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DERIVATIONS IN MATRIX SUBRINGS

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This note is an abstract of the author's papers [1], [2] and [3]. Let R be a ring with identity and let $M_n(R)$ denotes the ring of $n \times n$ matrices over $R$. We say that a subring $P$ of $M_n(R)$ is special with the relation $\omega$ if $P$ is of the form

$$P = \{ A \in M_n(R); A_{ij} = 0, \text{ for } (i,j) \notin \omega \},$$

where $\omega$ is a relation (reflexive and transitive) on the set $\{1,\ldots,n\}$.

We describe in this note all derivations, $R$-derivations and higher $R$-derivations of the ring $P$.

I. DERIVATIONS AND HIGHER DERIVATIONS IN A RING

1. Derivations. Let $P$ be a ring with identity. An additive mapping $d: P \rightarrow P$ is called a derivation (or an ordinary derivation) of $P$ if $d(xy) = d(x)y + xd(y)$, for any $x, y \in P$. We denote by $D(P)$ the set of all derivations of $P$. If $d$ and $d'$ are derivations of $P$ then the mapping $d + d'$ is also a derivation of $P$, so $D(P)$ is an abelian group.

Let $a \in P$ and let $d_a: P \rightarrow P$ be a mapping defined by $d_a(x) = ax - xa$, for any $x \in P$. Then $d_a$ is a derivation of $P$.

Let $d \in D(P)$. If there exists an element $a \in P$ such that $d = d_a$ then $d$ is called an inner derivation (with respect to $a$) of $P$. We denote by $ID(P)$ the set of all inner derivations of $P$. $ID(P)$ is a (normal) subgroup of $D(P)$.

We shall say that $P$ is an NS-ring if $ID(P) = D(P)$, that is, a ring $P$ is an NS-ring if and only if every derivation of $P$ is inner.
2. Higher derivations. Let $P$ be a ring with identity and let $S$ be a segment of $N = \{0, 1, \ldots \}$, that is, $S = N$ or $S = \{0, 1, \ldots, s\}$ for some $s \geq 0$. A family $d = (d_m)_{m \in S}$ of mappings $d_m : P \rightarrow P$ is called a derivation of order $s$ of $P$ (where $s = \sup(S) : S = \{0, 1, \ldots, s\}$) if the following properties are satisfied:

1. $d_m(x + y) = d_m(x) + d_m(y),$
2. $d_m(xy) = \sum_{i+j=m} d_i(x)d_j(y),$
3. $d_0 = \text{id}_P,$

for any $x, y \in P$, $m \in S$.

The set of derivations of order $s$ of $P$, denoted by $D_s(P)$, is the group under the multiplication $\ast$ defined by the formula

$$d \ast d' = \sum_{i+j=m} d_i \circ d'_j,$$

where $d, d' \in D_s(P)$ and $m \in S$.

Let $\delta : P \rightarrow P$ be a mapping. Then $\delta$ is an ordinary derivation of $P$ if and only if $(\text{id}_P, \delta)$ is a derivation of order 1 of $P$. Therefore we may identify: $D(P) = D_1(P)$.

3. Examples of higher derivations. Let $P$ and $S$ be as in Section 2.

Example 3.1. Let $a \in P$, $d_0 = \text{id}_P$, and $d_m(x) = a^m x - a^{m-1} xa$, for $m \geq 1, x \in P$. Then $d = (d_m)_{m \in S}$ belongs to $D_s(P)$.

Example 3.2. Let $d \in D_s(P)$, $k \in S - \{0\}$ and let $\delta = (\delta_m)_{m \in S}$ be the family of mappings from $P$ to $P$ defined by

$$\delta = \begin{cases} 0, & \text{if } k \nmid m \\ d_r, & \text{if } m = rk. \end{cases}$$

Then $\delta \in D_s(P)$.

The derivation $d$ (of order $s$) from Example 3.1 will be denoted by $[a, 1]$. The derivation $\mathbf{d}$ (of order $s$) from Example 3.2, for $d = [a, 1]$
will be denoted by \([a,k]\).

