Generalized reflected backward stochastic differential equations driven by a fractional Brownian motion

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Outline

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Fractional Brownian motion

A Fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$ is a zero mean Gaussian process $B^H = \{B^H_t,\ t \geq 0\}$ with the covariance function

$$R_H(s,t) = E\left(B^H_s B^H_t\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

The process B^H

- is not a semimartingale when $H \neq 1/2$
- for $H = 1/2$, B^H is a standard Wiener process
- has homogeneous increments, i.e. $B^H_{t+s} - B^H_s$ has the sam law of B^H_t
- is a self-similar, i.e. B^H_{at} has the same law as $a^H B^H_t$ for any $a > 0$
- has continuous trajectories

Fractional calculus

Let $H > 1/2$. Denote

$$\langle \xi, \eta \rangle_t = H(2H-1) \int_0^t \int_0^t |u-v|^{2H-2} \xi(u)\eta(v) du dv,$$

$$\|\xi\|_t^2 = \langle \xi, \xi \rangle_t.$$

$\mathcal{P}_T$ – the set of all polynomials of the fBm, i.e.

$$F(\omega) = f\left(\int_0^T \xi_1(t) dB_t^H, \ldots, \int_0^T \xi_k(t) dB_t^H \right)$$

where $(\xi_n)_n$ – a sequence such that $\langle \xi_i, \xi_j \rangle_T = \delta_{ij}$.
The Malliavin derivative operator D_s^H of $F \in \mathcal{P}_T$ is defined as follows:

$$D_s^H F = \sum_{i=1}^k \frac{\partial f}{\partial x_i} \left(\int_0^T \xi_1(t) dB_t^H, \ldots, \int_0^T \xi_k(t) dB_t^H \right) \xi_i(s), \;\; s \in [0, T].$$

Fractional calculus

$\mathbb{D}_{1,2}$ – the Banach space being a completition of $\mathcal{P}_T$ with the norm:

$$\|F\|_{1,2}^2 = E|F|^2 + E\|D_s^H F\|_T^2.$$

Denote

$$\mathbb{D}_t^H F = H(2H-1)\int_0^T |t-s|^{2H-2} D_s^H F ds.$$

THEOREM.

$$E\left(\int_0^T F_s dB_s^H\right) = 0$$

and

$$E\left(\int_0^T F_s dB_s^H\right)^2 = E\left(\|F\|_T^2 + \int_0^T\int_0^T \mathbb{D}_s^H F_t \mathbb{D}_t^H F_s ds dt\right).$$

Fractional calculus

$$X_t = X_0 + \int_0^t g_s^1 ds + \int_0^t f_s^1 dB_s^H, \; Y_t = Y_0 + \int_0^t g_s^2 ds + \int_0^t f_s^2 dB_s^H,$$

$t \in [0, T]$ and $F \in C^{1,2}$. Then

$$F(t, X_t) = F(0, X_0) + \int_0^t \frac{\partial F}{\partial s}(s, X_s)ds + \int_0^t \frac{\partial F}{\partial x}(s, X_s)dX_s + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(s, X_s)\frac{d}{ds}\left(\|f^1\|_s^2\right) ds$$

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s g_s^2 ds + \int_0^t X_s f_s^2 dB_s^H + \int_0^t Y_s g_s^1 ds + \int_0^t Y_s f_s^1 dB_s^H + \int_0^t \mathbb{D}_s^H X_s f_s^2 ds + \int_0^t \mathbb{D}_s^H Y_s f_s^1 ds.$$

Let

$$\eta_t = \eta_0 + \int_0^t \sigma(s) dB_s^H, \quad t \in [0, T],$$

where $\sigma : [0, T] \to \mathbb{R}$ – a deterministic continuous differentiable function, $\sigma(t) \neq 0$, η_0 – a constant.
For functions $\varphi, \psi : \mathbb{R} \to (-\infty, \infty]$ we use the following notations:

$$\partial\varphi(y) = \{\hat{y} \in \mathbb{R}; \quad \hat{y} \cdot (v - y) + \varphi(y) \leq \varphi(v), \forall v \in \mathbb{R}\}$$

$$\mathrm{Dom}\varphi = \{y \in \mathbb{R}; \quad \varphi(y) < \infty\}, \qquad \mathrm{Dom}(\partial\varphi) = \{y \in \mathbb{R}; \quad \partial\varphi(y) \neq \emptyset\}$$

$$(y, \hat{y}) \in \partial\varphi \Leftrightarrow y \in \mathrm{Dom}(\partial\varphi), \; \hat{y} \in \partial\varphi(y).$$

(analogously for ψ).

