t}}(\varphi) h_{\bar{t}}$ and $V_n(\varphi)$ being the solution of the cohomological equation $\mathfrak{r}_{a,n}(\varphi) = V_n(\varphi) \circ T - V_n(\varphi)$. \square

Remark 6.9. In view of the second part of Theorem 5.6, the regularity of the solution v for the equation $\varphi = v \circ T - v$ with $\varphi \in C^{n+P_a}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ proved in Theorem 6.8 is optimal. Indeed, let $r_0 = \mathfrak{o}(\bar{t}_0) > 0$ for some $\bar{t}_0 \in \mathcal{T}_{a,n}$. Let $\varphi \in C^{n+P_a}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ such that $\mathfrak{f}_{\bar{t}_0}(\varphi) \neq 0$ and $\mathfrak{f}_{\bar{t}}(\varphi) = 0$ for all $\bar{t} \in \mathcal{T}_{a,n}$ with $\mathfrak{o}(\bar{t}) < r_0$. By Theorem 6.8, the solution v of the cohomological equation belongs to $C^r(I)$ for any $r < r_0$. On the other hand, by Theorem 5.6, $v \notin C^r(I)$ for any $r > r_0$. Hence, the exponent $r_0 = \mathfrak{o}(\bar{t}_0)$ is a threshold for the regularity of the solution.

Similarly, if $r_0 = n - a$, $C_{\alpha,n}^{a,\pm}(\varphi) \neq 0$ for some $\alpha \in \mathcal{A}$ and $\mathfrak{f}_{\bar{t}}(\varphi) = 0$ for all $\bar{t} \in \mathcal{T}_{a,n}$ with $\mathfrak{o}(\bar{t}) < r_0$ (in fact, by (5.19), for any $\bar{t} \in \mathcal{T}_{a,n}$) then $v \in C^r(I)$ for every $r < r_0$ and $v \notin C^r(I)$ for every $r > r_0$.

7. PROOFS OF THE MAIN THEOREMS

In this last section, we construct generalized Forni's invariant distributions $\mathfrak{F}_{\bar{t}}$ on function spaces on a compact surface M . Roughly speaking, $\mathfrak{F}_{\bar{t}}$ is achieved by composing the operator $f \mapsto \varphi_f$ with the functional $\mathfrak{f}_{\bar{t}}$. Since the invariant distributions $\mathfrak{f}_{\bar{t}}$ are on C^{m+P_a} , we need to perform in Section 7.1 an additional correction of φ_f so that the resulting function belongs to C^{m+P_a} .

Finally, in Section 7.2, we apply the tools developed in [12] to make a transition from cohomological equations over IETs to equations for locally Hamiltonian flows on any minimal component $M' \subset M$. Then by combining them with the cohomological results over IETs in Section 6, optimal regularity of solutions to cohomological equations $Xu = f$ is obtained. The regularity is determined by the order (or the hat-order) of three different types of invariant distributions $\mathfrak{C}_{\sigma,l}^k$, $\mathfrak{D}_{\sigma,j}^k$ and $\mathfrak{F}_{\bar{t}}$.

7.1. Counterparts of Forni's invariant distributions. Let M be a compact connected orientable C^∞ -surface. Let $\psi_{\mathbb{R}}$ be a locally Hamiltonian C^∞ -flow on M with isolated fixed points and such that all its saddles are perfect and all saddle connections are loops. Let $M' \subset M$ be a minimal component of the flow and let $I \subset M'$ be a transversal curve. The corresponding IET $T : I \rightarrow I$ exchanges the intervals $\{I_\alpha : \alpha \in \mathcal{A}\}$. Let $\tau : I \rightarrow \mathbb{R}_{>0}$ be the first return time map. Let us consider the operator $f \mapsto \varphi_f$ defined for every integrable map $f : M \rightarrow \mathbb{R}$ as follows:

$$\varphi_f(x) = \int_0^{\tau(x)} f(\psi_t x) dt \text{ for every } x \in I.$$

If f is a smooth function on M then φ_f is also smooth on every $\text{Int } I_\alpha$, $\alpha \in \mathcal{A}$. The function φ_f may be discontinuous at the ends of the intervals or may have singularities. A detailed description of the behavior around the ends of the exchanged intervals is described in details in [12].