4. Inner derivations. Let \(P\) and \(S\) be as in Section 2 and let \(a = (a_m)\)
(where \(m \in S\)) be a sequence in \(P\). Denote by \(\Delta(a)\) the element in \(D_s(P)\)
defined by

\[
\Delta(a)_m = ([a_1,1] \star [a_2,2] \star \ldots \star [a_m,m])_m,
\]

for any \(m \in S\).

**Definition 4.1.** Let \(d \in D_s(P)\). If there exists a sequence \(a = (a_m)\)
of elements of \(P\) such that \(d = \Delta(a)\) then \(d\) is called an **inner derivation**
of order \(s\) of \(P\).

Denote by \(\text{ID}_s(P)\) the set of inner derivations of order \(s\) of \(P\).

**Proposition 4.2.** \(\text{ID}_s(P)\) is a normal subgroup of \(D_s(P)\).

**Proposition 4.3.** The following properties are equivalent

1. \(P\) is an NS-ring,

2. \(\text{ID}_s(P) = D_s(P)\), for any \(0 < s \leq \infty\),

3. \(\text{ID}_s(P) = D_s(P)\), for some \(0 < s \leq \infty\).

5. R-derivations. Let \(R \subseteq P\) be rings with identity and let \(S\) be
a segment of \(N\). If a derivation (of order \(s\)) \(d \in D_s(P)\) satisfies the
condition

\[
d_m(r) = 0,
\]

for all \(m \in S - \{0\}\), \(r \in R\), then \(d\) is called **R-derivation of order \(s\)**
of \(P\), and the set of all such derivations is denoted by \(D^R_s(P)\).

We define similarly an ordinary \(R\)-derivation, an inner \(R\)-derivation,
an inner \(R\)-derivation of order \(s\) and also, we define similarly the groups
\(D^R(P)\), \(\text{ID}^R(P)\) and \(\text{ID}^R_s(P)\).

The group \(D^R_s(P)\) is a subgroup of \(D_s(P)\) and the group \(\text{ID}^R_s(P)\) is a
normal subgroup of \(D^R_s(P)\).

\(\text{\(\ldots\)}\)
We shall say that $P$ is an NS-ring over $R$ if $D^R_s(P) = D^R_s(P)$.

Proposition 5.1. The following properties are equivalent

(1) $P$ is an NS-ring over $R$,

(2) $D^R_s(P) = D^R_s(P)$, for any $0 < s \leq \infty$,

(3) $D^R_s(P) = D^R_s(P)$, for some $0 < s \leq \infty$.

II. SPECIAL SUBRINGS OF MATRIX RINGS

6. Notices. Let $R$ be a ring with identity, $n$ a fixed natural number and $\omega$ a reflexive and transitive relation on the set $I_n = \{1, \ldots, n\}$. We denote by $M_n(R)$ the ring of $n \times n$ matrices over $R$ and by $Z(R)$ the center of $R$. Moreover, we use the following conventions:

- $F(R)$ = the set of mappings from $R$ to $R$,
- $\bar{\omega}$ = the smallest equivalence relation on $I_n$ containing $\omega$,
- $T_\omega$ = a fixed set of representatives of equivalence classes of $\bar{\omega}$,
- $A_{ij}$ = $ij$-coefficient of a matrix $A$,
- $E_{ij}$ = the element of the standard basis of $M_n(R)$,
- $M_n(R)_{\omega} = \{A \in M_n(R); A_{ij} = 0, \text{ for } (i,j) \notin \omega \}$.

The set $P = M_n(R)_{\omega}$ is a subring of $M_n(R)$ called a special subring with the relation $\omega$. Every special subring contains the ring $R$ (via injection $r \mapsto \bar{r}$, where $\bar{r}$ is the diagonal matrix whose all coefficients on the diagonal are equal to $r \in R$).

