ON THE RING OF CONSTANTS
FOR DERIVATIONS OF POWER SERIES RINGS
IN TWO VARIABLES

BY
LEONID MAKAR - LIMANOV (DETROIT) AND
ANDRZEJ NOWICKI (TORUŃ)

Abstract. Let \( k[[x, y]] \) be the formal power series ring in two variables over a field \( k \) of characteristic zero and let \( d \) be a nonzero derivation of \( k[[x, y]] \). We prove that if \( \text{Ker}(d) \neq k \) then \( \text{Ker}(d) = \text{Ker}(\delta) \), where \( \delta \) is a jacobian derivation of \( k[[x, y]] \). Moreover, we prove that \( \text{Ker}(d) \) is of the form \( k[[h]] \), for some \( h \in k[[x, y]] \).

1. Introduction. Let \( k[x, y] \) be the ring of polynomials in two variables over a field \( k \) of characteristic zero. Let \( d \) be a nonzero derivation of \( k[[x, y]] \) and let \( A \) be its ring of constants. It is well known (see for example [4]) that if \( A \neq k \) then \( A = \text{Ker}(\delta) \), where \( \delta \) is a jacobian derivation of \( k[[x, y]] \). It is also well known ([5] or [4]) that \( A \) is of the form \( k[[h]] \), for some \( h \in k[[x, y]] \).

In 1975, A. Płoski ([6]) proved similar facts for derivations in convergent power series ring in two variables. In this paper we show that the above facts are also true for derivations in formal power series ring in two variables.

2. Preliminaries. If \( F \) is a nonzero power series from \( k[[x, y]] \) then we denote by \( \omega(F) \) the lowest homogeneous form of \( F \), and we denote by \( o(F) \) the order of \( F \), that is, \( o(F) = \deg \omega(F) \). Moreover, we assume that \( \omega(0) = 0 \) and \( o(0) = \infty \). It is clear that \( o(FG) = o(F) + o(G) \) for all \( F, G \in k[[x, y]] \).

By a derivation of \( k[[x, y]] \) we mean every \( k \)-linear mapping \( d : k[[x, y]] \to k[[x, y]] \) such that \( d(FG) = Fd(G) + Gd(F) \) for \( F, G \in k[[x, y]] \). Let us recall (see for example [2], [4]) that each derivation \( d \) of \( k[[x, y]] \) has a unique presentation of the form \( d = F \frac{\partial}{\partial x} + G \frac{\partial}{\partial y} \), where \( F, G \in k[[x, y]] \). Recall also that if \( d \) is a derivation of \( k[[x, y]] \) then its kernel \( k[[x, y]]^d = \{ F \in k[[x, y]]; d(F) = 0 \} \) is a subring of \( k[[x, y]] \) containing \( k \).

If \( F, G \) are two power series from \( k[[x, y]] \) then we denote by \( J(F, G) \) the jacobian of \( F, G \), that is,

\[
J(F, G) = \frac{\partial F}{\partial x} \frac{\partial G}{\partial y} - \frac{\partial F}{\partial y} \frac{\partial G}{\partial x}.
\]

If \( F \in k[[x, y]] \) is fixed then a mapping \( \delta : k[[x, y]] \to k[[x, y]] \) defined as

\[
\delta(G) = J(F, G), \quad \text{for} \quad G \in k[[x, y]],
\]

is a derivation of \( k[[x, y]] \); we call it a jacobian derivation.

2000 Mathematics Subject Classification: Primary 12H05, Secondary 13F25
Supported by NSF Grant DMS-9700894 and KBN Grant 2 PO3A 017 16.
Assume that $\deg f = sn$, $\deg g = sm$, where $\gcd(n, m) = 1$. If $J(f, g) = 0$ then there exist a homogeneous polynomial $h \in k[x, y]$ of degree $s$ and nonzero elements $a, b \in k$ such that $f = ah^n$ and $g = bh^m$.

**Proof.** See, for example, [1]. □

### 3. Jacobian derivations

In this section we prove the following

**Theorem 3.2.** Let $d$ be a nonzero derivation of $k[[x, y]]$. If $F \in k[[x, y]]^d \setminus k$, then $k[[x, y]]^d = k[[x, y]]^d$, where $\delta = J(F, -)$.

For the proof of this theorem we need two lemmas.

**Lemma 3.3.** Let $d$ be a nonzero derivation of $k[[x, y]]$. If $F, G \in k[[x, y]]^d$ then the polynomials $\omega(F), \omega(G)$ are algebraically dependent over $k$.

**Proof.** It is obvious when $d(x) = 0$ or $d(y) = 0$ or $\omega(F) \in k$ or $\omega(G) \in k$.

So we may assume that $d(x) \neq 0$, $d(y) \neq 0$, $\deg \omega(F) \geq 1$ and $\deg \omega(G) \geq 1$.

