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THE FOURTEENTH PROBLEM OF HILBERT FOR POLYNOMIAL DERIVATIONS

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Abstract. We present some facts, observations and remarks concerning the problem of finiteness of the rings of constants for derivations of polynomial rings over a commutative ring k containing the field $\mathbb{Q}$ of rational numbers.

1. Definitions and basic facts. Throughout this paper k is a commutative ring containing $\mathbb{Q}$. Let A be a commutative algebra over k with identity. A k-linear mapping $d : A \to A$ is called a *k-derivation* of A if it satisfies the Leibniz rule: $d(ab) = ad(b)+bd(a)$.

Let d be a k-derivation of A. W denote by A^d the kernel of d, that is,

$$A^d = \operatorname{Ker} d = \{a \in A;\ d(a) = 0\}.$$

This set is a k-subalgebra of A which we call the *ring of constants* of d. If A is a domain and k is a field, then we denote by A_0 the field of quotients of A and we denote also by d the unique extension of d to A_0. In this case A_0^d is a subfield of A_0 containing k.

Let D be a family of k-derivations of A. Then we have the ring of constants $A^D = \bigcap_{d\in D} A^d = \{a \in A;\ d(a) = 0 \ \text{ for all } \ d \in D\}$. We will mostly consider derivations of a polynomial ring in a finite set of variables. In such a case the rings of the form A^D are not interesting for us. It is known ([36], [35]) that in this case every ring A^D is of the form A^d for some k-derivation d of A.

We say that a k-subalgebra B of A is *d-stable* if $d^{-1}(B) \subseteq B$, that is, if for any $a \in A$ we have: $d(a) \in B \Rightarrow a \in B$. The algebra A is of course d-stable and the intersection of d-stable subalgebras is d-stable. So always there exists the smallest d-stable subalgebra of A. It is easy to see ([41]) that this subalgebra is equal to

$$\operatorname{Nil}(d) := \{a \in A;\ \underset{n\geqslant 0}{\exists}\, d^n(a) = 0\}.$$

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We say that d is *locally nilpotent* if $A = \mathrm{Nil}(d)$. Thus, d is locally nilpotent if and only if A has no proper d-stable subalgebras.

Assume now that $A = k[X] := k[x_1, \dots, x_n]$ is the polynomial ring over k. In this case we know a description of all k-derivations of A. If d is a k-derivation of $k[X]$, then we have the polynomials $f_1 := d(x_1), \dots, f_n := d(x_n)$, belonging to $k[X]$, and then

$$d = f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}.$$

Every k-derivation d of $k[X]$ is uniquely determined by a sequence $(f_1, \dots, f_n)$ of polynomials from $k[X]$. If d is a k-derivation of $k[X]$, then d is locally nilpotent if and only if $x_1, \dots, x_n \in \mathrm{Nil}(d)$. If $n = 2$ and k is reduced (that is, without nonzero nilpotents), then we have the following useful criterion due to van den Essen and Berson ([16] 33, [2]).

THEOREM 1.1. *Let d be a k-derivation of $k[x, y]$, where k is a reduced ring containing $\mathbb{Q}$. Put*

$$m := \max\left\{\deg_x d(x),\ \deg_y d(x),\ \deg_x d(y),\ \deg_y d(y)\right\}.$$

Then d is locally nilpotent if and only if $d^{m+2}(x) = d^{m+2}(y) = 0$.

Locally nilpotent derivations of polynomial rings play an important role in algebra and algebraic geometry. It is well known that many open famous problems may be formulated using locally nilpotent derivations and their rings of constants. Various facts and results concerning this subject we may find in the following books and papers: van den Essen ([16]), Makar-Limanov ([29]), Kraft ([28]), Drensky ([14]), Berson ([1]), Maubach ([31]), van Rossum ([43]), Wang ([49]), the author ([35]), and in the papers of Daigle, Derksen, Deveney, Finston, Freudenburg, Hadas, Janssen, Matsumura, Miyanishi, Stein, Tyc, Zurkowski, and many others.