7. Transitive mappings and regular relations. Let $G$ be an abelian group. A mapping $f: \omega \longrightarrow G$ will be called transitive iff

\[ f(a,c) = f(a,b) + f(b,c), \]

for any $a,b$ and $b,c$.

If $f: \omega \longrightarrow G$ is a transitive mapping then we denote by $[f,\_]$
(in the case $G = \mathbb{R}$) the mapping from $\omega$ to $F(\mathbb{R})$ defined by

$$[f,\_](a,b)(r) = f(a,b)r - rf(a,b),$$

for $a \neq b$ and $r \in \mathbb{R}$. Clearly, $[f,\_]$ is transitive too.

We shall say that $f$ is **trivial** if there exists a mapping

$$\sigma: I_n \rightarrow G$$

such that

$$f(a,b) = \sigma(a) - \sigma(b),$$

for any $a \neq b$. Moreover, we shall say that $f$ is **quasi-trivial** (in the case $G = \mathbb{R}$) if $[f,\_]$ is trivial.

Every trivial transitive mapping from $\omega$ to $\mathbb{R}$ is quasi-trivial, but the converse is not necessarily true.

**Proposition 7.1.** Let $f: \omega \rightarrow \mathbb{R}$ be a quasi-trivial transitive mapping. Then there exists a unique mapping $\tau: I_n \rightarrow F(\mathbb{R})$ such that

1. $[f,\_](i,j) = \tau(i) - \tau(j)$, for all $i \neq j$,

2. $\tau(t) = 0$, for $t \in T_\omega$.

Moreover, $\tau(1), \ldots, \tau(n)$ are inner derivations of $\mathbb{R}$.

**Definition 7.2.** The relation $\omega$ is called **regular over an abelian group** $G$ if every transitive mapping from $\omega$ to $G$ is trivial.

8. The graph $\Gamma(\omega)$ and homology groups. Let $\equiv$ be the equivalence relation on $I_n$ defined by: $x \equiv y$ iff $xwy$ and $ywx$. Denote by $[x]$ the equivalence class of $x \in I_n$ with respect to $\equiv$ and let $I'_n$ be the set of all equivalence classes. We define a relation $\omega'$ of partial order on $I'_n$ as follows:

$[x] \omega' [y] \iff xwy.$

We will denote the pair $(I'_n, \omega')$ by $\Gamma(\omega)$ and call it the **graph of $\omega$**.

Elements of $I'_n$ we call **vertices** of $\Gamma(\omega)$ and pairs $(a,b)$, where $aw'b$ and $a \neq b$, **arrows** of $\Gamma(\omega)$.

Let us imbed the set of the vertices of $\Gamma(\omega)$ in an Euclidean space of a sufficiently high dimension so that the vertices will be
linearly independent.

If \( a_0, a_1, \ldots, a_k \) are elements of \( I_n \) such that \( a_i \neq a_{i+1} \) for \( i=0,1,\ldots,k-1 \), then by \( (a_0, a_1, \ldots, a_k) \) we denote the \( k \)-dimensional simplex with vertices \( a_0, \ldots, a_k \). The union of all 0, 1, 2 or 3-dimensional such simplicies we will denote also by \( \Gamma(\omega) \). Therefore, \( \Gamma(\omega) \) is a simplicial complex of dimension \( \leq 3 \).

Let \( C_k(\omega) \), for \( k=0,1,2,3 \), be the free abelian group whose free generators are \( k \)-dimensional simplicies of \( \Gamma(\omega) \). We have the following standard complex of abelian groups:

\[
0 \longrightarrow C_3(\omega) \overset{\partial_3}{\longrightarrow} C_2(\omega) \overset{\partial_2}{\longrightarrow} C_1(\omega) \overset{\partial_1}{\longrightarrow} C_0(\omega) \longrightarrow 0
\]

where

\[
\partial_1(a,b) = (b) - (a),
\partial_2(a,b,c) = (b,c) - (a,c) + (a,b),
\partial_3(a,b,c,d) = (b,c,d) - (a,c,d) + (a,b,d) - (a,b,c).
\]

Then \( H_1(\Gamma(\omega)) = \text{Ker}\partial_1/\text{Im}\partial_2 \), \( H_2(\Gamma(\omega)) = \text{Ker}\partial_2/\text{Im}\partial_3 \) and (by the Künneth formulas)

\[
H^1(\Gamma(\omega), G) = \text{Hom}(H_1(\Gamma(\omega)), G),
\]

for an arbitrary abelian group \( G \).