GRBSDE with respect to a fBm

We consider the following generalized backward stochastic variational inequality (GBSVI) driven by a fBm: $Y_T = \xi$ and

$$dY_t + f(t,\eta_t,Y_t,Z_t)dt + g(t,\eta_t,Y_t)d\Lambda_t - Z_t dB_t^H \in \partial\varphi(Y_t)dt + \partial\psi(Y_t)d\Lambda_t.$$

DEFINITION. A solution of a GBSVI driven by a fBm B^H associated with data (ξ, f, g, Λ) is a quadruple (Y, Z, U, V) satisfying GRBSDE

$$\begin{aligned} Y_t = \xi &+ \int_t^T f(s,\eta_s,Y_s,Z_s)ds + \int_t^T g(s,\eta_s,Y_s)d\Lambda_s - \int_t^T Z_s dB_s^H \\ &- \int_t^T U_s ds - \int_t^T V_s d\Lambda_s, \qquad t \in [0,T] \end{aligned} \tag{1}$$

and such that $(Y_t, U_t) \in \partial\varphi$, $(Y_t, V_t) \in \partial\psi$ and $Y \in \tilde{\mathcal{V}}_{[0,T]}^{1/2} \cap \tilde{\mathcal{V}}_{[0,T]}^{1/2,\Lambda}$, $Z \in \tilde{\mathcal{V}}_{[0,T]}^{H}$, $U, V \in \tilde{\mathcal{V}}_{[0,T]}^{H} \cap \tilde{\mathcal{V}}_{[0,T]}^{H,\Lambda}$.

THEOREM. There exists a solution of (1) such that

$$E\left(e^{\nu\Lambda_t}|Y_t|^2+\int_t^T e^{\nu\Lambda_s}s^{2H-1}|Z_s|^2ds+\int_t^T e^{\nu\Lambda_s}|Y_s|^2d\Lambda_s\right)<\infty.$$

Sketch of proof. We use the Itô formula for $e^{\nu\Lambda_t}|Y_t|^2$ and facts that

$$\mathbb{D}_s^H Y_s=\int_0^s \phi(s-r)D_r^H Y_s dr=\frac{\hat{\sigma}(s)}{\sigma(s)}Z_s$$

and there exists $M>0$ such that for all $t\in[0,T]$,

$$\frac{t^{2H-1}}{M}\leq\frac{\hat{\sigma}(t)}{\sigma(t)}\leq Mt^{2H-1}.$$

Penalization scheme

We approximate functions φ and ψ by sequences of convex, C^1 class functions φ_ε, ψ_ε respectively, $\varepsilon > 0$, defined by

$$\varphi_\varepsilon(y) = \inf\left\{\frac{1}{2\varepsilon}|y-v|^2 + \varphi(v);\ v \in \mathbb{R}\right\} = \frac{1}{2\varepsilon}|y - J_\varepsilon(y)|^2 + \varphi(J_\varepsilon(y)),$$

$$\psi_\varepsilon(y) = \inf\left\{\frac{1}{2\varepsilon}|y-v|^2 + \psi(v);\ v \in \mathbb{R}\right\} = \frac{1}{2\varepsilon}|y - \tilde{J}_\varepsilon(y)|^2 + \psi(\tilde{J}_\varepsilon(y)),$$

where $J_\varepsilon(y) = y - \varepsilon\nabla\varphi_\varepsilon(y)$ and $\tilde{J}_\varepsilon(y) = y - \varepsilon\nabla\psi_\varepsilon(y)$.