2. The fourteenth problem of Hilbert. Let k be a field of characteristic zero, $n \geqslant 1$, $k[X] := k[x_1, \dots, x_n]$ the polynomial ring over k, and $k(X) := k(x_1, \dots, x_n)$ the field of rational functions over k. Assume that L is a subfield of $k(X)$ containing k. The fourteenth problem of Hilbert is the following question ([33], [34]).

QUESTION 2.1. *Is the ring $L \cap k[X]$ finitely generated over k?*

In 1954 Zariski ([52]) proved that the answer is affirmative if $\mathrm{tr.deg}_k L \leqslant 2$. It is known, by a famous counterexample of Nagata ([33], [34]), that if $\mathrm{tr.deg}_k L \geqslant 4$, then it is possible to obtain a negative answer. If $\mathrm{tr.deg}_k L = 3$, then the problem is still open.

Assume that d is a k-derivation of $k[X]$. Then we have the field $L := \left(k[X]^d\right)_0$, the field of quotients of the constant ring $k[X]^d$, which is a subfield of $k(X)$ containing k. The intersection $L \cap k[X]$ is equal to $k[X]^d$. So, in this case, the fourteenth problem of Hilbert is the following question.

QUESTION 2.2. *Is the ring $k[X]^d$ finitely generated over k?*

In this question k is of characteristic zero. If $\mathrm{char}(k) > 0$, then it is easy to show ([37]) that the answer is affirmative. So, let again $\mathrm{char}(k) = 0$. Using the above mentioned result of Zariski it is not difficult to show (see [37]) that if $n \leqslant 3$, then the answer is affirmative. What does happen for $n \geqslant 4$?

In 1993 Derksen ([11]) showed that the ring from the Nagata counterexample is of the form $k[X]^d$ for some derivation d of $k[X]$ with $n = 32$. Thus, he proved:

THEOREM 2.3 (Derksen). *There exists a k-derivation d of $k[X] := k[x_1, \dots, x_{32}]$ such that $k[X]^d$ is not finitely generated over k.*

If G is a subgroup of $\mathrm{Aut}_k(k[X])$, the group of all k-automorphisms of $k[X]$, then we denote by $k[X]^G$ the subalgebra of invariants of G, that is,

$$k[X]^G = \{f \in k[X];\ \sigma(f) = f \text{ for all } \sigma \in G\}.$$

In 1994 the author, inspired by the above Derksen theorem, proved:

THEOREM 2.4. *If $G \subseteq \mathrm{GL}_n(k)$ is a connected algebraic group, then there exists a k derivation d of $k[X]$ such that $k[X]^G = k[X]^d$.*

In 1990 Roberts ([42]) gave a new counterexample to the fourteenth problem of Hilbert with $n = 7$. In 1994 Deveney and Finston ([13]) realized the Roberts counterexample in the following form.

THEOREM 2.5 (Deveney and Finston). *Let d be the k-derivation of $k[X] := k[x_1, x_2, x_3, y_1, y_2, y_3, y_4]$ defined by $d(x_1) = d(x_2) = d(x_3) = 0$ and*

$$d(y_1) = x_1^3, \quad d(y_2) = x_2^3, \quad d(y_3) = x_3^3, \quad d(y_4) = (x_1x_2x_3)^2.$$

Then the ring $k[X]^d$ is not finitely generated over k.

Using this theorem one can easily deduce that if $n \geqslant 7$, then there always exists a k-derivation d of $k[X]$ such that the ring of constants $k[X]^d$ is not finitely generated over k.

Observe that the derivation d from the above theorem is locally nilpotent. This derivation has no slice. We say that a locally nilpotent k-derivation d of a k-algebra A has a *slice* if there exists an element $s \in A$ such that $d(s) = 1$. It is well known (see for example [16] or [35]) that if A is a finitely generated k-algebra and d is a locally nilpotent k-derivation of A having a slice, then the ring of constants A^d is finitely generated over k.

Similar examples of derivations for $n \geqslant 7$ one can find, for instance, in [27] and [19]. Later, in 1998, Freudenburg ([22]) constructed a locally nilpotent derivation with the same property for $n = 6$.

