Feynman-Kac Formula and Restoration of High ISO Images[1]

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The Inverse Problem for RGB Images

$u : \bar{D} \to \mathbb{R}^3$ – original colour image,
$u_0 : \bar{D} \to \mathbb{R}^3$ – noisy colour image,

$$u_0 = u + \eta,$$

η – Gaussian noise.

We are given u_0, the problem is to reconstruct u.

The Inverse Problem

- linear filtering
- wevelets theory
- variational mathods
- PDE-based methods
- stochastic modelling
- non local means approach

Non Local Means

Let $v = \{u_0(i) | i \in I = \mathbf{Z}^2 \cap D\}$ be a discrete noisy image

$$NL(v)(i) = \sum_{j \in I} w(N_i, N_j) v(j).$$

The weight $w(N_i, N_j)$ depends on the similarity of the intensity gray level or colour vectors of neighbourhoods N_i, N_j centred at pixels i and j

Algorithm for Non Local Means

Let $B_{i,F}$ be a window of size $F \times F$ centred at i

$u(i) \leftarrow 0$; $Z \leftarrow 0$;
for $j \in B_{i,F}$ **do**
  $u(i) \leftarrow u(i) + w(B_{i,r}, B_{j,r}) \cdot v(j)$;
  $Z \leftarrow Z + w(B_{i,r}, B_{j,r})$;
end for
$u(i) \leftarrow \frac{u(i)}{Z}$;

Buades et al. (2005)

$$w(B_{i,r}, B_{j,r}) = \exp\left(-\frac{\max\left(\frac{\left\|B_{i,r}-B_{j,r}\right\|^2}{3(2r+1)} - 2\rho^2, 0\right)}{s^2}\right)$$

Buades et al. (2011)

Feynman-Kac Formula

$$\frac{\partial u}{\partial t} = Au$$

$$u(t, x) = \mathbf{E}\left[u_0(X_t)\right]$$

Diffusion process

$$X_0 = x, \qquad \text{Borkowski (2013)}$$
$$H_k = \Pi_D \left(X_{k-1} + h \cdot (\Phi_{0,1}, \Phi_{0,1}) \right)$$
$$X_k = \begin{cases} H_k, & \text{if } \Theta, \\ X_{k-1}, & \text{elsewhere}, \end{cases} \qquad k = 1, 2, ..., \tau_m,$$

$\Pi_D(x)$ denotes a projection of x on the set D
$\Phi_{0,1}$ is a random number generator from the normal distribution with mean 0 and standard deviation 1.
By Θ we mean the condition

$$\|(G_\delta * u_0)(H_k) - (G_\delta * u_0)(X_{k-1})\| \leq 0.8 \cdot \rho$$

where G_δ is 3×3 Gaussian mask and by τ_m

$$\tau_m = \min\{k; k \geq m \text{ and } \Theta \text{ is true } m \text{ times}\}.$$

Discretization of the Feynman-Kac formula

$$u(x) \approx \sum_{i=1}^{M} \frac{1}{M} u_0(X_{\tau_m(\omega_i)}(\omega_i)),$$

Algorithm for Stochastic Anisotropic Diffusion

$u(i) \leftarrow 0$; $X(0) \leftarrow i$;
for $m = 1 \dots M$ **do**
  $k \leftarrow 1$;
  while $k \leq K$ **do**
    $X(k) \leftarrow \Pi_D \left(X(k-1) + h \cdot (\Phi_{0,1}, \Phi_{0,1}) \right)$;
    if $|G_\delta * u_0(X(k)) - G_\delta * u_0(X(k-1))| \leq 0.8 \cdot \rho$ **then**
      $k \leftarrow k + 1$;
    end if
  end while
  $u(i) \leftarrow u(i) + \frac{1}{M} \cdot u_0(X(K))$;
end for

Feynman-Kac Formula and Non Local Means

$$u(x) = \frac{1}{Z}\sum_{i=1}^{M} w(B_{x,r}, B_{X_{\tau_m(\omega_i)}(\omega_i),r})u_0(X_{\tau_m(\omega_i)}(\omega_i)).$$

New Algorithm

$u(i) \leftarrow 0$; $X(0) \leftarrow i$; $Z \leftarrow 0$;
for $m = 1 \ldots M$ **do**
  $k \leftarrow 1$;
  while $k \leq K$ **do**
    $X(k) \leftarrow \Pi_D \left(X(k-1) + h \cdot (\Phi_{0,1}, \Phi_{0,1}) \right)$;
    if $\left| G_\delta * u_0(X(k)) - G_\delta * u_0(X(k-1)) \right| \leq 0.8 \cdot \rho$ **then**
      $k \leftarrow k + 1$;
    end if
  end while
  $u(i) \leftarrow u(i) + w\left(B_{i,r}, B_{X(K),r} \right) \cdot u_0(X(K))$;
  $Z \leftarrow Z + w\left(B_{i,r}, B_{X(K),r} \right)$;
end for
$u(i) \leftarrow \frac{u(i)}{Z}$;

a)  b) 