III DERIVATIONS IN SPECIAL SUBRINGS

9. Examples of derivations. Let \( P = M_n(R) \) be a special subring of \( M_n(R) \).

Example 9.1. Assume that \( f: \omega \longrightarrow R \) is a quasi-trivial transitive mapping and denote by \( \Delta^f \) the mapping from \( P \to P \) defined by

\[
\Delta^f_{pq}(B) = B_{pq} f(p,q) + \tau_f(p)(B_{pq}),
\]

for \( B \in P, p \neq q \), where \( \tau_f \) is the mapping \( \tau \) from Proposition 7.1.

Then \( \Delta^f \) is a derivation of \( P \). Moreover \( \Delta^f \) is inner if and only if \( f \) is trivial.
Example 9.2. Let $\delta = \{\delta_t; t \in T_\omega\}$ be a set of derivations of $R$. Denote by $\Theta_\delta$ the mapping from $P$ to $P$ defined by

$$\Theta_\delta(B)_{pq} = \delta_t(B)_{pq},$$

for $B \in P, p\omega q$, where $t \in T_\omega$ such that $p\omega t$.

Then $\Theta_\delta$ is a derivation of $P$. Moreover, $\Theta_\delta$ is inner if and only if $\delta_t$ is inner for any $t \in T_\omega$.

10. A description of $D(P)$. Let $P = M_n(R)_\omega$ be a special subring of $M_n(R)$. The following theorem describes all derivations of $P$.

Theorem 10.1. Every derivation $d$ of $P$ has a unique representation:

$$d = d_A + \Delta^f + \Theta_\delta,$$

where

1. $d_A$ is an inner derivation of $P$ with respect to a matrix $A \in P$ such that $A_{pp} = 0$, for $p = 1, \ldots, n$,

2. $f: \omega \longrightarrow R$ is a quasi-trivial transitive mapping and $\Delta^f$ is the derivation from Example 9.1,

3. $\delta = \{\delta_t; t \in T_\omega\}$ is a set of derivations of $R$ and $\Theta_\delta$ is the derivation from Example 9.2.

The next theorem describes special subrings which are NS-rings.

Theorem 10.2. The following conditions are equivalent

1. $P$ is an NS-ring,

2. $R$ is an SN-ring and the relation $\omega$ is regular over $Z(R)$.

11. $R$-derivations of $M_n(R)_\omega$. Let $P = M_n(R)_\omega$ be a special subring of $M_n(R)$.

Example 11.1. Let $f: \omega \longrightarrow Z(R)$ be a transitive mapping and denote by $\Delta^f$ the mapping from $P$ to $P$ defined by $\Delta^f(B)_{pq} = f(p,q)B_{pq}$, for $B \in P$ and $p\omega q$. Then $\Delta^f$ is an $R$-derivation of $P$. Moreover $\Delta^f$ is
inner if and only if \( f \) is trivial.

**Theorem 11.2.** Any R-derivation \( d \) of \( P \) has a unique representation
\[
d = d_A + \Delta^f,
\]
where (1) \( d_A \) is an inner derivation of \( P \) with respect to a matrix \( A \in P \) such that \( A_{ij} \in Z(R) \) for \( i,j=1,\ldots,n \), and \( A_{ii} = 0 \) for \( i=1,\ldots,n \),
(2) \( f : \omega \rightarrow Z(R) \) is a transitive mapping and \( \Delta^f \) is the derivation from Example 11.1.