THEOREM 2.6 (Freudenburg). *Let d be the k-derivation of $k[X] := k[x_1, x_2, y_1, y_2, y_3, y_4]$ defined by $d(x_1) = d(x_2) = 0$ and*

$$d(y_1) = x_1^3, \quad d(y_2) = x_2^3 y_1, \quad d(y_3) = x_2^3 y_2, \quad d(y_4) = x_1^2 x_2^2.$$

Then the ring $k[X]^d$ is not finitely generated over k.

In 1999, Daigle and Freudenburg ([8], see also [23]) gave a similar example with $n = 5$.

THEOREM 2.7 (Daigle and Freudenburg). *Let d be the k-derivation of the polynomial ring $k[X] := k[a, b, x, y, z]$ defined by $d(a) = d(b) = 0$ and*

$$d(x) = a^2, \quad d(y) = ax + b, \quad d(z) = y.$$

Then the ring $k[X]^d$ is not finitely generated over k.

Observe that also in this case the derivation d is locally nilpotent. So, if $n \leqslant 3$, then $k[X]^d$ is always finitely generated over k, and if $n \geqslant 5$, then there exists a k-derivation (even locally nilpotent) of $k[X]$ with non-finitely generated ring of constants. For $n = 4$ the problem is open. In this case there is no counterexample for arbitrary derivations and we do not know if $k[X]^d$ is finitely generated for locally nilpotent derivations. In the last case we know, by the result of Maubach ([30]), that $k[X]^d$ is finitely generated if d is triangular and monomial. More precisely,

THEOREM 2.8 (Maubach). *Let d be a k-derivation of $k[X] := k[x_1, x_2, x_3, x_4]$ such that $d(x_1)$, $d(x_2)$, $d(x_3)$, $d(x_4)$ are monomials and $d(x_1) \in k$, $d(x_2) \in k[x_1]$, $d(x_3) \in k[x_1, x_2]$ and $d(x_4) \in k[x_1, x_2, x_3]$. Then the ring $k[X]^d$ is generated by at most 4 elements.*

Recently Daigle and Freudenburg ([10]) proved that the ring of constants of any triangular derivation of $k[x_1, x_2, x_3, x_4]$ is finitely generated over k. This is obtained as a corollary of the following more general result concerning triangular R-derivations of $R[x, y, z]$ for certain rings R.

THEOREM 2.9 (Daigle and Freudenburg). *Let R be a k-affine Dedekind domain or a localization of such a ring. If d is any triangular R-derivation of $R[x, y, z]$, then the ring of constants $R[x, y, z]^d$ is finitely generated as an R-algebra.*

Let $n, m \geq 1$ and let $k[X, Y] := k[x_1, \ldots, x_n, y_1, \ldots, y_m]$ be the polynomial ring over a field k of characteristic zero. Let $k[X] := k[x_1, \ldots, x_n]$. A k-derivation $d : k[X, Y] \to k[X, Y]$ is called *elementary monomial* if it is of the form

$$d = a_1 \frac{\partial}{\partial y_1} + \cdots + a_m \frac{\partial}{\partial y_m},$$

where $a_1, \ldots, a_m$ are monomials belonging to $k[X]$. The derivation d from Theorem 2.5 is elementary monomial. Here $n = 3$ and $m = 4$, and the ring of constants $k[X, Y]^d$ is not finitely generated over k. A. van den Essen and T. Janssen, in [19], proved that if $n+m \leq 5$, then the ring of constants of every elementary monomial derivation is a finitely generated k-algebra. They proved also that if $m \geq 4$, then for any $n \geq 3$ there exists a elementary monomial derivation which the ring of constants is not finitely generated over k. Some interesting facts concerning elementary monomial derivations one can find in [49].

Recently Khoury ([26]) proved that if $n + m = 6$, then the ring of constants of every elementary monomial derivation is finitely generated over k. Moreover, he shows that, in this case, such rings of constants are generated by at most six elements which are either the variables $x_1, \ldots, x_n$ or linear with respect to $y_1, \ldots, y_m$.

