ON SOME COHEN-MACAULAY SUBSETS OF A PARTIALLY ORDERED ABELIAN GROUP

To the memory of Professor Gishiro Maruyama

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This paper is about some Cohen-Macaulay subsets of a partially ordered abelian group which are useful in the study of Galois extensions of higher derivation type (cf. Remark and [4]).

Let $N = \{1, \ldots, n\}$, and $Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. Now, if $G = (f_1) \times \cdots \times (f_n)$ is an abelian group which is the direct product of infinite cyclic groups $(f_i)$ generated by $f_i$ then $G$ becomes a partially ordered group by

$$(\#) : \quad \prod_{i=1}^{n} f_i^i \supseteq \prod_{i=1}^{n} f_i^i \quad \Leftrightarrow \quad \sum_{j=1}^{n} s_j \supseteq \sum_{j=1}^{n} t_j \text{ for all } k \in N.$$

This partially ordered group $G$ will be denoted by $(G, \#)$. Clearly $(G, \#)$ can be regarded as the partially ordered additive group $(Z^n, \#) = Z_1 \times \cdots \times Z_n$, where $Z_i = Z$ for all $i \in N$. As it is seen later on, $(Z^n, \#)$ is a modular lattice.

For $u_i, v_i \in Z$ with $u_i \leq v_i \ (i \in N)$, we set

$$\Delta = \prod_{i=1}^{n} [u_i, v_i] = \{(a_1, \ldots, a_n) : u_i \leq a_i \leq v_i, a_i \in Z\}$$

which is a subposet of $(Z^n, \#)$.

Our purpose of this note is to prove that $\Delta$ is a modular lattice under the ordering in $(Z^n, \#)$ (Theorem 7), and if, in particular, $u_i < v_i$ for all $i \in N$ then $\Delta$ is a modular sublattice of $(Z^n, \#)$ (Theorem 6).

In what follows, we shall use the following conventions:

Let $A$ be a poset with order $\geq$ and $\Delta$ a subposet of $A$. Then, for $a$, $b \in A$ and $c, d \in \Delta$,

- $a > b$ if and only if $a \geq b$ and $a \neq b$.
- $a \gg b$ (resp. $c \gg d$) if and only if $a > b$ (resp. $c > d$) and there are not elements $e$ in $A$ (resp. $e'$ in $\Delta$) such that $a > e > b$ (resp. $c > e' > d$).

For $a = (a_1, \ldots, a_n) \in (Z^n, \#)$, this is sometimes abbreviated to $a = (a_i)$, and for any $k \in N$, $\sum_{j=1}^{n} a_j$ is denoted by $t_k(a)$.

Let $(Z^n, \ast)$ be a vector group with order $\geq$ defined by

$$(\ast) : \quad (a_i) \geq (b_i) \quad \Leftrightarrow \quad a_i \geq b_i \text{ for all } i \in N \text{ (cf. [3])}.$$

Then, one will easily see that $(Z^n, \ast)$ is a modular lattice. We consider
here the mapping
\[ \phi: (\mathbb{Z}^n, \#) \longrightarrow (\mathbb{Z}^n, *) \]
defined by \( \phi(a) = (t_i(a)) \). Clearly \( \phi(a+b) = \phi(a) + \phi(b) \) for \( a, b \in (\mathbb{Z}^n, \#) \).
By our definition, \( \phi \) is injective. Moreover, since, for \( (x_i) \in (\mathbb{Z}^n, *) \),
\[ \phi((x_1 - x_2, x_2 - x_3, \ldots, x_{n-1} - x_n, x_n)) = (x_1, \ldots, x_n), \]
\( \phi \) is surjective. Hence \( \phi \) is a group isomorphism which preserves orders, and so, \((\mathbb{Z}^n, \#)\) is a modular lattice.

