Backward stochastic differential equations driven by multidimensional fractional Brownian motion[1]

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Introduction

We study the existence and uniqueness of the backward stochastic differential equations driven by multidimensional fractional Brownian motion with Hurst parameters H_k greater than $1/2$, $k=1,\dots,m$. The stochastic integral used throughout the paper is the divergence type integral.

Fractional calculus - multidimensional fBM

Assume that $B^{H_1}, B^{H_2}, \dots, B^{H_m}$ are independent fractional Brownian motions with Hurst parameters $H_1, H_2, \dots, H_m$ respectively, where $H_k \in \left(\frac{1}{2}, 1\right)$, $k=1,2,\dots,m$. It is well known that $EB_t^{H_k}=0$ and

$$E\left(B_t^{H_k}\cdot B_s^{H_j}\right)=\frac{1}{2}\left(|t|^{2H_k}+|s|^{2H_k}-|t-s|^{2H_k}\right)\delta_{kj},$$

$1\le k,j,\le m$, where $\delta_{kj}=1$ for $k=j$ and $\delta_{kj}=0$ for $k\neq j$.
Set $X_t=(X_t^1,\dots,X_t^d)$ where

$$X_t^i=X_0^i+\int_0^t b_i(s)ds+\sum_{k=1}^m\int_0^t\sigma_{ik}(s)dB_s^{H_k},\qquad t\in[0,T],\quad i=1,\dots,d,$$

$X_0=(X_0^1,\dots,X_0^d)$ is a constant vector and b_i are deterministic continuous functions with $\int_0^T|b_i(s)|ds<\infty$, $i=1,\dots d$.

Theorem 1 (multidimensional fractional Itô formula) *Let $\sigma_{ik}\in L^2([0,T])$, $i=1,\dots d$, $k=1,\dots m$ be deterministic functions. Let F be continuously differentiable with respect to t and twice continuously differentiable with respect to x. Then*

$$\begin{aligned}F(t,X_t)&=F(0,X_0)+\int_0^t\frac{\partial F}{\partial s}(s,X_s)ds+\sum_{i=1}^d\int_0^t\frac{\partial F}{\partial x_i}(s,X_s)b_i(s)ds\\&+\sum_{i=1}^d\sum_{k=1}^m\int_0^t\frac{\partial F}{\partial x_i}(s,X_s)\sigma_{ik}(s)dB_s^{H_k}\\&+\frac{1}{2}\sum_{i,j=1}^d\int_0^t\frac{\partial^2 F}{\partial x_i x_j}(s,X_s)\sum_{k=1}^m\frac{d}{ds}(\langle\sigma_{ik},\sigma_{jk}\rangle_{k,s})\,ds,\quad t\in[0,T].\end{aligned}$$

Theorem 2 (fractional Itô chain rule) *Let $E\int_0^T\int_0^T|\mathbb{D}_t^{H_j}\sigma_{ij}(s)|^2dsdt<\infty$. Then*

$$\begin{aligned}X_t^1X_t^2&=X_0^1X_0^2+\int_0^t X_s^1b_2(s)ds+\sum_{k=1}^m\int_0^t X_s^1\sigma_{2k}(s)dB_s^{H_k}\\&+\int_0^t X_s^2b_1(s)ds+\sum_{k=1}^m\int_0^t X_s^2\sigma_{1k}(s)dB_s^{H_k}\\&+\sum_{k=1}^m\left(\int_0^t\mathbb{D}_s^{H_k}X_s^1\sigma_{2k}(s)ds+\int_0^t\mathbb{D}_s^{H_k}X_s^2\sigma_{1k}(s)ds\right).\end{aligned}$$

BSDE with respect to multidimensional fBm

For a given vector constant $\eta_0=(\eta_0^1,\dots,\eta_0^d)$ consider $\eta_t^i=\eta_0^i+\sum_{k=1}^m\int_0^t\sigma_{ik}(s)dB_s^{H_k}$, $t\in[0,T]$, where for $i=1,\dots,d$, $k=1,\dots,m$

(H_1) $\sigma_{ik}:[0,T]\to\mathbb{R}$ are deterministic, continuous, differentiable functions such that $\sigma_{ik}(t)\neq 0$, for all $t\in[0,T]$.

Since $\langle\sigma_{ik},\sigma_{jk}\rangle_{k,t}=H_k(2H_k-1)\int_0^t\int_0^t|v-u|^{2H_k-2}\sigma_{ik}(v)\sigma_{jk}(u)dvdu$,

we have $\frac{d}{dt}(\langle\sigma_{ik},\sigma_{jk}\rangle_{k,t})=2\sigma_{ik}(t)\hat{\sigma}_{jk}(t)>0$, where $\hat{\sigma}_{jk}(t)=\int_0^t\phi_k(t-u)\sigma_{jk}(u)du$.

(H_2) $\xi=h(\eta_T)$ for some function h with bounded derivative and such that $E|\xi|^2<\infty$

(H_3) $f:[0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^m\to\mathbb{R}$ is a continuous function and there exists a positive constant L such that for all $t\in[0,T]$, $x\in\mathbb{R}^d$, $y,y'\in\mathbb{R}$, $z,z'\in\mathbb{R}^m$,

$$|f(t,x,y,z)-f(t,x,y',z')|\le L(|y-y'|+\|z-z'\|)$$

and

$$E\left(\int_0^T|f(t,\eta_t,0,0)|^2dt\right)<\infty.$$

Consider the set $\mathscr{V}_T=\{Y=\phi(\cdot,\eta(\cdot)):\phi\in C_{pol}^{1,3}([0,T]\times\mathbb{R})$ and $\frac{\partial\phi}{\partial t}$ is bounded$\}$