3. The minimal number of generators. Let d be a k-derivation of $k[X] = k[x_1, \ldots, x_n]$, where k is a ring containing $\mathbb{Q}$. If $k[X]^d \neq k$ and $k[X]^d$ is finitely generated over k then we denote by $\gamma(d)$ the minimal number of polynomials in $k[X] \setminus k$ which generate $k[X]^d$ over k. Moreover, we assume that $\gamma(d) = 0$ iff $k[X]^d = k$, and $\gamma(d) = \infty$ iff $k[X]^d$ is not finitely generated over k. We already know from the previous section that there exist a natural number n and a k-derivation d of $k[X]$ such that $\gamma(d) = \infty$.

If k is not a domain, then there always exist k-derivations of $k[X]$ (even for $n = 1$) with non-finitely generated ring of constants. It follows from the following proposition which we may find in [16] or [43].

Proposition 3.1. *Assume that k contains two nonzero elements a and b such that $ab = 0$. Let d be the k-derivation of $k[X]$ defined by*

$$d(x_1) = ax_1, \quad d(x_2) = 0, \quad \dots, \quad d(x_n) = 0.$$

Then $\gamma(d) = \infty$. More exactly, $k[X]^d = k[x_2, x_3, \dots, x_n, bx_1, bx_1^2, bx_1^3, \dots]$.

If $d = 0$, then it is clear that $\gamma(d) = n$. If $n = 1$, k is a domain and $d \neq 0$, then of course $\gamma(d) = 0$. Now we assume that k is a domain, $d \neq 0$ and $n \geqslant 2$.

It is known ([51], [15]) that if k is a field then every Dedekind k-subalgebra of $k[X]$ is a polynomial ring in one variable over k. As a consequence of this fact we obtain

Theorem 3.2. *Let d be a nonzero k-derivation of $k[X] = k[x_1, \dots, x_n]$, where k is a field of characteristic zero. If $\operatorname{tr.deg}_k(k[X]^d) \leqslant 1$, then $\gamma(d) \leqslant 1$.*

Note also the following similar result of Russell and Sathaye ([44]) for locally nilpotent derivations.

Theorem 3.3 (Russell and Sathaye). *Let k be a* UFD *and let d be a nonzero locally nilpotent k-derivation of $k[X]$. Then $\gamma(d) = 1$.*

As a consequence of Theorem 3.2 we get

Corollary 3.4. *Let k be a field of characteristic zero. If d is a nonzero k-derivation of $k[x, y]$, then $\gamma(d) \leqslant 1$.*

The same corollary is true when k is a UFD. If d is locally nilpotent, then it follows from Theorem 3.3 (see [7]). If d is arbitrary, then it is a result of Berson ([1], see also [16] 21).

Theorem 3.5 (Berson). *Let k be a* UFD *containing $\mathbb{Z}$. If d is a nonzero k-derivation of $k[x, y]$, then $\gamma(d) \leqslant 1$.*

If k is not a UFD (even if k is noetherian), then the following example shows that Theorem 3.5 does not always hold.

Example 3.6 (Berson). Let $k := \mathbb{C}[t^2, t^3]$. Consider the k-derivation d of $k[x, y]$ defined by

$$d(x) = t^3, \quad d(y) = -t^2.$$

Then k is not a UFD, and $k[x, y]^d$ is not finitely generated over k.

Observe that the derivation d from the above example is locally nilpotent. Similar examples of locally nilpotent derivations of $k[x, y]$ with non-finitely generated rings of constants one can find in [3] and [31].

We say that a k-derivation d of $k[X]$ is *irreducible* if the image $d(k[X])$ is not contained in a principal proper ideal of $k[X]$. The following theorem of Bhatwadekar and Dutta ([3]) gives a necessary and sufficient condition for the ring of constants of an irreducible locally nilpotent derivation of $k[x, y]$ to be isomorphic to a polynomial ring in one variable over k.

Theorem 3.7 (Bhatwadekar and Dutta). *Let k be a noetherian domain containing $\mathbb{Q}$, and let d be a locally nilpotent k-derivation of $k[x,y]$. The following conditions are equivalent.*

(1) *d is irreducible and $\gamma(d) = 1$.*

(2) *The polynomials $d(x)$ and $d(y)$ either form a regular $k[x,y]$-sequence or are comaximal in $k[x,y]$.*

Bhatwadekar and Dutta ([3]) proved also other interesting facts concerning locally nilpotent k-derivations of $k[x,y]$ and the number $\gamma(d)$. Now we present some of these facts.