**Theorem 11.3.** The following conditions are equivalent
(1) \( P \) is an NS-ring over \( R \),
(2) \( \omega \) is regular over \( Z(R) \).

**Corollary 11.4.** If \( d \) and \( \delta \) are R-derivations of \( R \) then the derivation \( d \delta - \delta d \) is inner.

**Corollary 11.5.** If \( d \) is an R-derivation of \( P \) then \( d(Z(R)) = 0 \).

**Corollary 11.6.** If \( d \) is an R-derivation of \( P \) and \( U \) is an ideal of \( P \) then \( D(U) = U \).

12. An example of non-inner R-derivation. For \( n \leq 3 \) every relation \( \omega \) (reflexive and transitive) on \( I_n \) is regular over any group. Therefore (by Theorem 11.3), in this case any special subring of \( M_n(R) \) has only inner R-derivations. For \( n=4 \) it is not true. Let \( \omega_0 \) be the relation on \( I_4 = \{1,2,3,4\} \) defined by the graph
\[
\begin{array}{c}
1 \rightarrow 3 \\
\downarrow \\
4 \leftarrow 2
\end{array}
\]
that is, \( \omega_0 = \{(1,1),(2,2),(3,3),(4,4),(1,3),(1,4),(2,3),(2,4)\} \).
Denote by \( S_4(R) \) the special subring of \( M_4(R) \) with the relation \( \omega_0 \).
Then we have

$$S_4(R) = \begin{bmatrix} R & 0 & R & R \\ 0 & R & R & R \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & R \end{bmatrix}.$$ 

Consider the mapping $d: S_4(R) \rightarrow S_4(R)$ defined by

$$d(\begin{bmatrix} x_{11} & 0 & x_{13} & x_{14} \\ 0 & x_{22} & x_{23} & x_{24} \\ 0 & 0 & x_{33} & 0 \\ 0 & 0 & 0 & x_{44} \end{bmatrix}) = \begin{bmatrix} 0 & 0 & x_{13} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$ 

Then $d$ is an $R$-derivation of $S_4(R)$ and $d$ is not inner.

In [1] there is a description of the group $D^R(S_4(R))$. Note one of the properties of $R$-derivations of the ring $S_4(R)$.

**Corollary 12.1.** If $d_1$ and $d_2$ are $R$-derivations of $S_4(R)$ then the composition $d_1d_2$ is also $R$-derivation of $S_4(R)$.

13. A description of regular relations. Let $P = M_n(R)$ be a special subring of $M_n(R)$.

We know, by Theorem 10.2, that $P$ is an NS-ring if and only if $R$ is an NS-ring and the relation $\omega$ is regular over $Z(R)$. We know also, by Theorem 11.3, that $P$ is an NS-ring over $R$ if and only if the relation $\omega$ is regular over $Z(R)$.

In this section we give some sufficient and necessary conditions for the relation $\omega$ to be regular over an abelian group.

We may reduce our consideration to the case where $\omega$ is connected (that is, for any $a, b \in I_n$ there exist elements $a_1, \ldots, a_r \in I_n$ such that $a = a_1$, $b = a_r$ and $a_i = a_{i+1}$ or $a_{i+1} = a_i$, for $i = 1, \ldots, r-1$), because it is easy to show the following

**Proposition 13.1.** Let $G$ be an abelian group. The relation $\omega$ is regular over $G$ if and only if every connected component of $\omega$ is regular over $G$. 


The next proposition says that we may also reduce our consideration to the case where \( \omega \) is a partial order.

**Proposition 13.2.** \( \omega \) is regular over \( G \) if and only if \( \omega' \) (see Section 8) is regular over \( G \).

Now we may give a description of regular relations.

