Rings and fields of constants
for derivations in characteristic zero

by

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Dedicated to Professor Takashi Nagahara for his sixtieth birthday.

Abstract

Let $k$ be a field of characteristic zero and $A$ a finitely generated $k$-algebra. We give a description of all $k$-subalgebras of $A$ which are rings of constants for derivations of $A$. Moreover we show some applications of our description.

Introduction. In this paper we prove, among others, the followings theorems:

(a) (An extension of results of Suzuki [11] and Derksen [2]). Let $K \subseteq L$ be fields of characteristic zero. Then $K$ is the field of constants of a derivation of $L$ if and only if $K$ is algebraically closed in $L$.

(b) Let $A$ be a finitely generated algebra over a field of characteristic zero and let $D$ be a family of $k$-derivations of $A$. Then there exists a one $k$-derivation $d$ of $A$ such that the ring of constants with respect to $D$ is the ring of constants with respect do $d$.

(c) Let $G \subseteq GL_n(k)$ be a connected algebraic group which acts on $k[x_1, \ldots, x_n]$, the polynomial ring over a field $k$ of characteristic zero. Then there exists a $k$-derivation $d$ of $k[x_1, \ldots, x_n]$ such that the invariant ring $k[x_1, \ldots, x_n]^G$ is equal to the ring of constants with respect to $d$.

We show that theorems (b) and (c) are consequences of the more general result (Theorem 4.4) which we prove using (a).

All rings in this paper are assumed to be commutative. If a $k$-algebra $A$ has no zero divisors then we say that $A$ is a $k$-domain and we denote by $A_0$ its field of fractions.
1 Preliminaries

Throughout this paper $k$ is a field of characteristic zero.

Let $A$ be a commutative $k$-algebra and $D$ a family of $k$-derivations of $A$. We denote by $A^D$ the set of constants of $A$ with respect to $D$, that is,

$$A^D = \{ a \in A; \; d(a) = 0, \; \text{for every} \; d \in D \}.$$  

If $D$ has only one element $d$ then we write $A^d$ instead of $A^D$.

The set $A^D$ is a $k$-subalgebra of $A$. If $A$ is a field then $A^D$ is a subfield of $A$ containing $k$.

It is easy to prove the following two propositions

**Proposition 1.1** If $D$ is a family of $k$-derivations of a $k$-domain $A$ then the ring $A^D$ is integrally closed in $A$.

**Proposition 1.2** If $D$ is a family of $k$-derivations of a field $L$ of characteristic zero then the field $L^D$ is algebraically closed in $L$.

2 Derivations with trivial fields of constants for purely transcendental field extensions

Let $S$ be a set of algebraically independent elements over $k$. Denote by $|S|$ the cardinality of $S$ and consider the field $k(S)$, the pure transcendental extension of $k$.

In this section we present $k$-derivations $d$ of $k(S)$ such that $k(S)^d = k$. Let us start from known examples for $|S| < \infty$.

**Proposition 2.1** Let $d_1$, $d_2$, $d_3$, $d_4$ be $k$-derivations of $k(x_1, \ldots, x_n)$ defined as follows

$$d_1 = \frac{\partial}{\partial x_1} + (x_1 x_2 + 1) \frac{\partial}{\partial x_2} + (x_2 x_3 + 1) \frac{\partial}{\partial x_3} + \ldots + (x_{n-1} x_n + 1) \frac{\partial}{\partial x_n}, \quad (1)$$

$$d_2 = \frac{\partial}{\partial x_1} + \frac{1}{x_1} \frac{\partial}{\partial x_2} + \frac{1}{x_2} \frac{\partial}{\partial x_3} + \ldots + \frac{1}{x_{n-1}} \frac{\partial}{\partial x_n}, \quad (2)$$

$$d_3 = \frac{\partial}{\partial x_1} + \frac{1}{x_1} \frac{\partial}{\partial x_2} + \frac{1}{x_1 x_2} \frac{\partial}{\partial x_3} + \ldots + \frac{1}{x_1 \ldots x_{n-1}} \frac{\partial}{\partial x_n}, \quad (3)$$

$$d_4 = \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_2 x_3 \frac{\partial}{\partial x_3} + \ldots + x_2 \ldots x_n \frac{\partial}{\partial x_n}, \quad (4)$$

Then $k(x_1, \ldots, x_n)^{d_i} = k$, for $i = 1, 2, 3, 4$.  