Consider a sequence of generalized BSDEs

$$Y_t^\varepsilon = \xi + \int_t^T f(s,\eta,Y_s^\varepsilon,Z_s^\varepsilon)ds + \int_t^T g(s,\eta,Y_s^\varepsilon)d\Lambda_s - \int_t^T Z_s^\varepsilon dB_s^H$$
$$- \int_t^T \nabla\varphi_\varepsilon(Y_s^\varepsilon)ds - \int_t^T \nabla\psi_\varepsilon(Y_s^\varepsilon)d\Lambda_s, \quad t \in [0, T].$$

Penalization scheme

$$(a)\ E\int_t^T e^{\nu\Lambda_s}s^{2H-1}\left(|\nabla\varphi_\varepsilon(Y_s^\varepsilon)|^2 ds + |\nabla\psi_\varepsilon(Y_s^\varepsilon)|^2 d\Lambda_s\right) \le C\Theta_2(t,T)$$

$$(b)\ E\int_t^T e^{\nu\Lambda_s}s^{2H-1}\left(\varphi\left(J_\varepsilon(Y_s^\varepsilon)\right) + \psi\left(\tilde{J}_\varepsilon(Y_s^\varepsilon)\right)\right) d\Lambda_s \le C\Theta_2(t,T)$$

$$(c)\ Ee^{\nu\Lambda_t}t^{2H-1}\left(|Y_t^\varepsilon - J_\varepsilon(Y_t^\varepsilon)|^2 + \left|Y_t^\varepsilon - \tilde{J}_\varepsilon(Y_t^\varepsilon)\right|^2\right) \le \varepsilon\cdot C\Theta_2(t,T)$$

$$(d)\ Ee^{\nu\Lambda_t}t^{2H-1}\left(\varphi\left(J_\varepsilon(Y_t^\varepsilon)\right) + \psi\left(\tilde{J}_\varepsilon(Y_t^\varepsilon)\right)\right) \le C\Theta_2(t,T)$$

$$(e)\ E\int_t^T e^{\nu\Lambda_s}s^{2H-1}\left(|Y_s^\varepsilon - J_\varepsilon(Y_s^\varepsilon)|^2 ds + \left|Y_s^\varepsilon - \tilde{J}_\varepsilon(Y_s^\varepsilon)\right|^2 d\Lambda_s\right) \le \varepsilon^2\cdot C\Theta_2(t,T).$$

Penalization scheme

where

$$\Theta_2(t,T) = E\Bigg(T^{2H-1} e^{\nu\Lambda_T} \left(\varphi(\xi) + \psi(\xi)\right) + \int_t^T e^{\nu\Lambda_s} s^{2H-1} \left(|\eta_s|^2 + |Y_s^\varepsilon|^2 + |Z_s^\varepsilon|^2 \right) ds + \int_t^T e^{\nu\Lambda_s} \left(|\eta_s|^2 + |Y_s^\varepsilon|^2 \right) d\Lambda_s \Bigg).$$

Penalization scheme

THEOREM. $(Y^{\varepsilon}, Z^{\varepsilon})$ is a Cauchy sequence, i.e. for $\varepsilon, \delta > 0$

$$E\Bigg(e^{\nu\Lambda_t} t^{2H-1}|Y_t^{\varepsilon} - Y_t^{\delta}|^2 + \int_t^T e^{\nu\Lambda_t} s^{2H-2}|Y_s^{\varepsilon} - Y_s^{\delta}|^2 (ds + d\Lambda_s)$$

$$+ \int_t^T e^{\nu\Lambda_t} s^{2(2H-1)}|Z_s^{\varepsilon} - Z_s^{\delta}|^2 ds\Bigg)$$

$$\leq C \cdot (\varepsilon + \delta) \cdot \Theta_2(t, T).$$

Existence

$$U_t \cdot (u_t - Y_t) + \varphi(Y_t) \le \varphi(u_t) \quad \text{and} \quad V_t \cdot (v_t - Y_t) + \psi(Y_t) \le \psi(v_t),$$

mean that $(Y_t, U_t) \in \partial\varphi$ and $(Y_t, V_t) \in \partial\psi$, $t \in [0, T]$.

$(Y^\varepsilon, Z^\varepsilon, \nabla\varphi_\varepsilon(Y^\varepsilon), \nabla\psi_\varepsilon(Y^\varepsilon))$ converges to a solution of (1).

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