Image Denoising using Backward Stochastic Differential Equations[1]

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Problem

Given $u_0 : D \to \mathbf{R}^3$ being an observed image we want to reconstruct the original RGB image $u : D \to \mathbf{R}^3$, where

$$u_0 = u + \eta,$$

η is a white Gaussian noise with standard deviation σ added independently to each coordinate and $D \subset \mathbf{R}^2$ is a closed rectangle.

Continuous Model

$$\begin{cases} X_t = x + \int_0^t \sigma(s, X_s)\, dW_s + K_t^{\overline{D}}, & t \in [0,T], \\ Y_t = u_0(X_S) + \int_t^T c(s)(Y_s - u_0(X_s))ds - \int_t^T Z_s\, dW_s, & t \in [0,T], \end{cases}$$

where $S < T$,

$$\sigma(s, X_s) = \left[\left(1 - \tfrac{c(s)}{c}\right) \theta_-(G_\gamma * u_0, X_s),\ \tfrac{c(s)}{c}\theta_+(G_\gamma * u_0, X_s) \right]$$

$$c(t) = \begin{cases} 0 & \text{if } t < S \quad \text{or} \quad N(G_\gamma * u_0, x) < d, \\ c & \text{if } t \geq S \quad \text{and } N(G_\gamma * u_0, x) \geq d. \end{cases}$$

where $\{X_t\}_{t\in[0,T]}$ is a stochastic diffusion process with reflection with values in the domain of the image $\overline{D}$. The term $\{K_t^{\overline{D}}\}_{t\in[0,T]}$ is the minimal push needed to keep process X in $\overline{D}$. $\{Y_t\}_{t\in[0,T]}$ (the first component of the solution to the BSDE) is a backward stochastic diffusion process with values in the codomain of the image $\mathbf{R}^n$. $\{Z_t\}_{t\in[0,T]}$, the second component of the solution to the BSDE determines the measurability of the process Y. The process Y is constructed so that it starts at $t = T$ until $t = 0$. The value of the process Y at time $t = 0$ is the reconstructed pixel $u(x)$.

Approximation

Theorem 1 *Let $S < T$, $u_0 : \overline{D} \to \mathbf{R}^n$, $x \in \overline{D}$. Assume that $f(t,y) = c(t)(y - u_0(X_t))$, $c(t) = 0$ for $t < S$ and $\xi = u_0(X_S)$, where X is two-dimensional diffusion process with reflection with values in $\overline{D}$ and starting from x. If (Y,Z) is a solution to BSDE*

$$Y_t = \xi + \int_t^T f(s, Y_s)ds - \int_t^T Z_s\, dW_s, \ \ t \in [0,T],$$

then

$$\lim_{m\to+\infty} Y_0^m = Y_0,$$

where

$$Y_0^m = \sum_{k=j}^{m-1} a_k \mathbf{E}\left[u_0(X_{t_k})\right],$$

and j is a index of discretisation, such that

$$0 = t_0 < t_1 < \cdots < t_j \leq S < t_{j+1} < \cdots < t_m = T, \quad t_{i+1} - t_i = \frac{T}{m},$$

$$a_j = \prod_{r=j}^{m-1}\left(1 + \frac{c(t_r)T}{m}\right) - \frac{c(t_j)T}{m},$$

$$a_k = -\frac{c(t_k)T}{m}\prod_{r=j}^{k-1}\left(1 + \frac{c(t_r)T}{m}\right), \quad k = j+1, j+2, \ldots, m-1,$$

and by $\mathbf{E}$ we denote the expected value.

The coefficients a_k, $k = j, j+1, \ldots, m-1$ define the weights used in the process reconstruction and satisfy the useful condition $\sum_{k=j}^{m-1} a_k = 1$.

[1]This research was supported by the National Science Centre (Poland) under decision number DEC-2012/07/D/ST6/02534

We obtain the following discrete inverse formula for RGB images.