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Proof. (1) is a consequence of Shamsuddin’s result [10] mentioned in [4]. For (2) and (3) see Suzuki [11]. For (4) see Derksen [2]. It follows also from (3) because $d_4 = x_1 \ldots x_{n-1}d_3$ (under a permutation of variables).

Applying the same argument as Suzuki [11] in his proof for the derivation $d_2$ we obtain (for countable $S$):

**Proposition 2.2** Let $d$ be the $k$-derivation of $L = k(x_1, x_2, \ldots)$ defined by $d(x_i) = \frac{1}{x_{i-1}}$, for $i = 1, 2, \ldots$, where $x_0 = 1$. Then $L^d = k$.

There exists a simpler example of $k$-derivation in $k(x_1, x_2, \ldots)$ with the trivial field of constants. It is not difficult to prove the following

**Proposition 2.3** Let $d$ be the $k$-derivation of $L = k(x_1, x_2, \ldots)$ defined by $d(x_i) = x_{i+1}$, for $i = 1, 2, \ldots$. Then $L^d = k$.

A similar derivation, as in Proposition 2.3, may be constructed for any infinite cardinality of $S$.

**Proposition 2.4** Let $S$ be an infinite set of algebraically independent elements over $k$ and let $L = k(S)$. Then there exists a $k$-derivation $d$ of $L$ such that $L^d = k$.

**Proof.** Since $S$ is infinite, there exists a well-order $\leq$ on $S$ without maximal element. If $s \in S$ then denote by $s^*$ the next element of $s$, that is,

$$s^* = \min\{t \in S; \ s < t\}.$$ 

Now let $d$ be the $k$-derivation of $L$ defined by $d(s) = s^*$, for any $s \in S$. It is clear that $L^d = k$.

From the above propositions we get the following

**Theorem 2.5** If $k \subset L$ is a purely transcendental field extension of characteristic zero then there exists a derivation $d$ of $L$ such that $L^d = k$.

## 3 Algebraically closed subfields and fields of constants

Suzuki, in [11], and Derksen, in [2], have showed that if $k \subset L$ is an extension of fields (of characteristic zero) of finite transcendence degree then every intermediate field, which is algebraically closed in $L$, is the field of constants for a $k$-derivation of $L$. In the proofs they used the derivations from Proposition 2.1 and moreover they used the following evident
Lemma 3.1 Let \( k \subset L \) be an algebraic field extension, \( d : k \to k \) a derivation, and \( \delta : L \to L \) the derivation which is the unique extension of \( d \) to \( L \). If the field \( k^d \) is algebraically closed in \( L \) then \( L^\delta = k^d \).

Thanks to Lemma 3.1 and Theorem 2.5 we see that the proofs of Suzuki and Derksen are valid for arbitrary field extension (without any assumption on the transcendence degree).

Theorem 3.2 Let \( K \subseteq L \) be fields of characteristic zero. The following conditions are equivalent:

(1) There exists a derivation \( d \) of \( L \) such that \( L^d = K \);
(2) \( K \) is algebraically closed in \( L \).

Proof. \((1) \implies (2)\) See Proposition 1.2.

\((2) \implies (1)\). Let \( S \) be a transcendence basis of \( L \) over \( K \). Then the extension \( K(S) \subseteq L \) is algebraic. Let \( d' : K(S) \to K(S) \) be a derivation such that \( K(S)^{d'} = K \) (Theorem 2.5), and let \( d : L \to L \) be the unique extension of \( d' \) to \( L \). Then, by Lemma 3.1, \( L^d = K(S)^{d'} = K \).

Applying this theorem and Proposition 1.2 one can prove, for instance, the following

Theorem 3.3 Let \( D \) be a family of derivations of a field \( L \) of characteristic zero. Then there exists a one derivation \( d \) of \( L \) such that \( L^D = L^d \).

4 Integrally closed subrings and rings of constants

We see, by Theorem 3.2, that the converse of Proposition 1.2 is also true. Now let us return to Proposition 1.1. Let \( A \) be a \( k \)-domain and let \( B \) be a \( k \)-subalgebra of \( A \) which is integrally closed in \( A \). We may ask the following

Question 4.1 Is \( B \) a ring of constants with respect to \( k \)-derivations of \( A \) ?