**Theorem 13.3.** Assume that \( \omega \) is a connected partial order. The following properties are equivalent:

1. \( \omega \) is regular over some non-zero group,
2. \( \omega \) is regular over every torsion-free group,
3. \( \omega \) is regular over some torsion-free group,
4. \( \omega \) is regular over \( \mathbb{Z} \),
5. \( H_1(\Gamma(\omega)) \) is finite,
6. \( H^1(\Gamma(\omega), G) = 0 \), for any torsion-free group \( G \).

**Theorem 13.4.** Assume that \( \omega \) is connected partial order. The following properties are equivalent:

1. \( \omega \) is regular over any group,
2. \( \omega \) is regular over \( \mathbb{Q}/\mathbb{Z} \),
3. \( H_1(\Gamma(\omega)) = 0 \),
4. \( H^1(\Gamma(\omega), G) = 0 \), for any group \( G \).

**Theorem 13.5.** Assume that \( \omega \) is connected partial order, such that the order of the group \( H_1(\Gamma(\omega)) \) is equal to \( m > 1 \). Let \( G \) be an abelian group. The following properties are equivalent:

1. \( \omega \) is regular over \( G \),
2. \( G \) is an \( m \)-torsion-free group,
3. \( H^1(\Gamma(\omega), G) = 0 \).

**Corollary 13.6.** Let \( P = \mathcal{M}_n(\mathbb{R})_\omega \) be a special subring of \( \mathcal{M}_n(\mathbb{R}) \). The
following properties are equivalent

(1) Every $R$-derivation of $P$ is inner,

(2) The relation $\omega$ is regular over $Z(R)$,

(3) $H^1(\Gamma(\omega), Z(R)) = 0$.

14. Examples. Let $P = M_n(R)$ where

a) $n \leq 3$, or

b) the graph $\Gamma(\omega)$ is a tree, or

c) the graph $\Gamma(\omega)$ is a cone (that is, there exists $b \in I_n$ such that $b \omega a$ or $a \omega b$ for any $a \in I_n$), in particular $P = M_n(R)$ is the ring of $n \times n$ matrices over $R$, or $P$ is the ring of triangular $n \times n$ matrices over $R$.

Then every $R$-derivation (or every derivation, if every derivation of $R$ is inner) of $P$ is inner.

By Theorem 13.5 it follows that there exist relations $\omega$ which are regular over some groups and which are not regular over another groups. In the paper 1 there is an example of such a relation $\omega$ (for $n=17$) that if $R$ is 2-torsion-free ring then $P = M_n(R)$ is an NS-ring over $R$, and if $\text{char}(R) = 2$ then $P = M_n(R)$ is not an NS-ring over $R$.

IV HIGHER DERIVATIONS IN SPECIAL SUBRINGS

15. An example of higher derivations. Let $P = M_n(R)$ be a special subring of $M_n(R)$, $S$ a segment of $N$, and let $d = \{d(t); t \in T_\omega\}$ be a family of derivations of order $s$ (where $s = \text{sup}(S)$) of the ring $R$.

Denote by $\mathcal{O}(d)$ the sequence $(d_m)_{m \in S}$ of mappings from $P$ to $P$ defined by

$$d_m(A)_{ij} = d_m((v(i))(A_{ij})),$$

for $m \in S$, $A \in P$, where $v: I_n \longrightarrow T_\omega$ is the mapping: $(j)p = t$ iff $p \omega t$.
Then $\Theta(d)$ is a derivation of order $s$ of $P$. If $d \neq 0$ then the derivation $\Theta(d)$ is not an $R$-derivation.

In the next sections of this note we shall interesting only in $R$-derivations of order $s$ of $P$.

16. **Transitive mappings of order** $s$. A sequence $f = \{f_m\}_{m \in S}$ of mappings $f_m : \omega \longrightarrow Z(R)$ is called a **transitive mapping of order** $s$ (from $\omega$ to $R$) if the following properties are satisfied:

1. $f_0(p, q) = 1$, for all $p \omega q$,

2. $f_m(p, r) = \sum_{i+j=m} f_i(p, q)f_j(q, r)$, for all $m \in S$ and $p \omega q$ and $q \omega r$.