Let I be an ideal of a noetherian domain R. Let $\operatorname{Ass}_R(R/I) = \{P_1,\dots,P_r\}$ be the set of all the associated prime ideals of I and let S be the multiplicative subset of R defined as $S := R \smallsetminus (P_1 \cup \ldots \cup P_r)$. Then the *$n$-th symbolic power* $I^{(n)}$ of I is defined to by the ideal

$$I^{(n)} := R \cap I^n(S^{-1}R).$$

We denote by $\mathcal{R}(I)$ the *symbolic Rees algebra* of I, that is, $\mathcal{R}(I)$ is the R-subalgebra of the polynomial algebra $R[t]$ (in one variable t over R) such that each element u of $\mathcal{R}(I)$ is of the form

$$u = a_n t^n + \cdots + a_1 t + a_0,$$

with $a_0 \in I^{(0)} = R$, $a_1 \in I^{(1)}$, …, $a_n \in I^{(n)}$. In other words, $\mathcal{R}(I)$ is the graded algebra

$$\mathcal{R}(I) = \bigoplus_{n\geqslant 0} I^{(n)} t^n.$$

Let us recall (see for example [25]) the notion of a reflexive R-module. Let R be a ring and M an R-module. Let $M^* := \operatorname{Hom}_R(M,R)$ denote the *dual* R-module of M. Consider also the R-module

$$M^{**} = (M^*)^* = \operatorname{Hom}_R(M^*,R) = \operatorname{Hom}_R(\operatorname{Hom}_R(M,R),R).$$

Then we have the R-homomorphism $\Phi_M : M \to M^{**}$ defined by

$$\Phi_M(m)(\varphi) := \varphi(m),$$

for all $m \in M$ and $\varphi \in M^*$. We say that M is *reflexive* if Φ_M is an isomorphism of R-modules. It is well-known that if R is a field, then every finite dimensional vector space over R is a reflexive R-module.

Theorem 3.8 (Bhatwadekar and Dutta). *Let k be a noetherian normal domain containing $\mathbb{Q}$ and let d be a nonzero locally nilpotent k-derivation of $k[x,y]$.*

Then $k[x,y]^d$ has the structure of a graded ring

$$k[x,y]^d = \bigoplus_{n\geqslant 0} A_n$$

with $A_0 = k$ and A_n a finite reflexive k-module of rank one for every n.

In fact, when k is not a field, there exists an ideal I of unmixed height one in k such that $k[x,y]^d$ is isomorphic to the symbolic Rees algebra $\mathcal{R}(I)$ as a graded k-algebra.

Conversely, let k be as above and let I be an unmixed ideal of height one in k. Then there exists a locally nilpotent k-derivation d of $k[x,y]$ whose ring of constants $k[x,y]^d$ is isomorphic to $\mathcal{R}(I)$ as a graded k-algebra.

As a consequence of this theorem we obtain:

THEOREM 3.9 (Bhatwadekar and Dutta). *Let k be a one-dimensional noetherian normal domain containing $\mathbb{Q}$. Then for any locally nilpotent k-derivation d of $k[x,y]$ the ring of constants $k[x,y]^d$ is finitely generated over k.*

If k is a domain, then we denote by $\mathrm{Cl}(k)$ and $\mathrm{Pic}(k)$, the class group of k and its Picard subgroup, respectively.

THEOREM 3.10 (Bhatwadekar and Dutta). *Let k be a noetherian normal domain containing $\mathbb{Q}$ such that the group $Cl(k)/Pic(k)$ is torsion. Then for any locally nilpotent k-derivation d of $k[x,y]$ the ring of constants $k[x,y]^d$ is finitely generated over k.*

Using a result of Cowsik [5] and the above facts we get

THEOREM 3.11 (Bhatwadekar and Dutta). *Let k be a noetherian normal domain containing $\mathbb{Q}$ with* $\mathrm{Krull\text{-}dim}(k) = 2$. *Then the following two conditions are equivalent.*

(1) *The group $Cl(k)/Pic(k)$ is torsion.*

(2) *For any locally nilpotent k-derivation d of $k[x,y]$ the ring of constants $k[x,y]^d$ is finitely generated over k.*

If $\mathrm{Krull\text{-}dim}(R) > 2$, then the following example shows that the above condition (1) is not necessary for the kernel of every locally nilpotent R-derivation of $k[x,y]$ to be finitely generated over k.