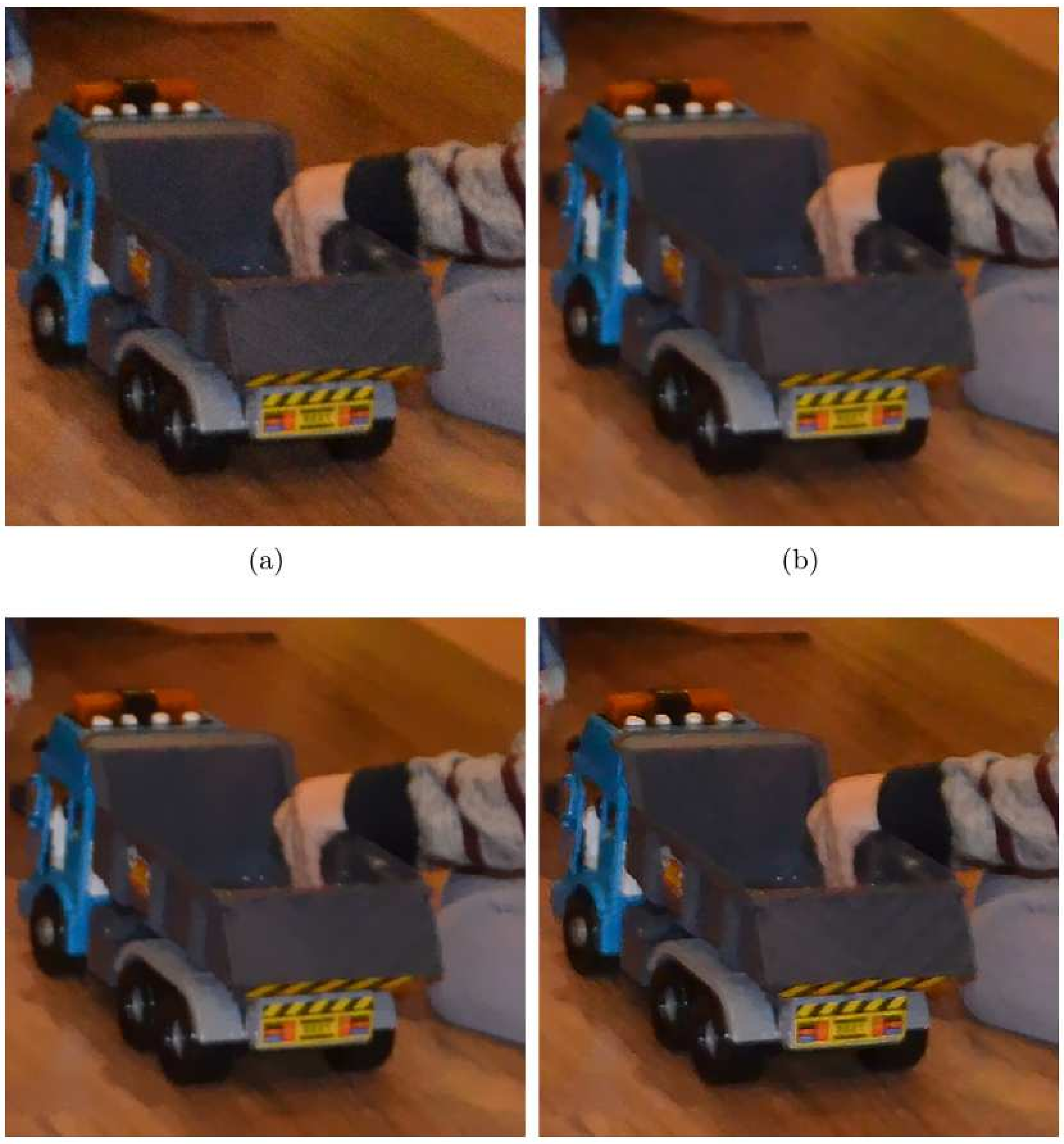
$$u(x) \approx \begin{cases} \dfrac{1}{N_0}\sum_{n_0=1}^{N_0} u_0(X_{t_j}^x(\omega_{n_0})) & \text{if } N(G_\gamma * u_0, x) < d, \\ \frac{1}{N_0}\sum_{n_0=1}^{N_0}\Big[a_j u_0\left(X_{t_j}^x(\omega_{n_0})\right) & \\ \quad + \frac{1}{N_1}\sum_{n_1=1}^{N_1}\sum_{k=j+1}^{m-1} a_k u_0\left(X_{t_k}^{X_{t_j}^x(\omega_{n_0})}(\omega_{n_1})\right)\Big] & \text{if } N(G_\gamma * u_0, x) \geq d \end{cases}$$

where $X_t^a(\omega)$ is the value at time t of ω-trajectory of the process X starting from a point a.

Experimental Results

Lenna (PSNR)

ρ	non local means	new method ($c = 1.5$)	total variation
10	**34.7758**	33.4106	33.2549
15	**33.1034**	32.2666	31.7692
20	**31.9212**	31.3412	30.7777
25	**30.8765**	30.6641	29.9996
30	**30.1963**	30.0058	29.3863
35	**29.5205**	29.4037	28.8702
40	28.8700	**28.8767**	28.4306
45	28.3743	**28.3912**	28.0531
50	27.7861	**27.9544**	27.7119
55	27.3753	**27.5024**	27.4102



(a) (b) (c) (d)

(a) Input high ISO image (b) Result of reconstruction using total variation (c) Result of reconstruction using non local means (d) Result of reconstruction using the new method