By $\tilde{\mathscr{V}}_T^H$ denote the completition of the set of processes from $\mathscr{V}_T$ with the norm

$$\|Y\|_H=\left(\int_0^T t^{2H-1}E|Y_t|^2dt\right)^{1/2}=\left(\int_0^T t^{2H-1}E|\phi(t,\eta_t)|^2dt\right)^{1/2},\qquad H\ge 1/2.$$

Definition 1 *A solution of the backward stochastic differential equation driven by multidimensional fBm is a pair of processes $(Y,Z)=(Y_t,Z_t)_{t\in[0,T]}$ satisfying*

$$Y_t=\xi+\int_t^T f(s,\eta_s,Y_s,Z_s)ds-\sum_{k=1}^m\int_t^T Z_s^k dB_s^{H_k}\tag{1}$$

and such that $Y\in\tilde{\mathscr{V}}_T^{1/2}$ and $Z=(Z^1,\dots,Z^m)$, where $Z^k\in\tilde{\mathscr{V}}_T^{H_k}$.

Theorem 3 *Assume $(H_1)-(H_3)$. For $(U,V)\in\mathscr{V}_T\times\mathscr{V}_T^m$ let $(Y,Z)\in\mathscr{V}_T\times\mathscr{V}_T^m$ be the unique solution of the following BSDE:*

$$Y_t=\xi+\int_t^T f(s,\eta_s,U_s,V_s)ds-\sum_{k=1}^m\int_t^T Z_s^k dB_s^{H_k},\qquad t\in[0,T].\tag{2}$$

Let $\xi=0$. Then for all $\beta>0$ there exists $C(\beta)=C(\beta,T,L)>0$ such that

$$\begin{aligned}&\sup_{t\in[0,T]}e^{\beta t}E|Y_t|^2+\int_0^T e^{\beta s}E|Y_s|^2ds+\sum_{k=1}^m\int_0^T e^{\beta s}s^{2H_k-1}E|Z_s^k|^2ds\\&\le C(\beta)\left(\int_0^T e^{\beta s}E|U_s|^2ds+\sum_{k=1}^m\int_0^T e^{\beta s}s^{2H_k-1}E|V_s^k|^2ds+\int_0^T e^{\beta s}E|f(s,\eta_s,0,0)|^2ds\right).\end{aligned}$$

Moreover, $C(\beta)$ can be chosen such that $\lim_{\beta\to\infty}C(\beta)=0$.

Theorem 4 *Assume $(H_1)-(H_3)$. There exists a unique solution of (1).*

Proof. We use a fixed point theorem. Consider the mapping $\Phi:\mathscr{V}_T\times\mathscr{V}_T^m\to\mathscr{V}_T\times\mathscr{V}_T^m$ given by $(U,V)\mapsto\Phi(U,V)=(Y,Z)$, where (Y,Z) is the unique solution of the equation (2). $\Phi:\mathscr{V}_{[0,T]}\times\mathscr{V}_{[0,T]}^m\to\mathscr{V}_{[0,T]}\times\mathscr{V}_{[0,T]}^m$ described by (2) is a contraction with respect to norm:

$$\|(u,v)\|_{1/2,H}=\|u\|_{1/2}+\|v\|_H=\|u\|_{1/2}+\sqrt{\sum_{k=1}^m\|v^k\|_{H_k}^2}.$$

Therefore a sequence $\{(Y^p,Z^p)\}_{p\in\mathbb{N}}$ defined by

$$Y_t^{p+1}=\xi+\int_t^T f(s,\eta_s,Y_s^p,Z_s^p)ds-\sum_{k=1}^m\int_t^T Z_s^{p+1,k}dB_s^{H_k},\qquad t\in[0,T]$$

is a Cauchy sequence in $\tilde{\mathscr{V}}_T^{1/2}\times\tilde{\mathscr{V}}_T^H$, $\tilde{\mathscr{V}}_T^H=\tilde{\mathscr{V}}_T^{H_1}\times\dots\times\tilde{\mathscr{V}}_T^{H_m}$. Hence there exists a pair (Y,Z) satsyfying (1) and such that $\lim_{p\to\infty}\|(Y^p-Y,Z^p-Z)\|_{1/2,H}=0$.

Applications

We can write the partial differential equation associated with (1) of the form:

$$\begin{aligned}u(T,x)=h(x),\qquad&\frac{\partial u}{\partial t}+\frac{1}{2}\sum_{i,j=1}^d\sum_{k=1}^m\frac{d}{dt}\langle\sigma_{ik},\sigma_{jk}\rangle_{k,t}\frac{\partial^2u}{\partial x_ix_j}\\&=-f\left(t,x,u,-\sum_{i=1}^d\sigma_{i1}\frac{\partial u}{\partial x_i},\dots,-\sum_{i=1}^d\sigma_{im}\frac{\partial u}{\partial x_i}\right)\end{aligned}\tag{3}$$

Theorem 5 *If the equation (3) has a solution $u(t,x)$ which is continuously differentiable with respect to t and twice continuously differentiable with respect to x, then $Y_t=u(t,\eta_t)$, $Z_t^k=\sum_{i=1}^d\sigma_{ik}(t)\frac{\partial u}{\partial x_i}(t,\eta_t)$, $k=1,\dots,m$.*

Theorem 6 *Assume that (1) has a solution (Y,Z) of the form $Y_t=u(t,\eta_t)$ and $Z_t=(Z_t^1,\dots,Z_t^m)$ where $Z_t^k=v_k(t,\eta_t)$, $k=1,\dots m$. Then $v_k(t,x)=\sum_{i=1}^d\sigma_{ik}(t)\frac{\partial u}{\partial x_i}(t,x)$.*

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