EXAMPLE 3.12 ([3]). Consider the ring

$$k := \mathbb{C}[x_1, x_2, x_3, x_4]/(x_1x_2 - x_3x_4).$$

It is well known that $\mathrm{Cl}(k) = \mathbb{Z}$ and $\mathrm{Pic}(k) = 0$. So, $\mathrm{Cl}(k)/\mathrm{Pic}(k)$ is not a torsion group and $\mathrm{Krull\text{-}dim}(k) = 3$. If I is an unmixed ideal of height one in k, then it is possible to show that $I^{(n)} = I^n$ for any $n \geqslant 0$. Hence the symbolic Rees algebra of I coincides with the ordinary Rees algebra of I, so $\mathcal{R}(I)$ is finitely generated over R. Now, by Theorem 3.8, the ring of constants of every locally nilpotent k-derivation of $k[x,y]$ is finitely generated over k. $\square$

Now let $n = 3$. We already know from the previous section that if k is a field, then $\gamma(d) < \infty$. However in this case the number $\gamma(d)$ is unbounded. Strelcyn and the author ([38]) proved:

THEOREM 3.13. *If $n \geqslant 3$ and $r \geqslant 0$, then there exists a k-derivation d of $k[X]$ such that $\gamma(d) = r$.*

In some cases the number $\gamma(d)$ is bounded. Note the following consequence of Theorem 3.5.

THEOREM 3.14. *Let d be a nonzero k-derivation of $k[x,y,z]$, where k is a* UFD *containing $\mathbb{Z}$. If $d(x) = 0$, then $\gamma(d) \leqslant 2$.*

For locally nilpotent derivations over a field the number $\gamma(d)$ is also bounded.

THEOREM 3.15 (Miyanishi [32]). *If d is a nonzero locally nilpotent derivation of $k[x, y, z]$, where k is a field of characteristic zero, then $k[x,y,z]^d = k[f,g]$ for some algebraically independent polynomials $f, g \in k[x,y,z]$.*

There exists also the following homogeneous version of this theorem.

THEOREM 3.16 ([53], [6]). *Let d be a nonzero homogeneous locally nilpotent derivation of $k[x, y, z]$. Assume that the degrees of x, y, z are positive. Then $k[x,y,z]^d = k[f,g]$, for some homogeneous polynomials $f, g \in k[x,y,z]$.*

This was first proved by Zurkowski [53] in the case k is algebraically closed; Zurkowski does not assert the homogeneity of f and g in his statement of the theorem, but this is included in his proof ([21] 172).

PROPOSITION 3.17 ([6]). *Let $f, g \in k[x,y,z]$, where k is a field of characteristic zero. Assume that $k[f,g]$ is the kernel of some locally nilpotent derivation of $k[x,y,z]$. Then the polynomials f and g are irreducible in $\overline{k}[x,y,z]$, where $\overline{k}$ denotes the algebraic closure of k.*

Let us end this section with the following two recent results for $n = 4$.

THEOREM 3.18 ([12]). *If k is a field of characteristic zero and d is a nonzero triangular derivation of $k[x_1, x_2, x_3, x_4]$ with a slice, then $\gamma(d) = 3$.*

THEOREM 3.19 ([9]). *For any integer $n \geqslant 3$ there exists a triangular derivation d of $k[x_1, x_2, x_3, x_4]$ (where k is a field of characteristic zero) such that $n \leqslant \gamma(d) \leqslant n+1$.*

4. The Weitzenböck theorem and its generalizations. Let k be a field of characteristic zero. The classical result of Weitzenböck [50] states that every linear action of the additive group $(k, +)$ on $\mathbb{A}^n$ has a finitely generated algebra of invariants. A modern proof of this result is due to Seshadri [45]; cf. [33] pp. 36-40 and Tyc ([48]). In the language of derivations this is equivalent to the following.