Now, let \( a = (a_i), b = (b_i) \in (\mathbb{Z}^n, \#) \), and \( a > b \). Then \( \{ c \in (\mathbb{Z}^n, \#); a \equiv c \equiv b \} \) is a finite set whose cardinal number is
\[ \prod_{i=1}^n (t_i(a) - t_i(b) + 1) = \prod_{i=1}^n (\sum_{j=1}^n (a_j - b_j) + 1). \]

By \( f(a) \), we denote \( \sum_{i=1}^n t_i(a) \). Then one will easily see that \( f(a) = \sum_{i=1}^n i a_i \).

Additionally, let \( a \gg b \), and \( \phi(a) = (x_1, \ldots, x_n) \). Then \( \phi(a) \gg \phi(b) \),
and whence
\[ \phi(b) = (x_1, \ldots, x_{i-1}, x_i - 1, x_{i+1}, \ldots, x_n) \]
for some \( 1 \leq i \leq n \). Hence, it follows that there holds either
\[ b = (a_1, \ldots, a_{i-2}, a_{i-1} + 1, a_i - 1, a_{i+1}, \ldots, a_n) (2 \leq i \leq n) \]
\[ b = (a_i - 1, a_2, \ldots, a_n). \]
Moreover, we see that \( f(b) = \sum_{i=1}^n t_i(b) = \sum_{i=1}^n x_i - 1 = f(a) - 1 \).

Our study starts with the following

**Lemma 1.** Let \( a = (a_i), b = (b_i) \in (\mathbb{Z}^n, \#), \) and \( a > b \). If
\[ a = a^{(0)} \gg a^{(1)} \gg \ldots \gg a^{(s)} = b \quad (a^{(s)} \in (\mathbb{Z}^n, \#)) \]
then \( p = \sum_{i=1}^n i (a_i - b_i) \), whence the length \( p \) is uniquely determined by \( a > b \).

**Proof.**
\[ \sum_{i=1}^n i (a_i - b_i) = \sum_{i=1}^n i a_i - \sum_{i=1}^n i b_i = f(a) - f(b) \]
\[ = \sum_{i=1}^n (f(a^{(i)}) - f(a^{(i+1)})) = p. \]

The above \( p \) will be denoted by \( | a > b | \).

**Lemma 2.** Let \( \Delta = \prod_{i=1}^n [u_i, v_i] \) where \( u_i, v_i \in \mathbb{Z} \) and \( u_i < v_i \) \((i \in N)\). If \( a, b \in \Delta \) and \( a \gg b \) then \( a > b \).
Proof. Let \( a = (a_i) \) and \( b = (b_i) \). One will easily see that our assertion is true for \((Z, \#)\). Hence we assume that our lemma holds for \((Z^{n-1}, \#)\). For any \( r = (r_1, \ldots, r_n) \in (Z^n, \#)\), we set \( C_2(r) = (r_2, \ldots, r_n) \), and \( C_2(\Delta) = \{ C_2(r) : r \in \Delta \} \). Then \( C_2(\Delta) \) can be regarded as a subset of \((Z^{n-1}, \#)\), and \( C_2(a), C_2(b) \subseteq C_2(\Delta) \). Clearly \( C_2(a) \cong C_2(b) \). In case \( C_2(a) = C_2(b) \), one will easily see that \( b = (a_1 - 1, a_2, \ldots, a_n) \) where \( a_1 > u_1 \), and so \( a \gg b \). Hence, let \( C_2(a) > C_2(b) \). Then, there exists an element \( e' \) in \( C_2(\Delta) \) such that

\[
C_2(a) \gg_{C_2(\Delta)} e' \cong C_2(b)
\]

By the induction assumption, we have \( C_2(a) \gg e' \). Hence \( e' \) coincides with either

\[
f' = (a_2, \ldots, a_{k-1}, a_{k-1} + 1, a_{k-1}, a_{k+1}, \ldots, a_n)
\]

where \( 3 \leq k \leq n, a_{k-1} < v_{k-1} \) and \( a_k > u_{k} \) (in case \( k = 3, \ f' \) is taken as \( (a_2 + 1, a_3 - 1, a_4, \ldots, a_n) \)), or

\[
g' = (a_2 - 1, a_3, \ldots, a_n)
\]

where \( a_2 > u_2 \). In case \( e' = f' \), we set

\[
f = (a_1, a_2, \ldots, a_{k-2}, a_{k-1} + 1, a_{k-1}, a_{k+1}, \ldots, a_n)
\]