This question has a negative answer in general

Example 4.2 Let \( A = k[x_1, \ldots, x_n] \) \( (n \geq 2) \) be the polynomial ring over \( k \) and let \( B \) be the integral closure of the ring \( k[x_1, x_1x_2] \) in \( A \). Then of course \( B \) is integrally closed in \( A \) and \( x_2 \notin B \) (see the example of Gustafson in [12] p.489). Therefore

\[
 k[x_1, x_1x_2] \subseteq B \subseteq k[x_1, x_2] \subseteq A.
\]

Suppose that \( D \) is a family of \( k \)-derivations of \( A \) such that \( A^D = B \). Let \( d \in D \). Then \( d(x_1) = 0 \) and \( 0 = d(x_1x_2) = x_1d(x_2) \). Hence \( x_2 \in B \) and we have a contradiction.

Observe that if \( B \) and \( A \) are as in Example 4.2, and \( B_0 \) is the field of fractions of \( B \), then \( B_0 = k(x_1, x_2) \), so \( B_0 \cap A = k[x_1, x_2] \neq B \).

Rings of constants have an additional property:
Proposition 4.3 Let $D$ be a family of $k$-derivations of a $k$-domain $A$ and let $B = A^D$. Then $B_0 \cap A = B$.

Proof. Denote by $D_0$ the set $\{d_0; \ d \in D\}$, where $d_0$ is the $k$-derivation of $A_0$ defined by $d_0(x) = (a \cdot y + a \cdot y \cdot b^{-2})$, for all $a, b \in A$ and $b \neq 0$. Let $M$ be the field $A_0^{D_0}$. Then it is clear that $B_0 \subseteq M$ and $M \cap A = B$. Hence $B \subseteq B_0 \cap A \subseteq M \cap A = B$, that is, $B_0 \cap A = B$.

Now we are able to prove the following description of all $k$-subalgebras of a finitely generated $k$-domain, which are rings of constants with respect to $k$-derivations.

Theorem 4.4 Let $A$ be a finitely generated $k$-domain, where $k$ is a field of characteristic zero. Let $B$ be a $k$-subalgebra of $A$. The following conditions are equivalent:

1. There exists a $k$-derivation $d$ of $A$ such that $B = A^d$;
2. The ring $B$ is integrally closed in $A$ and $B_0 \cap A = B$.

Proof. (1) $\implies$ (2) follows from Propositions 1.1 and 4.3.

(2) $\implies$ (1). Let $M$ be the algebraic closure of the field $B_0$ in the field $A_0$.

By Theorem 3.2 there exists a $k$-derivation $\delta : A_0 \to A_0$ such that $M = A_0^\delta$. Since $A$ is finitely generated over $k$, $A = k[f_1, \ldots, f_s]$ and $A_0 = k(f_1, \ldots, f_s)$, for some $f_1, \ldots, f_s \in A$. Let $w$ be a nonzero element of $A$ such that the elements $w \delta(f_1), \ldots, w \delta(f_s)$ belong to $A$, and let $\delta' = w \delta$. Then $A_0^{\delta'} = A_0^\delta = M$ and $\delta'(A) \subseteq A$. Consider the $k$-derivation $d$ of $A$ which is the restriction of $\delta'$ to $A$.

We will show that $A^{d} = B$.

For this purpose observe, at first, that $A^{d} = M \cap A$.

In fact: If $x \in M \cap A$ then $x \in A$ and $\delta'(x) = 0$, hence $d(x) = \delta'(x) = 0$, i.e., $x \in A^{d}$.

If $x \in A^{d}$ then $x \in A$ and $\delta'(x) = d(x) = 0$, so $x \in M \cap A$.

Now we will prove that $M \cap A = B$. The inclusion $B \subseteq M \cap A$ is clear.

Assume that $x \in M \cap A$. Then $x \in A$ and $x$ is algebraic over $B_0$. So, there exists a natural number $n$ such that

$$\frac{p_n}{q_n}x^n + \frac{p_{n-1}}{q_{n-1}}x^{n-1} + \ldots + \frac{p_1}{q_1}x + \frac{p_0}{q_0} = 0$$

where $p_0, \ldots, p_n; q_0, \ldots, q_n \in B$, $p_n \neq 0$ and $q_0 q_1 \ldots q_n \neq 0$.

Multiplying the two sides of the above equality by $q_0 q_1 \ldots q_n$ we have

$$c_n x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0,$$

where $c_0, \ldots, c_n \in B$ and $c_n \neq 0$.