THEOREM 4.1 (Weitzenböck). *Let d be a k-derivation of $k[X] = k[x_1, \ldots, x_n]$ such that*

$$d(x_i) = \sum_{j=1}^{n} a_{ij} x_j \quad \text{for } i = 1, \ldots, n, \quad \text{with } a_{ij} \in k.$$

If the matrix $[a_{ij}]$ is nilpotent, then $k[X]^d$ is finitely generated over k.

It turns out that the assumption of nilpotency of the matrix $[a_{ij}]$ is not essential. It is true also for an arbitrary matrix ([33], [47]). We may prove also the following theorem.

THEOREM 4.2 ([39]). *If d is a k-derivation of $k[X] = k[x_1, \ldots, x_n]$ such that*

$$d(x_i) = \sum_{j=1}^{m} a_{ij} x_j + b_i \quad \text{for } i = 1, \ldots, n, \quad \text{with } a_{ij} \in k \quad \text{and} \quad b_i \in k,$$

then $k[X]^d$ is finitely generated over k.

If, in the above theorem, k is replaced by a UFD, then in general it is not true that the ring of constants is finitely generated (see Theorem 2.7).

All the known proofs of the Weitzenböck theorem are not constructive. Given a linear k-derivation it is not easy to describe its ring of constants even if we assume that its matrix is nilpotent. In 1989 L. Tan in [46] gave a method of computing generators of the ring of constants for some linear locally nilpotent k-derivations of $k[X]$. Later, in 1992, van den Essen ([17], [18]) showed that this method works in a more general case. He gave an algorithm computing a finite set of generators of the ring of constants for any locally nilpotent k-derivation (of a finitely generated k-domain), in the case when the finiteness of the ring of constants is known. An important role in his algorithm play Gröbner bases.

Let δ be the Weitzenböck derivation of $k[X] := k[x_1, \dots, x_n]$, that is,

$$\delta(x_1) = 0,\ \delta(x_2) = x_1,\ \delta(x_3) = x_2,\ \dots,\ \delta(x_n) = x_{n-1}.$$

An algebraic description of the rings of the form $k[X]^\delta$, for $n \leqslant 6$, is given in [40]. In [20] we may find a finite set of polynomials which is proposed as a generating set of $k[X]^\delta$. But, as shows L. Tan in [46], it is not a generating set. Full lists of generators, for $n \leqslant 5$, are described (in a slightly modified form) in [46]. The generators for $n \leqslant 6$ can be found in the tables of Cerezo [4]. The author, in [35], using the van den Essen algorithm, presented a similar list for $n \leqslant 6$. Cerezo also computed a system of generators of minimal length for the case $n = 7$ (the minimal length is 23, the degrees of these generators go up to 18).

Assume that $n \geqslant 2$ and let $\alpha = (\alpha_1, \dots, \alpha_{n-1})$ be a sequence of positive integers. Let d be the derivation of $k[X]$ defined by

$$d(x_1) = 0,\ d(x_2) = x_1^{\alpha_1},\ d(x_3) = x_2^{\alpha_2},\ \cdots,\ d(x_n) = x_{n-1}^{\alpha_{n-1}}.$$

This derivation we call the *generalized Weitzenböck derivation of type* α. If $\alpha = (1, \dots, 1)$ then, by the Weitzenböck theorem, the ring of constants of the above derivation is finitely generated over k.

Question 4.3. *Is the ring of constants of every generalized Weitzenböck derivation of $k[X]$ finitely generated over k?*

We do not know any answer to this question. There are no counterexamples. If $n \leqslant 3$, then the answer is affirmative, because we know that, in this case, every derivation of $k[X]$ has a finitely generated ring of constants. If $n = 4$, the answer is also affirmative by a result of Maubach [30] (see Theorem 2.8). For $n \geqslant 5$ the problem is open.