If $f = \{f_m\}_{m \in S}$ is a transitive mapping of order $s$ then

$$f_1(p, r) = f_1(p, q) + f_1(q, r),$$

for any $p \omega q \omega r$ so, $f_1: \omega \longrightarrow Z(R)$ is a transitive mapping in the sense of Section 7.

17. **$R$-derivations of order** $s$. In this section we give a description of the group $D_s^R(P)$.

**Example 17.1.** Let $f = \{f_m\}_{m \in S}$ be a transitive mapping of order $s$ from $\omega$ to $Z(R)$. Denote by $\Delta^f$ the sequence $\{\Delta^f_m\}_{m \in S}$ of mappings $\Delta^f_m : P \longrightarrow P$ defined by the following formula:

$$\Delta^f_m(A)_{pq} = f_m(p, q)A_{pq},$$

for all $A \in P$ and $p \omega q$.

Then $\Delta^f$ is an $R$-derivation of order $s$ of $P$.

**Theorem 17.2.** Every $R$-derivation $d$ of order $s$ of $P$ has a unique representation:

$$d = \Delta(A) * \Delta^f,$$

where
(1) $A = (A^{(m)})_{m \in S} - (0)$ is a sequence of matrices $A^{(m)} \in P \cap M_n(Z(R))$ such that $A^{(m)}_{ii} = 0$, for $i = 1, \ldots, n$, and $\Delta(A)$ is the inner derivation of order $s$ with respect to $A$.

(2) $f$ is a transitive mapping of order $s$ from $\omega$ to $R$ and $\Delta^f$ is the $R$-derivation from Example 17.1.

**Corollary 17.3.** If $d \in D^R_S(P)$ and $U$ is an ideal of $P$ then $d_{m}(U) \subseteq U$, for all $m \in S$.

**Corollary 17.4.** If $d \in D^R_S(P)$, then $d_{m}(Z(R)) = 0$, for all $m \in S - \{0\}$.

**Corollary 17.5.** Assume that there do not exist three different elements $e, b, c \in I$ such that $a \omega b \omega c$. Let $d = (d_{m})_{m \in S}$ be a sequence of mappings from $P$ to $P$ such that $d_{0} = id_{P}$. Then $d$ is an $R$-derivation of order $s$ of $P$ if and only if every mapping $d_{m}$ (for $m \in S - \{0\}$) is an ordinary $R$-derivation of $P$.

18. **Integrable derivations.** Let $S = \{0, 1, \ldots, s\}$, where $s < \omega$. Assume that $S'$ is a segment of $N$ such that $S \subseteq S'$. We say that an $R$-derivation $d \in D^R_S(P)$ is $s'$-integrable (where $s' = \sup(S') \leq \omega$) if there exists an $R$-derivation $d' = (d'_{m})_{m \in S'}$ of order $s'$ of $P$ such that $d'_{m} = d_{m}$, for all $m \in S$.

In the paper [3] there are some necessary conditions for any $R$-derivation of order $s$ of $P$ to be $s'$-integrable, and there is an example of non-integrable $R$-derivation (In this example $n=17$ and $R = Z_{2}$).

In this paper there are also proofs of the following two partial results:

**Theorem 18.1.** Let $s < s' \leq \beta$. If $H_{2}(\Gamma(\omega)) = 0$ and $H_{1}(\Gamma(\omega))$ is a free abelian group then every $R$-derivation of order $s$ of $P$ is $s'$-integrable.

**Theorem 18.2.** Assume that the homology group $H_{1}(\Gamma(\omega))$ is free abelian. Then
(1) Every $R$-derivation of order $s < 3$ of $P$ is $3$-integrable.

(2) If $R$ is $2$-torsion-free then every $R$-derivation of order $< 5$ of $P$
is $5$-integrable.

(3) If $R$ is $6$-torsion-free then every $R$-derivation of order $< 7$ of $P$
is $7$-integrable.

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