Figure : Original test images: 512×512 a) Peppers b) Lenna

PSNR

Image	Noise ρ	NL-means	Stoch. anis. diff.	New method
Peppers	10	32.9404	32.3572	**33.1289**
	20	30.2984	30.6057	**31.0984**
	30	27.3031	29.2140	**29.8178**
	40	26.6428	28.1988	**28.6991**
	50	25.9941	27.2549	**27.8611**
	60	25.5353	26.5094	**27.0802**
Lenna	10	34.0127	33.0964	**34.3550**
	20	31.5780	31.1301	**31.8386**
	30	29.6376	29.6041	**30.2871**
	40	28.6433	28.5068	**29.0773**
	50	27.7377	27.5843	**28.1308**
	60	27.0094	26.7234	**27.3205**

SSIM

Image	Noise ρ	NL-means	Stoch. anis. diff.	New method
Peppers	10	0.9536	0.9470	**0.9548**
	20	0.9194	0.9146	**0.9232**
	30	0.8898	0.8866	**0.8994**
	40	0.8645	0.8645	**0.8759**
	50	0.8383	0.8447	**0.8564**
	60	0.8284	0.8229	**0.8362**
Lenna	10	**0.9588**	0.9467	0.9573
	20	0.9224	0.9115	**0.9232**
	30	0.8929	0.8790	**0.8930**
	40	0.8640	0.8528	**0.8648**
	50	0.8377	0.8289	**0.8414**
	60	0.8174	0.8050	**0.8176**

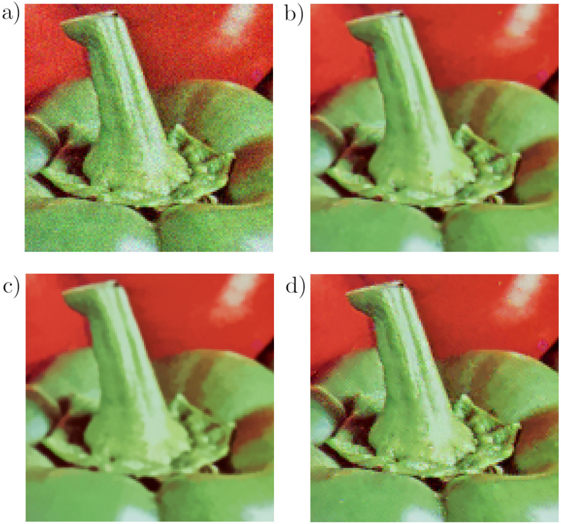


Figure : a) Noisy image $\rho = 10$ b) New method c) Anisotropic diffusion d) NL-means

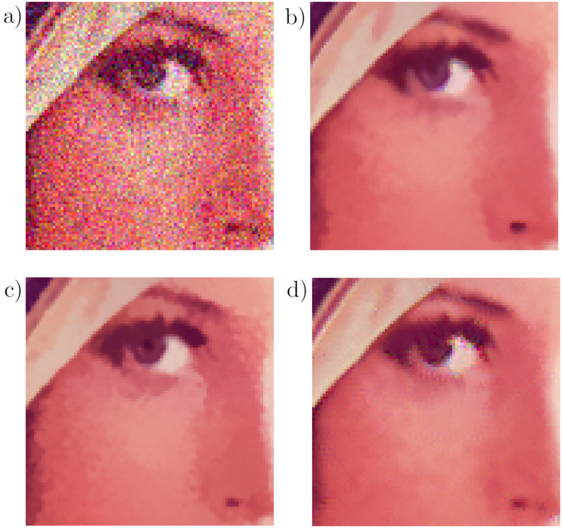
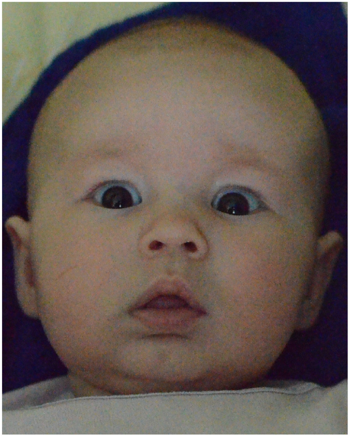


Figure : a) Noisy image $\rho = 30$ b) New method c) Anisotropic diffusion d) NL-means

a) 

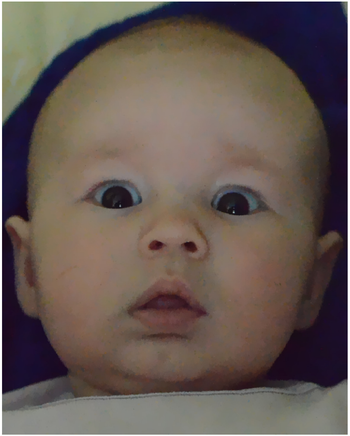
b) 

Figure : a) Input high ISO image b) Result of the reconstruction using new method

Conclusion

Applying the Feynman-Kac formula to express anisotropic diffusion in stochastic terms allows to adapt the idea from non local means, giving what is the best from these two classical approaches.