Then \( f \in \Delta \), \( t_i(f) = t_i(a) \cong t_i(b) \). \( C_2(f) = e' \cong C_2(b) \). and whence \( a > f \cong b \) in \( \Delta \). Hence \( f = b \), and so \( a \gg b \). In case \( e' = g' \), we set

\[
g = (a_1 + 1, a_2 - 1, a_3, \ldots, a_n)
\]

If \( a_1 < v_1 \) then there exists a chain \( a > g \cong b \) in \( \Delta \), whence \( g = b \), and so, \( a \gg b \). If \( a_1 = v_1 \) then there is a chain

\[
a > (a_1 - 1, a_2, \ldots, a_n) > (a_1, a_2 - 1, a_3, \ldots, a_n) \cong b \text{ in } \Delta
\]

which is a contradiction. This completes the proof.

By virtue of the results of Lemma 1 and Lemma 2, we obtain the following

**Theorem 3.** Let \( \Delta = \Pi_{i=1}^n [u_i, v_i] \) where \( u_i, v_i \in Z \) and \( u_i < v_i \) (i \( \in \) \( N \)). Then, if \( a = (a_i), b = (b_i) \in \Delta, a \gg b \), and

\[
a = a^{(0)} \gg a^{(1)} \gg \cdots \gg a^{(q)} = b \quad (a^{(i)} \in \Delta)
\]

then \( q = |a \gg b| = \sum_{i=1}^n i(a_i - b_i) \), and \( a^{(t)} \gg a^{(t+1)} \) for \( t = 0, 1, \ldots, q - 1 \).

**Lemma 4.** Let \( \Delta = \Pi_{i=1}^n [u_i, v_i] \) where \( u_i, v_i \in Z \) and \( u_i < v_i \) (i \( \in \) \( N \)). Let \( b = (b_i), c, d \in \Delta, c \neq d, c \gg b, d \gg b, \text{ and } e = c \cup d \in \text{ the lattice } (Z^n, \#). \) Then

\[
e \in \Delta, e \gg c \gg b, e \gg d \gg b,
\]
and for \( r \in \Delta \) with \( r \geq c \) and \( r \geq d \),
\[
r \geq e \text{ and } |r \geq e| = |r > b| - 2.
\]

**Proof.** We have the two cases (1) and (2):

(1) \( c = (b_1 + 1, b_2, \ldots, b_n) \) and
\[
d = (b_1, \ldots, b_{j-2}, b_{j-1} - 1, b_j + 1, b_{j+1}, \ldots, b_n) \quad (2 \leq j \leq n);
\]

(2) \( c = (b_1, \ldots, b_{i-2}, b_{i-1} - 1, b_i + 1, b_{i+1}, \ldots, b_n) \quad (2 \leq i \leq n) \) and
\[
d = (b_1, \ldots, b_{j-2}, b_{j-1} - 1, b_j + 1, b_{j+1}, \ldots, b_n) \quad (i \leq j - 1 < n).
\]

In case (1) and \( j - 1 > 1 \), we set
\[
e' = (b_1 + 1, b_2, \ldots, b_{j-2}, b_{j-1} - 1, b_j + 1, b_{j+2}, \ldots, b_n).
\]

In case (1) and \( j - 1 = 1 \), we set
\[
e' = (b_1, b_2 + 1, b_3, \ldots, b_n).
\]

In case (2) and \( j - 1 > i \), we set
\[
e' = (b_1, \ldots, b_{i-2}, b_{i-1} - 1, b_i + 1, b_{i+1}, \ldots, b_{j-2}, b_{j-1} - 1, b_j + 1, b_{j+1}, \ldots, b_n).
\]