Put $d(x) = U$, $d(y) = V$, and let $F = F - \omega(F)$, $G = G - \omega(G)$, $U = U - \omega(U)$, $V = V - \omega(V)$. Then:

\[
\begin{align*}
0 = d(F) &= \left( \frac{\partial \omega(F)}{\partial x} + \frac{\partial F}{\partial x} \right) (\omega(U) + U) \\
&\quad + \left( \frac{\partial \omega(F)}{\partial y} + \frac{\partial F}{\partial y} \right) (\omega(V) + V), \\
0 = d(G) &= \left( \frac{\partial \omega(G)}{\partial x} + \frac{\partial G}{\partial x} \right) (\omega(U) + U) \\
&\quad + \left( \frac{\partial \omega(G)}{\partial y} + \frac{\partial G}{\partial y} \right) (\omega(V) + V).
\end{align*}
\]

Assume that $\deg \omega(U) < \deg \omega(V)$. Then comparing the lowest forms in (1) we have:

\[
\frac{\partial \omega(F)}{\partial x} \omega(U) = 0 \quad \text{and} \quad \frac{\partial \omega(G)}{\partial x} \omega(U) = 0.
\]

Hence $\frac{\partial \omega(F)}{\partial x} = \frac{\partial \omega(G)}{\partial x} = 0$ and the polynomials $\omega(F)$ and $\omega(G)$ are algebraically dependent because they belong to $k[y]$. If $\deg \omega(U) > \deg \omega(V)$ then analogously $\omega(F)$ and $\omega(G)$ are in $k[x]$ and are also dependent.

Assume now that $\deg \omega(U) = \deg \omega(V)$. Then comparing the lowest forms in (1) we get the following system of equations:

\[
\begin{align*}
\frac{\partial \omega(F)}{\partial x} \omega(U) + \frac{\partial \omega(F)}{\partial y} \omega(V) &= 0, \\
\frac{\partial \omega(G)}{\partial x} \omega(U) + \frac{\partial \omega(G)}{\partial y} \omega(V) &= 0.
\end{align*}
\]

Since $(\omega(U), \omega(V)) \neq (0, 0)$, this system has a nonzero solution. This means that the jacobian $J(\omega(F), \omega(G))$ is equal to zero and so, the polynomials $\omega(F)$ and $\omega(G)$ are algebraically dependent. □
Lemma 3.4. Let $d_1$ and $d_2$ be nonzero derivations of $k[[x,y]]$. Assume that the rings $k[[x,y]]^{d_1}$ and $k[[x,y]]^{d_2}$ both contain a series $F \in k[[x,y]] \setminus k$. Then $k[[x,y]]^{d_1} = k[[x,y]]^{d_2}$.

Proof. We can assume that $\omega(F) \notin k$. By Lemma 3.3 and 2.1 we know that $\omega(F)$ is a polynomial of $h_1$ and a polynomial of $h_2$ (for some $h_1, h_2 \in k[x,y]$). So we may assume that $h_1 = h_2 = h$.

Let us take any $G \in k[[x,y]]$ for which $\omega(G)$ is not a polynomial of $h$. Then $d_1(G) \neq 0$ and $d_2(G) \neq 0$. Let us consider the derivation

$$d_3 = d_2(G)d_1 - d_1(G)d_2.$$ 

It is clear that $d_3(F) = d_3(G) = 0$. But $\omega(F)$ and $\omega(G)$ are algebraically independent. By Lemma 3.3 it is possible only if $d_3 = 0$. So $d_2(G)d_1 = d_1(G)d_2$ and the kernels are the same. \qed

Proof of Theorem 3.2. It is a simple consequence of Lemma 3.4 because $d \neq 0$, $\delta \neq 0$ and the rings $k[[x,y]]^d$ and $k[[x,y]]^{\delta}$ contain $F \in k[[x,y]] \setminus k$. \qed

4. A generator of the ring of constants Let $d : k[[x,y]] \to k[[x,y]]$ be a nonzero derivation and let $\mathcal{A} = k[[x,y]]^d$. We want to show that the ring $\mathcal{A}$ is of the form $k[[F]]$ for some series $F \in k[[x,y]]$. If $\mathcal{A} = k$, then $\mathcal{A} = k[[F]]$ for $F = 0$. Let us assume that $\mathcal{A} \neq k$.