Denote $y = c_n x$. Then $y \in A$ and

$$y^n + c_{n-1}y^{n-1} + c_{n-2}y^{n-2} + \ldots + c_1 y^{n-1} + c_0 y^{n-1} = 0.$$

This means that $y$ is an element of $A$ which is integral over $B$. So $y \in B$, because $B$ is integrally closed in $A$. Hence $x = yc_n^{-1} \in B_0$ and hence $x \in B_0 \cap A = B$. Therefore $M \cap A = B$ and we have $A^{d} = M \cap A = B$. 

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The following is an immediate consequence of Theorem 4.4, Proposition 1.1 and Proposition 4.3.

**Theorem 4.5** Let $A$ be a finitely generated algebra over a field $k$ of characteristic zero and let $D$ be a family of $k$-derivations of $A$. Then there exists a one $k$-derivation $d$ of $A$ such that $A^D = A^d$.

5 Rings of invariants

Let $A$ be a $k$-domain and $G$ a subgroup of $\text{Aut}_k(A)$, the group of all $k$-automorphisms of $A$. Denote

$$A^G = \{a \in A; \sigma(a) = a, \text{ for any } \sigma \in G\}.$$  

The set $A^G$ is a $k$-subalgebra of $A$. We may ask the following

**Question 5.1** Is $A^G$ of the form $A^d$, for some $k$-derivation $d$ of $A$?

It is evident that if $B = A^G$ then $B_0 \cap A = B$. Therefore, by Theorem 4.4, our question reduces to

**Question 5.2** Is $A^G$ integrally closed in $A$?

If $G$ is finite then it is well known that $A$ is integral over $A^G$ (see, for instance, [1] Exercises in Section 5). This means that our questions have negative answers in general.

**Proposition 5.3** Let $A$ be a $k$-domain and $G \subseteq \text{Aut}_k(A)$ a group. Assume that $G$ does not have any proper subgroup of finite index. Then $A^G$ is integrally closed in $A$.

**Proof.** Let us denote $B = A^G$.

Assume that $a \in A$ is an integral element over $B$ and $f \in B[t]$ is a monic polynomial such that $f(a) = 0$. Then

$$f(\sigma(a)) = \sigma(f(a)) = \sigma(0) = 0,$$

for any $\sigma \in G$, hence $S = \{\sigma(a); \sigma \in G\}$ is a set of roots of $f$. Since $B$ is a $k$-domain, the polynomial $f$ has only a finite set of roots. Let $\{r_1 = a, r_2, \ldots, r_s\}$ be the set of all roots of $f$ belonging to $S$ and let

$$G_i = \{\sigma \in G; \sigma(a) = r_i\},$$

for $i = 1, \ldots, s$.

Then $G = G_1 \cup \ldots \cup G_s$ and $G_i \cap G_j = \emptyset$, for $i \neq j$, and we see that $G_1$ is a subgroup of $G$ and its index is equal to $s < \infty$. So $s = 1$ and hence $\sigma(a) = a$, for any $\sigma \in G$. Therefore $a \in B = A^G$.
Assume now that $G \subseteq \text{GL}_n(k)$ is an algebraic group which acts on $k[x_1, \ldots, x_n]$, the polynomial ring over $k$. If $G$ is connected then $G$ has no closed proper subgroup of finite index (see for instance [3] p.53). Repeating the argument of the proof of Proposition 5.3, we see that (if $G$ is connected) the ring $k[x_1, \ldots, x_n]^G$ is integrally closed in $k[x_1, \ldots, x_n]$. Therefore, by Theorem 4.4, we obtain

**Theorem 5.4** Let $k$ be a field of characteristic zero, and $G \subseteq \text{GL}_n(k)$ a connected algebraic group. Then there exists a $k$-derivation $d$ of $k[x_1, \ldots, x_n]$ such that $k[x_1, \ldots, x_n]^G = k[x_1, \ldots, x_n]^d$.

Look now at the Nagata’s counterexample [7] to the fourteenth problem of Hilbert. As a simple consequence of Theorem 5.4 we have

**Corollary 5.5 (Derksen [2])** Let $A = k[x_1, \ldots, x_n]$, where $n = 2r^2$, $r = 4, 5, \ldots$. There exists a $k$-derivation $d$ of $A$ such that the ring $A^d$ is not finitely generated over $k$.

The above corollary is also a consequence of Theorem 4.5 because it is clear that the ring in the Nagata’s counterexample is of the form $A^D$, where $D$ is a family of locally nilpotent $k$-derivations of $A = k[x_1, \ldots, x_n]$.