Suppose that the equation $\varphi_f = v \circ T - v$ has a smooth solution $v : I \rightarrow \mathbb{R}$. This is a necessary condition for the existence of a smooth solution to the equation $Xu = f$. In a sense, this is also a sufficient condition for the existence of a smooth solution to the equation $Xu = f$. We can define $u_{v,f} : M' \setminus (\text{Sd}(\psi_{\mathbb{R}}) \cup \text{SL}(\psi_{\mathbb{R}})) \rightarrow \mathbb{R}$ as follows: if $\psi_t x \in I$ for some $t \in \mathbb{R}$ then

$$u_{v,f}(x) := v(\psi_t x) - \int_0^t f(\psi_s x) ds.$$

The map $u_{v,f}$ is a smooth solution of $Xu = f$, but only on $M' \setminus (\text{Sd}(\psi_{\mathbb{R}}) \cup \text{SL}(\psi_{\mathbb{R}}))$ that is an open subset of M' . Usually $u_{v,f}$ cannot be smoothly extended to M' or even to the end compactification M'_e defined in [12]. As proven in [12, Theorem 1.2],

the vanishing of some invariant distributions $\mathfrak{d}_{\sigma,j}^k(f)$ and $\mathfrak{C}_{\sigma,l}^k(f)$ is the necessary and sufficient condition for the existence of a smooth solution (an extension of $u_{v,f}$) to $Xu = f$ on M'_e .

After [12], for any $[(\sigma, k, l)] \in \mathcal{TC}/\sim$ we define a map $\widehat{\xi}_{[(\sigma,k,l)]} : I \rightarrow \mathbb{R}$. For any closed interval $J \subset I_\alpha$ denote by $J^\tau \subset M$ the closure of the set of orbit segments starting from $\text{Int } J$ and running until the first return to I . For any $[(\sigma, k, l)] \in \mathcal{TC}/\sim$ there exists $\alpha \in \mathcal{A}$ and an interval J of the form $[l_\alpha, l_\alpha + \varepsilon]$ or $[r_\alpha - \varepsilon, r_\alpha]$ such that l_α or r_α is the first backward meeting point of a separatrix incoming to $\sigma \in \text{Sd}(\psi_{\mathbb{R}})$ and J^τ contains all angular sectors $U_{\sigma,l'}$ for which $(\sigma, k, l') \sim (\sigma, k, l)$. Let $\widehat{\xi}_{[(\sigma,k,l)]} : I \rightarrow \mathbb{R}$ be a map such that

- $\widehat{\xi}_{[(\sigma,k,l)]}$ is zero on any interval I_β with $\beta \neq \alpha$;
- if $J = [l_\alpha, l_\alpha + \varepsilon]$ then for any $s \in I_\alpha$,

$$\begin{aligned} \widehat{\xi}_{[(\sigma,k,l)]}(s) &= \frac{(s - l_\alpha)^{\frac{k - (m_\sigma - 2)}{m_\sigma}}}{m_\sigma^2 k!} \text{ if } k \neq m_\sigma - 2 \pmod{m_\sigma} \\ \widehat{\xi}_{[(\sigma,k,l)]}(s) &= -\frac{(s - l_\alpha)^{\frac{k - (m_\sigma - 2)}{m_\sigma}} \log(s - l_\alpha)}{m_\sigma^2 k!} \text{ if } k = m_\sigma - 2 \pmod{m_\sigma}; \end{aligned}$$