If a generalized Weitzenböck derivation of $k[X]$ of type $(1, \alpha_2, \dots, \alpha_{n-1})$ has a finitely generated ring of constants, then every generalized Weitzenböck derivation of $k[X]$ of type $(c, \alpha_2, \dots, \alpha_{n-1})$, for any integer $c \geqslant 1$, has a finitely generated ring of constants. It was proved by Maubach ([31] for $n = 4$, [16] 236 Ex. 6). Using the same idea as in the Maubach proof we may prove the following proposition.

Proposition 4.4. *Let k be a ring containing $\mathbb{Q}$ and let $k[t, u, X] := k[t, u, x_1, \dots, x_n]$ and $k[u, X] := k[u, x_1, \dots, x_n]$ be the polynomial rings over k. Let $\beta(t)$ be a monic polynomial from $k[t]$ of degree $c \geqslant 1$, and let $f_1, \dots, f_n$ be polynomials belonging to $k[u, X]$.*

Consider two derivations d and δ of $k[t,u,X]$ defined by

$$\begin{cases} d(t) &= 0, \\ d(u) &= t, \\ d(x_1) &= f_1, \\ & \vdots \\ d(x_n) &= f_n, \end{cases} \qquad \begin{cases} \delta(t) &= 0, \\ \delta(u) &= \beta(t), \\ \delta(x_1) &= f_1, \\ & \vdots \\ \delta(x_n) &= f_n. \end{cases}$$

If $k[t,u,X]^d$ is finitely generated over k, then $k[t,u,X]^\delta$ is also finitely generated over k.

More exactly, assume that $k[t,u,X]^d = k[g_1,\dots,g_m]$. Then

$$(*) \qquad k[t,u,X]^\delta = k[t][\varphi(g_1),\dots,\varphi(g_m)],$$

where $\varphi : k[t,u,X] \to k[t,u,X]$ is the k-algebra endomorphism such that $\varphi(t) = \beta(t)$, $\varphi(u) = u$ and $\varphi(x_i) = x_i$ for all $i = 1,\dots,n$.

Proof. Let φ be the endomorphism as in this proposition. Observe that φ is injective and $\delta\varphi = \varphi d$. It is enough to prove the equality $(*)$. The inclusion $\supseteq$ is obvious.

Assume that $h \in k[t,u,X]^\delta$ and consider the unique presentation

$$h = h_0 + h_1 t + \cdots + h_{c-1} t^{c-1},$$

in which the polynomials $h_0,\dots,h_{c-1}$ belong to $A := k[\beta(t),u,X] := k[\beta,u,x_1,\dots,x_n]$. Then we have

$$0 = \delta(h) = \delta(h_0) + \delta(h_1)t + \cdots + \delta(h_{c-1})t^{c-1}.$$

It is clear that $\delta(A) \subseteq A$. Since such a presentation is unique, we have

$$\delta(h_i) = 0 \quad \text{for all} \ \ i = 0,1,\dots,c-1.$$

The polynomials $h_0,\dots,h_{c-1}$ belong of course to the image of φ. Thus, there exist $H_0,\dots,H_{c-1} \in k[t,u,X]$ such that

$$h_i = \varphi(H_i), \quad \text{for all} \ \ i = 0,\dots,c-1.$$

Then $\varphi d(H_i) = \delta\varphi(H_i) = \delta(h_i) = 0$, and hence, since φ is injective, $d(H_i) = 0$ for all $i = 0,1,\dots,c-1$. Whence, each H_i belongs to $k[g_1,\dots,g_m]$, so $h_i = \varphi(H_i) \in k[\varphi(g_1),\dots,\varphi(g_m)]$, and consequently $h = \sum h_i t^i \in k[t][\varphi(g_1),\dots,\varphi(g_m)]$. □

Note added in proof. (1) In 2001, Gene Freudenburg ([24]) presented a survey of counterexamples to Hilbert's Fourteenth Problem, including recent counterexamples in low dimension constructed with locally nilpotent derivations.

(2) A part of the facts from this paper was presented at the Conference on Differential Galois Theory (Będlewo, Poland, May 28th–June 1st 2001) organized by Teresa Crespo and Zbigniew Hajto. The author thanks the organizers for the excellent conference.

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