We already know (by Lemmas 3.3 and 2.1) that all lowest homogeneous forms of nonzero elements in $\mathcal{A}$ are scalar multiples of powers of a homogeneous form $\varphi$. For each $F \in \mathcal{A} \setminus \{0\}$ let us denote by $\gamma(F)$ the degree of $\varphi$ in $\omega(F)$, that is, if $\omega(F) = a\varphi^n$ where $0 \neq a \in k$, then $\gamma(F) = n$. Assume moreover that $\gamma(0) = \infty$.

Let us consider the semi-group $\pi = \{\gamma(F); 0 \neq F \in \mathcal{A}\}$. Since $\mathcal{A} \neq k$ this semi-group contains positive numbers. Let $\gamma$ be the greatest common divisor of the elements of $\pi$.

Lemma 4.5. There exist $F, G \in \mathcal{A} \setminus \{0\}$ such that $\gamma = \gamma(F) - \gamma(G)$.

Proof. Since $\gamma$ is the greatest common divisor, there exist nonnegative integers $i_1, \ldots, i_n, j_1, \ldots, j_m$ and nonzero series $F_1, \ldots, F_n, G_1, \ldots, G_m$ from $\mathcal{A}$ such that

$$\gamma = i_1\gamma(F_1) + \cdots + i_n\gamma(F_n) - j_1\gamma(G_1) - \cdots - j_m\gamma(G_m).$$

Put $F = F_1^{i_1} \cdots F_n^{i_n}$ and $G = G_1^{j_1} \cdots G_m^{j_m}$. Then $F, G \in \mathcal{A} \setminus \{0\}$ and $\gamma = \gamma(F) - \gamma(G)$. \qed

Lemma 4.6. Let $F, G \in \mathcal{A} \setminus \{0\}$ be as in Lemma 4.5. Let $\gamma(G) = s\gamma$. Then $n\gamma \in \pi$, for any $n > s^2 - s - 1$. 

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Proof. Since \( \gamma = \gamma(F) - \gamma(G) \), we have \( \gamma(F) = (s + 1)\gamma \). Assume that \( n > s^2 - s - 1 \). Let \( n = us + r \), where \( u, r \) are integers such that \( 0 \leq r < s \). Put \( i = r, j = u - r \). Then \( i \geq 0 \) and \( j \geq 0 \) (since \( u \geq s - 1 \geq r \)), and moreover
\[
i(s + 1) + js = r(s + 1) + (u - r)s = us + r = n.
\]
Let \( H = F^iG^j \). Then \( 0 \neq H \in A \) and
\[
\gamma(H) = i\gamma(F) + j\gamma(G) = i(s + 1)\gamma + js\gamma = n\gamma,
\]
that is, \( n\gamma \in \pi \). □

We may extend the mapping \( \gamma(-) \) to \( \mathbb{A}_0 \setminus \{0\} \) (where \( \mathbb{A}_0 \) is the field of fractions of \( \mathbb{A} \)) by defining
\[
\gamma(A/B) = \gamma(A) - \gamma(B)
\]
for all nonzero \( A, B \in \mathbb{A} \).

Lemma 4.7. Let \( f, g \in \mathbb{A}_0 \setminus \{0\} \). If \( \gamma(f) = \gamma(g) \), then there exists \( c \in k \setminus \{0\} \) such that \( \gamma(f - cg) > \gamma(f) \).

Proof. It is clear if \( f, g \in \mathbb{A} \). Let \( f = A/B, g = C/D \), where \( A, B, C, D \in \mathbb{A} \setminus \{0\} \). Since \( \gamma(f) = \gamma(g) \), we have
\[
\gamma(AD) = \gamma(A) + \gamma(D) = \gamma(C) + \gamma(B) = \gamma(CB),
\]
and so, there exists nonzero \( c \in k \) such that \( \gamma(AD - CB) > \gamma(AD) \). Then we have
\[
\gamma(f - cg) = \gamma((AD - CB)/BD) = \gamma(AD - CB) - \gamma(BD)
\]
\[
> \gamma(AD) - \gamma(BD) = \gamma(A/B)
\]
\[
= \gamma(f),
\]
that is \( \gamma(f - cg) > \gamma(f) \). □

Consider now the fraction
\[
h = F/G,
\]
where \( F \) and \( G \) are such nonzero series from \( \mathbb{A} \) as in Lemma 4.5. We know that \( \gamma(h) = \gamma \). We want to show that \( h \in \mathbb{A} \).

Lemma 4.8. There exists a natural number \( n \) such that \( h^n \in \mathbb{A} \).

Proof. Let \( \gamma(G) = s\gamma, \gamma(F) = (s + 1)\gamma \) and let \( n \) be a natural number such that \( n > s^2 - s \). We shall show that \( h^n \in k[[x, y]] \).