- if $J = [r_\alpha - \varepsilon, r_\alpha]$ then for any $s \in I_\alpha$,

$$\begin{aligned} \widehat{\xi}_{[(\sigma,k,l)]}(s) &= \frac{(r_\alpha - s)^{\frac{k - (m_\sigma - 2)}{m_\sigma}}}{m_\sigma^2 k!} \text{ if } k \neq m_\sigma - 2 \pmod{m_\sigma} \\ \widehat{\xi}_{[(\sigma,k,l)]}(s) &= -\frac{(r_\alpha - s)^{\frac{k - (m_\sigma - 2)}{m_\sigma}} \log(r_\alpha - s)}{m_\sigma^2 k!} \text{ if } k = m_\sigma - 2 \pmod{m_\sigma}. \end{aligned}$$

As $\frac{k - (m_\sigma - 2)}{m_\sigma} = \mathfrak{o}(\sigma, k)$, we have $\widehat{\xi}_{[(\sigma,k,l)]} \in C^{n_{\sigma,k} + P_{a_{\sigma,k}} G}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ with $n_{\sigma,k} := \lceil \mathfrak{o}(\sigma, k) \rceil$ and $a_{\sigma,k} := n - \mathfrak{o}(\sigma, k)$, and exactly one of $C_\alpha^+(D^{n_{\sigma,k}} \widehat{\xi}_{[(\sigma,k,l)]})$, $C_\alpha^-(D^{n_{\sigma,k}} \widehat{\xi}_{[(\sigma,k,l)]})$ is non-zero.

Let us consider $\xi_{[(\sigma,k,l)]} \in C^{n_{\sigma,k} + P_{a_{\sigma,k}} G}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ given by

$$\begin{aligned} \xi_{[(\sigma,k,l)]} &:= \mathfrak{r}_{a_{\sigma,k}, n_{\sigma,k}}(\widehat{\xi}_{[(\sigma,k,l)]}) = \widehat{\xi}_{[(\sigma,k,l)]} - \sum_{\bar{t} \in \mathcal{T}_{a_{\sigma,k}, n_{\sigma,k}}^*} \mathfrak{f}_{\bar{t}}(\widehat{\xi}_{[(\sigma,k,l)]}) h_{\bar{t}} \\ &= \widehat{\xi}_{[(\sigma,k,l)]} - \sum_{\bar{t} \in \mathcal{T}^*, \mathfrak{o}(\bar{t}) < \mathfrak{o}(\sigma, k)} \mathfrak{f}_{\bar{t}}(\widehat{\xi}_{[(\sigma,k,l)]}) h_{\bar{t}}. \end{aligned}$$

In view of Lemma 5.7,

$$(7.1) \quad \mathfrak{f}_{\bar{t}}(\xi_{[(\sigma,k,l)]}) = 0 \text{ if } \mathfrak{o}(\bar{t}) < \mathfrak{o}(\sigma, k).$$

Since $C_\alpha^\pm(D^{n_{\sigma,k}} \xi_{[(\sigma,k,l)]}) = C_\alpha^\pm(D^{n_{\sigma,k}} \widehat{\xi}_{[(\sigma,k,l)]}) \neq 0$, by Theorem 5.6,

$$(7.2) \quad \begin{aligned} \lim_{j \rightarrow \infty} \frac{1}{j} \log \left(\|S(j)(\xi_{[(\sigma,k,l)]})\|_{L^1(I^{(j)})} / |I^{(j)}| \right) &= -\lambda_1(n_{\sigma,k} - a_{\sigma,k}) = -\lambda_1 \mathfrak{o}(\sigma, k) \\ \lim_{j \rightarrow \infty} \frac{1}{j} \log \|S(j)(\xi_{[(\sigma,k,l)]})\|_{\text{sup}} &= -\lambda_1 \mathfrak{o}(\sigma, k) \text{ if } \mathfrak{o}(\sigma, k) > 0. \end{aligned}$$