In case (2) and \( j - 1 = i \), we set
\[
e' = (b_1, \ldots, b_{i-2}, b_{i-1} - 1, b_i, b_{i+1} + 1, b_{i+2}, \ldots, b_n).
\]

Then we have
\[
e' \in \Delta, e' \gg c \gg b \text{ and } e' \gg d \gg b.
\]

Since \((Z^n, \#)\) is a lattice, \(e'\) coincides with \(c \cup d\) in \((Z^n, \#)\), that is, \(e' = e\). Moreover, for \(r \in \Delta\) with \(r \geq c\) and \(r \geq d\), we see that
\[
r \geq e' \text{ and } |r \geq e'| = |r > b| - 2.
\]

**Lemma 5.** Let \( \Delta = \prod_{i} [u_i, v_i] \) where \( u_i, v_i \in Z \) and \( u_i < v_i \) \((i \in N)\). Let \( c, d \in \Delta, e = c \cup d \), and \( f = c \cap d \) in the lattice \((Z^n, \#)\). Then \(e, f \in \Delta\).

**Proof.** Clearly \( u = (u_i) \in \Delta\), \( c \geq u\), and \( d \geq u\). Hence, we have \(e \geq u\) and a finite length \(|e \geq u|\). Let \(m(c, d)\) be the smallest integer in \(|e \geq w| ; c \geq w, d \geq w, w \in \Delta\). If \(m(c, d) = 0 \) then \(c = d = e\). By the induction with respect to \(m(c, d)\), we shall prove that \(e \in \Delta\). Hence, let \(m(c, d) = t \geq 1\), and assume that \(c' \cup d' \in \Delta\) for \(c', d' \in \Delta\) with \(m(c', d') < t\). If either \(c \geq d\) or \(d \geq c\) then our assertion holds trivially. Hence, let \(c \nleq d\) and \(d \nleq c\). Let \(w_0\) be an element of \(\Delta\) such that \(|e \geq w_0| = m(c, d)\), and consider the following chains (note Theorem 3):
\[
e \geq c = c^{(0)} \gg c^{(1)} \gg \cdots \gg c^{(q)} = w_0 \quad (c^{(q)} \in \Delta),
\]
\[
e \geq d = d^{(0)} \gg d^{(1)} \gg \cdots \gg d^{(q)} = w_0 \quad (d^{(q)} \in \Delta).
\]
Clearly \( c^{(q-1)} \neq d^{(q-1)} \). To prove \( e \in \Delta \), we shall distinguish the following cases:

1. \( c \gg w_0, \ d \gg w_0 \);
2. \( c \gg w_0, \ d \geq d^{(q-2)} \ (q \geq 2) \);
3. \( c \geq c^{(q-2)}, \ d \geq d^{(q-2)} \ (p \geq 2, \ q \geq 2) \).

In case (1), we have \( e = c \cup d \in \Delta \) by Lemma 4.

In case (2), we set \( w_1 = c \cup d^{(q-1)} \). Then, we have \( w_1 \in \Delta \) by Lemma 4, and \( w_1 \cup d = e \). Moreover, \( w_1 \geq d^{(q-1)}, \ d \geq d^{(q-1)} \), and so, \( m(w_1, d) < t \). Hence, we have \( w_1 \cup d \in \Delta \) by the induction assumption, that is, \( e \in \Delta \).

In case (3), we set \( w_1 = c^{(q-1)} \cup d^{(q-1)} \). Then, we have \( w_1 \in \Delta \) by Lemma 4, and \( (c \cup w_1) \cup (w_1 \cup d) = e \). Since \( m(c, w_1) < t \) and \( m(w_1, d) < t \), we have \( c \cup w_1 \in \Delta \) and \( w_1 \cup d \in \Delta \). Moreover, since \( m(c \cup w_1, w_1 \cup d) < t \), it follows that \( (c \cup w_1) \cup (w_1 \cup d) \in \Delta \), that is, \( e \in \Delta \). By the duality of the lattice \((Z^n, \#)\), we have \( f \in \Delta \). This completes the proof.