We know, by Lemma 4.6 and its proof, that there exist integers \( i_1 \geq 0, j_1 \geq 0 \) such that \( \gamma(H_1) = n\gamma \), where \( H_1 = F^{i_1}G^{j_1} \). Then \( \gamma(h^n) = n\gamma = \gamma(H_1) \), so (by Lemma 4.7) \( \gamma(h^n - c_1H_1) > \gamma(h^n) \) for some \( c_1 \in k \setminus \{0\} \).

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Put \( h_1 = h^n - c_1 H_1 \). If \( h_1 = 0 \) then \( h^n = c_1 H_1 \in \mathbb{A} \) and we are done. Assume that \( h_1 \neq 0 \). Then \( \gamma(h_1) = n_1 \gamma \), where \( n_1 > n \geq n_0 \). Using again Lemmas 4.6 and 4.7 we see that \( \gamma(h_1 - c_2 H_2) > \gamma(h_1) \), for some \( c_2 \in k \setminus \{0\} \), \( H_2 = F^{12} G^{j_2}, \ i_2 \geq 0, \ j_2 \geq 0 \). Put \( h_2 = h_1 - c_2 H_2 = h^n - c_1 H_1 - c_2 H_2 \). If \( h_2 = 0 \), then \( h^n \in \mathbb{A} \) and we are done, and so on.

If the above procedure has no end, then we obtain an infinite sequence \((c_m H_m)\) of nonzero elements from \( \mathbb{A} \) such that \( o(H_m) < o(H_{m+1}) \) for any natural \( m \) and \( o(h^n - U_{m+1}) > o(h^n - U_m) \), where \( U_m = c_1 H_1 + \cdots + c_m H_m \). This means that \( h^n \) is the limit of the convergent sequence \((U_m)\). Since each \( U_m \) belongs to the ring \( k[x, y] \) which is complete, the limit \( h^n \) also belongs to \( k[x, y] \). Therefore \( h^n \in k[x, y] \cap \mathbb{A}_0 = \mathbb{A} \). □

Lemma 4.9. \( h \in k[x, y] \).

Proof. The ring \( k[x, y] \) is a unique factorization domain (see, for example, [3] p.163), hence it is integrally closed. Lemma 4.8 implies that \( h \) integral over \( k[x, y] \), so \( h \in k[x, y] \). □

Lemma 4.10. \( \mathbb{A} = k[[h]] \).

Proof. We already know (by the previous lemma) that \( h \in k[[x, y]] \cap \mathbb{A}_0 = \mathbb{A} \). We know also that \( \gamma(h) = \gamma \geq 1 \), so \( o(h) \geq 1 \) (that is \( h \) has no constant term). It is clear that \( k[[h]] \subseteq \mathbb{A} \). Let \( U \in \mathbb{A} \setminus k \). We shall show that \( U \in k[[h]] \).

Since \( \gamma(U) = n_1 \gamma \) for some natural \( n_1 \), there exists a nonzero \( c_1 \in k \) such that \( \gamma(U_1) > \gamma(U) \) for \( U_1 = U - c_1 h^{n_1} \). If \( U_1 = 0 \) then \( U \in k[[h]] \) and we are done. Assume that \( U_1 \neq 0 \). Since \( U_1 \in \mathbb{A} \) there exist \( n_2 > n_1 \) and \( 0 \neq c_2 \in k \) such that \( \gamma(U_2) > \gamma(U_1) \) for \( U_2 = U_2 - c_2 h^{n_2} = U - c_1 h^{n_1} - c_2 h^{n_2} \). If \( U_2 = 0 \) then \( U \in k[[h]] \) and we are done, and so on.

If the above procedure has an end we see that \( U \in k[[h]] \). In the opposite case \( U \) is the limit of the infinite convergent sequence \((h_m)\), where each \( h_m \) is of the form

\[
h_m = c_1 h^{n_1} + \cdots + c_m h^{n_m},
\]

where \( c_1, \ldots, c_m \) are nonzero elements of \( k \) and \( n_1 < \cdots < n_m \). Therefore \( U \in k[[h]] \). □

From the above lemmas we get the following main result of our paper.

Theorem 4.11. If \( d \) is a nonzero derivation of \( k[x, y] \), then \( k[x, y]^d = k[[h]] \) for some \( h \in k[x, y] \). □

References


Department of Mathematics and Computer Science
Bar-Ilan University
52900 Ramat-Gan, Israel,
(e-mail: lml@macs.biu.ac.il);

Department of Mathematics,
Wayne State University
Detroit, Michigan 48202, USA,
(e-mail: lml@bmath.wayne.edu)

N. Copernicus University,
Faculty of Mathematics and Informatics,
87-100 Toruń, POLAND,
(e-mail: anow@mat.uni.torun.pl).