Lemma 7.1. *For any $r \geq -\frac{m-2}{m}$ let $n = \lceil r \rceil$ and $a = n - r$. Then for any $f \in C^{k_r}(M)$ we have*

$$(7.3) \quad \mathfrak{s}_r(f) = \varphi_f - \sum_{\substack{[(\sigma,k,l)] \in \mathcal{TC}/\sim \\ \mathfrak{o}(\sigma,k) < r}} \mathfrak{C}_{[(\sigma,k,l)]}(f) \xi_{[(\sigma,k,l)]} \in C^{m+\text{Pa}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$$

and the operator $\mathfrak{s}_r : C^{k_r}(M) \rightarrow C^{n+\text{Pa}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ is bounded.

Proof. By Theorem 5.6 in [12],

$$\widehat{\mathfrak{s}}_r(f) := \varphi_f - \sum_{\substack{[(\sigma,k,l)] \in \mathcal{TC}/\sim \\ \mathfrak{o}(\sigma,k) < r}} \mathfrak{C}_{[(\sigma,k,l)]}(f) \widehat{\xi}_{[(\sigma,k,l)]} \in C^{m+\text{Pa}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$$

and the operator $\widehat{\mathfrak{s}}_r : C^{k_r}(M) \rightarrow C^{n+\text{Pa}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ is bounded. Moreover,

$$\mathfrak{s}_r(f) = \widehat{\mathfrak{s}}_r(f) + \sum_{\substack{[(\sigma,k,l)] \in \mathcal{TC}/\sim \\ \mathfrak{o}(\sigma,k) < r}} \mathfrak{C}_{[(\sigma,k,l)]}(f) (\widehat{\xi}_{[(\sigma,k,l)]} - \xi_{[(\sigma,k,l)]}).$$

Since $\widehat{\xi}_{[(\sigma,k,l)]} - \xi_{[(\sigma,k,l)]}$ is a polynomial over any exchanged interval, this gives our claim. \square

Definition 8. Let any $r \geq -\frac{m-2}{m}$. For any $\bar{t} \in \mathcal{TF}^*$ with $\mathfrak{o}(\bar{t}) < r$ denote by $\mathfrak{F}_{\bar{t}} : C^{k_r}(M) \rightarrow \mathbb{C}$ the operator given by $\mathfrak{F}_{\bar{t}} := \mathfrak{f}_{\bar{t}} \circ \mathfrak{s}_r$. As $\mathfrak{s}_r : C^{k_r}(M) \rightarrow C^{n+\text{Pa}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ with $n = \lceil r \rceil$, $a = \lceil r \rceil - r$ and $\bar{t} \in \mathcal{T}_{a,n}^*$ by (5.19), the operator is well-defined and bounded.

Remark 7.2. Note that the definition of $\mathfrak{F}_{\bar{t}}$ does not depend on the choice of r . Indeed, suppose that $\mathfrak{o}(\bar{t}) < r_1 < r_2$. Then for every $f \in C^{k_{r_2}}(M)$,

$$\mathfrak{s}_{r_1}(f) - \mathfrak{s}_{r_2}(f) = \sum_{\substack{[(\sigma,k,l)] \in \mathcal{TC}/\sim \\ r_1 \leq \mathfrak{o}(\sigma,k) < r_2}} \mathfrak{C}_{[(\sigma,k,l)]}(f) \xi_{[(\sigma,k,l)]}.$$

In view of (7.1), it follows that $\mathfrak{f}_{\bar{t}}(\mathfrak{s}_{r_1}(f)) = \mathfrak{f}_{\bar{t}}(\mathfrak{s}_{r_2}(f))$, which yields our claim.