Now, by virtue of the result of Lemma 5, we obtain the following theorem which is our main result.

**Theorem 6.** Let \( \Delta = \prod_{i=1}^{n} [u_i, v_i] \) where \( u_i, v_i \in \mathbb{Z} \) and \( u_i < v_i \ (i \in N) \). Then \( \Delta \) is a modular sublattice of \((Z^n, \#)\).

By the general theory of modular lattices, it has been known that for any modular lattice with both chain conditions and for any chain \( a \geq b \) in its lattice, the composition chains \( a \gg \ldots \gg b \) has a unique length (cf. Lemma 1, Theorem 3, Theorem 6, [1], [2], and [5]). Now, we shall prove the following

**Theorem 7.** Let \( \Delta = \prod_{i=1}^{n} [u_i, v_i] \) where \( u_i, v_i \in \mathbb{Z} \) and \( u_i \leq v_i \ (i \in N) \). Then \( \Delta \) is a modular lattice under the ordering in \((Z^n, \#)\), and whence it is a Cohen-Macaulay poset. If \( a = (a_i), \ b = (b_i) \in \Delta \), \( a > b \) and

\[
 a = a^{(0)} \gg a^{(1)} \gg \ldots \gg a^{(q)} = b \ (a^{(l)} \in \Delta)
\]

then \( q = |a > b| - \sum_{i=1}^{n} (i(i-1)(a_i-b_i) = \sum_{i=1}^{n} (i-1)(a_i-b_i) \) where \( i(0) \) is the cardinal number of the set \( \{j \in N; \ u_j = v_j, \ j < i\} \).

**Proof.** Let \( \{\varepsilon(1), \ldots, \varepsilon(m)\} = \{i \in N; \ u_i < v_i\} \) where \( \varepsilon(1) < \cdots < \varepsilon(m) \). Then, there is an ordered isomorphism

\[
 \psi: \Delta = \prod_{i=1}^{n} [u_i, v_i] \longrightarrow \prod_{i=1}^{n} [u_{\varepsilon(i)}, v_{\varepsilon(i)}] \subset (Z^n, \#)
\]
such that \( \psi(d_1, \ldots, d_n) = (d_{\varepsilon(1)}, \ldots, d_{\varepsilon(m)}) \). By Theorem 6, \( \psi(\Delta) \) is a mod-
ular sublattice of \((\mathbb{Z}^n, \#)\). Hence \(\Delta\) is a modular lattice under the ordering in \((\mathbb{Z}^n, \#)\). Since \(u_{\varepsilon} \prec v_{\varepsilon}\) for \(j = 1, \ldots, m\) and \(\psi(a^{t+1}) \gg_{\psi \Delta} \psi(a^{t})\) in \(\psi(\Delta)\) for \(t = 0, 1, \ldots, q-1\), we have \(q = \sum_{t=1}^{n} \psi(a_{\varepsilon}^{t}) \geq \sum_{t=1}^{n} \psi(a_{\varepsilon}^{t-1})\) by Theorem 3. Hence, it follows from Lemma 1 that

\[
|a > b| = \sum_{t=1}^{n} \psi(a_{\varepsilon}^{t}) - \sum_{t=1}^{n} \psi(a_{\varepsilon}^{t-1}) + \sum_{t=1}^{n} \psi(a_{\varepsilon}^{t}) - \sum_{t=1}^{n} \psi(a_{\varepsilon}^{t-1}) = q + \sum_{t=1}^{n} \psi(0) (a_{\varepsilon}^{t} - b_{\varepsilon}^{t}).
\]

Remark. Let \(B\) be a ring with an identity 1 and \(A\) a subring of \(B\) with common identity 1 of \(B\). In [4], a sequence of additive \(A\)-endomorphisms \(\{f_0 = 1, f_1, \ldots, f_n\}\) of \(B\) is said to be a relative sequence of homomorphisms with \(\Psi\) if it satisfies the following conditions: for every \(j \in N = \{1, 2, \ldots, n\},\)

