Image Restoration using Anisotropic Stochastic Diffusion Collaborated with Non Local Means

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Problem

Let D be a bounded, convex domain in $\mathbf{R}^2$, $u : \overline{D} \to \mathbf{R}$ be an original image and $u_0 : \overline{D} \to \mathbf{R}$ be the observed image of the form

$$u_0 = u + \eta,$$

where η stands for a white Gaussian noise. We assume that u and u_0 are appropriately regular. We are given u_0, the problem is to reconstruct u.

Modified Diffusion with Random Terminal Time

Let

$$X_t = x + \int_0^t \begin{bmatrix} -\frac{(G_\gamma * u_0)_{x_2}(X_s)}{|\nabla(G_\gamma * u_0)(X_s)|}, & 0 \\ \frac{(G_\gamma * u_0)_{x_1}(X_s)}{|\nabla(G_\gamma * u_0)(X_s)|}, & 0 \end{bmatrix} dW_s + K_t^{\overline{D}},$$

where $u_{x_i}(y) = \frac{\partial u}{\partial x_i}(y)$. To avoid false detections due to noise, u_0 is convolved with a Gaussian kernel $G_\gamma(x) = \frac{1}{2\pi\gamma^2} e^{-\frac{|x|^2}{2\gamma^2}}$ (in practice a 3×3 Gaussian mask). The reconstruction pixel is given by

$$u(x) = \mathbf{E}\left[u_0(X_T)\right] \approx \frac{1}{M} \sum_{i=1}^{M} u_0(X_T^m(\omega_i)), \tag{1}$$

where $X^m(\omega_i)$ is the approximation of trajectory of stochastic process X and M is the number of Monte Carlo method iterations.

$$\begin{aligned} X_0^m &= X_0, \\ H_{t_k}^m &= \Pi_{\overline{D}}[X_{t_{k-1}}^m + \sigma(X_{t_{k-1}}^m)(W_{t_k} - W_{t_{k-1}})], \\ X_{t_k}^m &= \begin{cases} H_{t_k}^m, \text{ if } \Theta, \\ X_{t_{k-1}}^m, \text{ elsewhere}, \end{cases} \quad k = 1, 2, ..., \tau_m, \end{aligned}$$

where $\tau_m = \min\{k; k \geq m \text{ and } \Theta \text{ is true } m \text{ times}\}$.
Terminal time τ_m guarantees that the numerical simulation of the diffusion trajectory gives at least m values of $X_{t_k}^m$ which differ from the value in the previous step.

Non Local Means Algorithm

By $\|B_{i,r} - B_{j,r}\|_2$ we denote the Euclidean distance between $B_{i,r}$ and $B_{j,r}$, where patch $B_{i,r}$ means a neighbourhood of a size $2r+1 \times 2r+1$ pixels centred at i. For computational purposes the searching of similar windows can be restricted from all pixels in the image to some square window $B_{i,f}$. The denoising of an image v and a certain patch $B_{i,r}$ is equal to

$$\hat{B}_{i,r} = \frac{1}{Z} \sum_{j \in B_{i,f}} v(B_{j,r}) w(B_{i,r}, B_{j,r}),$$

where $Z = \sum_{j \in B_{i,f}} w(B_{i,r}, B_{j,r})$ and the weight function is given by

$$w(B, Q) = e^{-\frac{\max\left(\|B-Q\|_2^2 - 2\rho^2, 0.0\right)}{s^2}}.$$

Here by ρ we denoted the standard deviation of the noise and by s the filtering parameter set depending on the value of ρ. The weight function is chosen in order to average similar patches up to noise. That is, patches with square distances smaller than $2\rho^2$ are set to 1, while larger distances decrease rapidly accordingly to the exponential kernel.

By applying the procedure for all patches in the image, we will get $(2r+1)^2$ possible estimates for each pixel. These estimates can be finally averaged at each pixel location in order to build the final denoised image

$$NL(v)(i) = \frac{1}{(2r+1)^2} \sum_{j \in B_{i,r}} \hat{B}_{j,r}(i).$$

Anisotropic Stochastic Diffusion Collaborated with Non Local Means

In this section we propose a new method of the image reconstruction based on modified diffusion with random terminal time and patchwise implementation of non local means.
In the case of numerical scheme with random terminal time the reconstructed formula of anisotropic diffusion (1) can be written as

$$u(x) \approx \sum_{i=1}^{M} \frac{1}{M} u_0(X_{\tau_m}^m(\omega_i)),$$

which means that each pixel $u_0(X_{\tau_m}^m(\omega_i))$ is weighted with the value $\frac{1}{M}$. But since pixels have different intensities we may consider them with different weights depending on their neighbourhood. We follow NL-means algorithm and propose a new method of the image restoration based on modified diffusion but such that the weights depend on patches similarity:

$$u(x) = \frac{1}{Z} \sum_{i=1}^{M} u_0(X_{\tau_m}^m(\omega_i)) w(B_{x,r}, B_{X_{\tau_m}^m(\omega_i),r}).$$

In the above formula we used the following notations:

$$\begin{aligned} Z &= \sum_{i=1}^{M} w(B_{x,r}, B_{X_{\tau_m}^m(\omega_i),r}), \; w(B,Q) = e^{-\frac{\max\left(\|B-Q\|_2^2 - 2\rho^2, 0.0\right)}{s^2}} \\ X_0^m(\omega_i) &= x, \\ H_{t_k}^m(\omega_i) &= \Pi_{\overline{D}}[X_{t_{k-1}}^m(\omega_i) + \sigma(X_{t_{k-1}}^m(\omega_i))(W_{t_k} - W_{t_{k-1}})], \\ X_{t_k}^m(\omega_i) &= \begin{cases} H_{t_k}^m(\omega_i), \text{ if } \Theta, \\ X_{t_{k-1}}^m(\omega_i), \text{ elsewhere}, \end{cases} \quad k = 1, 2, ..., \tau_m. \end{aligned}$$

Experimental Results

Maximum values of PSNR

Image	Noise ρ	NL-means algorithm	Stoch. anisotropic diffusion	Perona-Malik	New method
Pirate	10	33.0394	32.0450	32.8268	**33.1379**
	15	30.7474	30.5158	30.8167	**30.8944**
	20	29.6731	29.3619	29.5095	**29.6914**
Cameraman	10	35.3814	35.2304	34.9002	**35.8238**
	15	32.9025	33.7815	32.8394	**33.8727**
	20	32.1931	**32.2683**	31.2357	32.1571

Maximum values of SSIM

Image	Noise ρ	NL-means algorithm	Stoch. anisotropic diffusion	Perona-Malik	New method
Pirate	10	**0.9562**	0.9533	0.9519	0.9558
	15	0.9245	0.9246	0.9204	**0.9257**
	20	0.8952	0.8935	0.8913	**0.8974**
Cameraman	10	0.9600	0.9583	0.9436	**0.9606**
	15	0.9334	0.9369	0.9173	**0.9393**
	20	**0.9214**	0.9144	0.8838	0.9196

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