6 Remarks

Let $A = k[x_1, \ldots, x_n]$, $L = k(x_1, \ldots, x_n)$. Let us look again at Proposition 2.1. We see four $k$-derivations $d$ of $L$ with the trivial field of constants. Observe that, in any case, $d(x_1) = 1$. There exists a useful method for constructions of such derivations in $L$. This method is based on the following two propositions.

**Proposition 6.1 (Suzuki [11] Lemma 4)** Let $k \subset k(x) \subseteq L$ be fields of characteristic zero, where $x \in L$ is a transcendental element over $k$. Let $d : k \rightarrow k$ be a derivation and let $t$ be an element from $k \setminus d(k)$. Assume that $\delta : k(x) \rightarrow k(x)$ is the unique derivation such that $\delta|k = d$ and $\delta(x) = t$. Then $k(x)^\delta = k^d$.

**Proposition 6.2** Let $S$ be a finite set of algebraically independent elements over a field $k$ of characteristic zero, and let $M$ be an overfield of $L = k(S)$. If $d : L \rightarrow M$ is a $k$-derivation then $d(L) \neq M$.

Note that if the set $S$ is infinite then Proposition 6.2 in general fails.

**Example 6.3** Let $Q$ be the field of rational numbers and let $L = Q(x_1, x_2, \ldots)$. Then $L = \{a_1, a_2, \ldots\}$ is a countable set. Consider the derivation $d : L \rightarrow L$ defined as $d(x_n) = a_n$, for $n = 1, 2, \ldots$. Then $d(L) = L$. 

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We do not know any example of $k$-derivation $d$ of $L = k(x_1, \ldots, x_n)$, $n \geq 4$, such that $L^d = k$ and $d(x_1), \ldots, d(x_n)$ are homogeneous polynomials of the same degree. There is a candidate:

$$d = x_1x_2 \frac{\partial}{\partial x_1} + x_2x_3 \frac{\partial}{\partial x_2} + \ldots + x_{n-1}x_n \frac{\partial}{\partial x_{n-1}} + x_n x_1 \frac{\partial}{\partial x_n}.$$  

We know only that $k[x_1, \ldots, x_n]^d = k$ (see [6]).

Our second candidate is the following generalization of the Jouanolou’s derivation ([5] page 159).

$$d = x_2^s \frac{\partial}{\partial x_1} + x_3^s \frac{\partial}{\partial x_2} + \ldots + x_{n-1}^s \frac{\partial}{\partial x_{n-1}} + x_n^s \frac{\partial}{\partial x_n},$$

where $s \geq 2, n \geq 3$.

It is known ([5], see also [6]) that if $s \geq 2$ and $n = 3$ then $k(x_1, \ldots, x_n)^d = k$, but in general case the problem seems to be difficult.

Let $A = k[x_1, \ldots, x_n]$, $d$ a $k$-derivation of $A$ and $B = A^d$. We showed ([9]) that if $n \geq 3$ then the minimal number of generators of $B$ over $k$ is unbounded. It is known (see [8] for details) that if $n \leq 3$ then $B$ is finitely generated over $k$. We see, by Corollary 5.5, that it is not true in general for $n = 32, 50, \ldots$. There is a still open question for remaining $n$, for instance, if $n = 4$.

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**References**


**Note added in proof.**

(1) We already know that the derivation $d = x_1 x_2 \frac{\partial}{\partial x_1} + \cdots + x_n x_1 \frac{\partial}{\partial x_n}$, mentioned in Section 7, has the trivial field of constants, that is, $k(x_1, . . . , x_n)^d = k$ (see [6]).

(2) There exist linear homogeneous $k$-derivations of $k[x_1, . . . , x_n]$ with trivial fields of constants. It is proved in the author’s paper *On the non-existence of rational first integrals for systems of linear differential equations*, to appear in Linear Algebra and Its Applications.

(3) Recently Deveney and Finston (in $G_a$–actions on $\mathbb{C}^3$ and $\mathbb{C}^7$, preprint 1993) showed that if $n = 7$ then there exist a $k$-derivation $d$ of $k[x_1, . . . , x_n]$ such that the ring $k[x_1, . . . , x_n]^d$ is not finitely generated over $k$.

(4) The results of this paper were presented at the Luminy Conference on the Polynomial Automorphisms in October 1992.