1. \(f_j f_k = f_k f_j\) and \(f_k(1) = 0\) for all \(k \in N\).
2. There exists \(\Psi_j = \{g(f_i, f_j) ; 0 \leq i \leq j\} \subset \text{End}(B)\) such that
   - \(g(f_i, f_j)(x)(y) = \sum_{t=0}^{j} g(f_j, f_i)(x)f_i(y)\) for \(x, y \in B\),
   - \(g(f_i, f_0) = f_i\),
   - \(g(f_i, f_j)\) is a ring isomorphism.

As is easily seen, for an \(A\)-automorphism \(\sigma\) of \(B\), if we put \(D = \sigma - 1\), then \(\Phi = \{D^0 = 1, D, \ldots, D^n\}\) becomes a relative sequence of homomorphisms with \(\Psi\) such that

\[
\Psi_j = \{g(D^i, D^j) = \binom{j}{i} \sigma^i D^{j-i} ; 0 \leq i \leq j\}.
\]

A subset \(\Phi = \{d_0 = 1, d_1, \ldots, d_n\}\) of \(\text{End}(B)\) is said to be an \(A\)-higher derivation of \(B\) if \(d_j(xy) = \sum_{t=0}^{j} d_{j-t}(x)d_t(y)\) for \(x, y \in B\). Then \(\Phi\) becomes a relative sequence of homomorphisms with \(\Psi\) such that \(\Psi_j = \{g(d_i, d_j) = d_{j-i} ; 0 \leq i \leq j\}\).

Now, for a relative sequence of homomorphisms \(\Phi = \{f_0 = 1, f_1, \ldots, f_n\}\) with \(\Psi\), we consider the multiplication subsemigroup \(L\) of \(\text{End}(B)\) which is generated by \(\Phi\), and assume the following conditions on \(L\):

1. There exists a positive integer \(q\) such that \((f_k)^q = 0\) and \((f_k)^s \neq 0\) for all \(k \in N\) and \(0 \leq s \leq q - 1\).
2. \(\Pi_{l=1}^{n} f_{l}^{s_l} \neq 0\) if \(0 \leq s_l \leq q - 1\) for all \(i \in N\).
3. If \(\Omega = \Pi_{l=1}^{n} f_{l}^{r_l}\) and \(\Lambda = \Pi_{l=1}^{n} f_{l}^{r_l} (0 \leq s_l, r_l \leq q - 1)\), then \(\Omega = \Lambda\) if and only if \(s_l = r_l\) for all \(i \in N\).

Then \(L = U \cup \{0\}\) where \(U = \{\Pi_{l=1}^{n} f_{l}^{s_l} ; 0 \leq s_l \leq q - 1\}\) becomes a
commutative finite multiplicative subsemigroup of \( \text{End}(B_s) \), and \( U \) becomes a modular lattice as is shown in Theorem 6. Further, in this case, we can see that
\[
(i)^* : \quad \mathcal{Q}(xy) = \sum_{r \leq \mathcal{Q}} g(\mathcal{Q}, \mathcal{I})(x)\mathcal{I}(y)
\]
for \( x, y \in B \) where \( g(\mathcal{Q}, \mathcal{I}) \) is obtained as a sum of products of \( g(f_{i_s} f_{o_s})'s \) with \( \prod_s f_{i_s} = \mathcal{Q} \) and \( \prod_s f_{o_s} = \mathcal{I} \). One of the authors made a study on Galois theory of \( B/A \) where \( A = B^t = \{ b \in B; \Lambda(b) = 0 \text{ for all } \Lambda(\neq 1) \in U \} \) in [4]. In that paper, \((i)^*\) and the uniqueness of \( |\mathcal{Q}| > 1 \) play important rôles. One of motivations of this paper comes from this study of Galois theory.

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