Remark 7.3. For any $\bar{t} \in \mathcal{TF}^*$ take $\mathfrak{o}(\bar{t}) < r < \mathfrak{o}(\bar{t}) + \frac{1}{m}$. By definition, $k_r \leq k_{\mathfrak{o}(\bar{t})} + 1$. It follows that the functional $\mathfrak{F}_{\bar{t}}$ is defined on $C^{k_{\mathfrak{o}(\bar{t})}+1}(M)$. If $\mathfrak{o}(\bar{t}) \notin \mathbb{Z}/m$ then the domain of $\mathfrak{F}_{\bar{t}}$ is enlarged to $C^{k_{\mathfrak{o}(\bar{t})}}(M)$.

7.2. Proofs of the main results.

Proof of Theorem 1.1. Choose $r_0 \in \mathbb{R}_{>0}$ which is the smallest element of $\{\mathfrak{o}(\sigma, k) : k \geq 0, \sigma \in \text{Sd}(\psi_{\mathbb{R}}) \cap M'\} \cup \{\mathfrak{o}(\bar{t}) : \bar{t} \in \mathcal{TF}\}$ larger than r . By assumption, T satisfies the FFDC, $f \in C^{k_r}(M) = C^{k_{r_0}}(M)$ and

- $\mathfrak{d}_{\sigma,j}^k(f) = 0$ for all $(\sigma, k, j) \in \mathcal{TD}$ with $\widehat{\mathfrak{o}}(\mathfrak{d}_{\sigma,j}^k) < r_0$;
- $\mathfrak{C}_{\sigma,l}^k(f) = 0$ for all $(\sigma, k, l) \in \mathcal{TC}$ with $\mathfrak{o}(\mathfrak{C}_{\sigma,l}^k) < r_0$;
- $\mathfrak{F}_{\bar{t}}(f) = 0$ for all $\bar{t} \in \mathcal{TF}$ with $\mathfrak{o}(\mathfrak{F}_{\bar{t}}) < r_0$.

By Theorem 1.1 in [12], $\varphi_f \in C^{n+\text{Pa}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ with $n = \lceil r_0 \rceil$ and $a = \lceil r_0 \rceil - r_0$. Moreover, there exists $C_r > 0$ such that $\|\varphi_f\|_{C^{n+\text{Pa}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)} \leq C_r \|f\|_{C^{k_r}(M)}$ for all $f \in C^{k_r}(M) \cap \ker(\mathfrak{C}_{\sigma,l}^k)$ for $(\sigma, k, l) \in \mathcal{TC}$ with $\mathfrak{o}(\mathfrak{C}_{\sigma,l}^k) < r$.

By assumption, in view of (7.3), $\varphi_f = \mathfrak{s}_{r_0}(f)$. It follows that $\mathfrak{f}_{\bar{t}}(\varphi_f) = \mathfrak{f}_{\bar{t}}(\mathfrak{s}_{r_0}(f)) = \mathfrak{F}_{\bar{t}}(f) = 0$ for all $\bar{t} \in \mathcal{TF}$ with $\mathfrak{o}(\bar{t}) < r$. As $r < r_0 = n - a$, in view of Theorem 6.8, there exists a solution $v \in C^r(I)$ of the cohomological equation $\varphi = v \circ T - v$ such that $v(0) = 0$. Moreover, there exists $C'_r > 0$ such that $\|v\|_{C^r(I)} \leq C'_r \|\varphi_f\|_{C^{n+\text{Pa}}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)}$.

By Theorem 1.2 in [12], there exists $u_{v,f} \in C^r(M'_e)$ satisfying $Xu_{v,f} = f$ on M'_e . Moreover, there exists a constant $C''_r > 0$ such that $\|u_{v,f}\|_{C^r(M'_e)} \leq C''_r(\|v\|_{C^r(I)} + \|f\|_{C^{kr}(M)})$. It follows that

$$\|u_{v,f}\|_{C^r(M'_e)} \leq C''_r(1 + C_r C'_r) \|f\|_{C^{kr}(M)},$$

which completes the proof. \square

Proof of Theorem 1.2. If $f \in C^{kr}(M)$ and there exists $u \in C^r(M'_e)$ such that $Xu = f$ on M'_e , then by Theorem 1.3 in [12], $\mathfrak{d}_{\sigma,j}^k(f) = 0$ for all $(\sigma, k, j) \in \mathcal{T}\mathcal{D}$ with $\widehat{\mathfrak{o}}(\mathfrak{d}_{\sigma,j}^k) < r$ and $\mathfrak{C}_{\sigma,l}^k(f) = 0$ for all $(\sigma, k, l) \in \mathcal{T}\mathcal{C}$ with $\mathfrak{o}(\mathfrak{C}_{\sigma,l}^k) < r$. In view of Theorem 1.1 in [12], $\varphi_f \in C^{m+P_a}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ with $n = \lceil r \rceil$ and $a = \lceil r \rceil - r$. By (7.3), it follows that $\varphi_f = \mathfrak{s}_r(f)$. Hence $\mathfrak{F}_{\bar{t}}(f) = \mathfrak{f}_{\bar{t}}(\mathfrak{s}_r(f)) = \mathfrak{f}_{\bar{t}}(\varphi_f)$ for all $\bar{t} \in \mathcal{T}\mathcal{F}$ with $\mathfrak{o}(\bar{t}) < r$.

On the other hand $\varphi_f = v \circ T - v$, where $v \in C^r(I)$ is the restriction of u to I . By Theorem 5.6, this gives $\mathfrak{f}_{\bar{t}}(\varphi_f) = 0$ for all $\bar{t} \in \mathcal{T}\mathcal{F}$ with $\mathfrak{o}(\bar{t}) < r$. Therefore, $\mathfrak{F}_{\bar{t}}(f) = 0$ for all $\bar{t} \in \mathcal{T}\mathcal{F}$ with $\mathfrak{o}(\bar{t}) < r$. \square

Proof of Theorem 1.3. In view of Lemma 7.1 and Theorem 5.6, for any $f \in C^{kr}(M)$,

$$\begin{aligned} \varphi_f &= \mathfrak{s}_r(f) + \sum_{\substack{[(\sigma,k,l)] \in \mathcal{T}\mathcal{C}/\sim \\ \mathfrak{o}(\sigma,k) < r}} \mathfrak{C}_{[(\sigma,k,l)]}(f) \xi_{[(\sigma,k,l)]} \\ &= \sum_{\substack{\bar{t} \in \mathcal{T}\mathcal{F}^* \\ \mathfrak{o}(\bar{t}) < r}} \mathfrak{f}_{\bar{t}}(\mathfrak{s}_r(f)) h_{\bar{t}} + \mathfrak{r}_{a,n}(\mathfrak{s}_r(f)) + \sum_{\substack{[(\sigma,k,l)] \in \mathcal{T}\mathcal{C}/\sim \\ \mathfrak{o}(\sigma,k) < r}} \mathfrak{C}_{[(\sigma,k,l)]}(f) \xi_{[(\sigma,k,l)]} \\ &= \sum_{\substack{\bar{t} \in \mathcal{T}\mathcal{F}^* \\ \mathfrak{o}(\bar{t}) < r}} \mathfrak{F}_{\bar{t}}(f) h_{\bar{t}} + \sum_{\substack{[(\sigma,k,l)] \in \mathcal{T}\mathcal{C}/\sim \\ \mathfrak{o}(\sigma,k) < r}} \mathfrak{C}_{[(\sigma,k,l)]}(f) \xi_{[(\sigma,k,l)]} + \mathfrak{r}_r(f) \end{aligned}$$

with $\mathfrak{r}_r := \mathfrak{r}_{a,n} \circ \mathfrak{s}_r$, where $n = \lceil r \rceil$ and $a = \lceil r \rceil - r$. Note that (1.3), (1.4) and (1.5) follow directly from (5.8) and (7.2). Moreover, (1.6) follows from (5.21) and (5.22). \square

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