Equivalent formulations of the Jacobian Conjecture

by

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Let $k$ be a field of characteristic zero and $k[x_1, \ldots, x_n]$ the ring of polynomials over $k$. Let $f_1, \ldots, f_n \in k[x_1, \ldots, x_n]$. We denote by $J(f_1, \ldots, f_n)$ the Jacobian of $(f_1, \ldots, f_n)$, that is, $J(f_1, \ldots, f_n) = \det(\partial f_i / \partial x_j)$. If $k[f_1, \ldots, f_n] = k[x_1, \ldots, x_n]$ then it is easy to see that $J(f_1, \ldots, f_n)$ is a non-zero constant. The Jacobian Conjecture is the converse:

(JC$_1$) If $J(f_1, \ldots, f_n)$ is a non-zero constant then $k[x_1, \ldots, x_n] = k[f_1, \ldots, f_n]$.

The problem was raised by Keller in 1939 [11]. For $n=1$ the conjecture holds trivially. For $n \geq 2$ the conjecture is still not settled.

1. Equivalent formulations for arbitrary $n$. Denote $X = (x_1, \ldots, x_n)$, $F = (f_1, \ldots, f_n)$ and let $k^* = k \setminus \{0\}$. Then the Jacobian Conjecture says that: if $J(F) \in k^*$ then $k[X] = k[F]$. This assertion is equivalent to

(JC$_2$) If $J(F) \in k^*$, then $k(X) = k(F)$.

Denote by $A_F$ the $k$-endomorphism of $k[X]$ defined by $A_F(x_i) = f_i$, for $i=1, \ldots, n$. We see that (JC$_1$) is equivalent to

(JC$_3$) If $J(F) \in k^*$, then $A_F$ is surjective.

If $J(F) \in k^*$ then it is easy to show that the polynomials $f_1, \ldots, f_n$
are algebraically independent over \( k \), that is, the map \( \varphi \) is injective. Therefore, \( (JC_3) \) is equivalent to

\[(JC_4) \text{ If } J(F) \in k^*, \text{ then } \varphi \text{ is } k\text{-automorphism of } k[x].\]

If \( f_1, \ldots, f_n \) are algebraically independent over \( k \) then \( k(F) \subseteq k(x) \) is an algebraic extension of fields. It is well-known ([6], [28], [26][31]) that \( (JC_4) \) is equivalent to the following conjecture

\[(JC_5) \text{ If } J(F) \in k^*, \text{ then } k(x) \text{ is Galois over } k(F).\]

We remark also on the following equivalent formulations of the Jacobian Conjecture.

\[(JC_6) \text{ ([30], [4]) If } J(F) \in k^*, \text{ then } k[x] \text{ is a finitely generated } k[F]\text{-module.}\]

\[(JC_7) \text{ ([30], [4]) If } J(F) \in k^*, \text{ then } k[x] \text{ is a projective } k[F]\text{-module.}\]

\[(JC_8) \text{ ([31], [33]) If } J(F) \in k^*, \text{ then } k[x] \text{ is integral over } k[F].\]

Observe that if \( f_1, \ldots, f_n \in k[x] \) and \( a_1, \ldots, a_n \in k \) then

\[J(f_1 + a_1, \ldots, f_n + a_n) = J(f_1, \ldots, f_n) \quad \text{and} \]

\[k[f_1 + a_1, \ldots, f_n + a_n] = k[f_1, \ldots, f_n].\]

Therefore, we may assume that the polynomials \( f_1, \ldots, f_n \) have no constant terms. Hence we have the \( k\)-endomorphism \( \overline{\varphi} \) of \( k[[x]] = k[[x_1, \ldots, x_n]] \) (the power series ring over \( k \)) such that \( \overline{\varphi}(x_i) = f_i \), for \( i = 1, \ldots, n \). It is well-known ([4], [8], [21]) the following Inverse Function
Theorem 1.1. If $J(F) \in k^*$, then $\bar{\Lambda}_F$ is a $k$-automorphism of $k[[x]]$.

Recall the proof of Theorem 1.1 given by Nousiainen and Sweedler [21].

Let $w = J(F) \in k^*$ and denote by $\bar{D}_1, \ldots, \bar{D}_n$ the $k$-derivations of $k[[x]]$ defined by

$$\bar{D}_i(h) = w^{-1} J(f_1, \ldots, f_{i-1}, h, f_{i+1}, \ldots, f_n),$$

for $h \in k[[x]]$ and $i = 1, \ldots, n$.

Consider the mapping $\bar{B}_F : k[[x]] \to k[[x]]$ defined by

$$\bar{B}_F(h) = \sum_{i_1, \ldots, i_n} (x_1 - f_1)^{i_1} \cdots (x_n - f_n)^{i_n} \bar{D}_{i_1} \cdots \bar{D}_{i_n}(h),$$

where

$$\bar{D}_{i_1} \cdots \bar{D}_{i_n}(h) = (i_1 ! \cdots i_n !)^{-1} \bar{D}_{i_1} \cdots \bar{D}_{i_n}(h).$$

It is easy to verify that $\bar{B}_F \bar{\Lambda}_F = \bar{\Lambda}_F \bar{B}_F = \text{id}$, and hence, $\bar{\Lambda}_F$ is a $k$-automorphism of $k[[x]]$.

Observe that the restriction of $\bar{\Lambda}_F$ to $k[x]$ is equal to $\Lambda_F$. Therefore the Jacobian Conjecture is equivalent to

$$(JC_9) \text{ If } J(F) \in k^*, \text{ then } \bar{B}_F(k[x]) \subseteq k[x].$$

Let $D_i = \bar{D}_i | k[x]$, for $i = 1, \ldots, n$. Then $D_1, \ldots, D_n$ are $k$-derivations of $k[x]$ and we have the next two equivalent formulations of the Jacobian Conjecture

$$(JC_{10}) \text{ ([33], [21]) If } J(F) \in k^*, \text{ then the derivations } D_1, \ldots, D_n \text{ are locally nilpotent.}$$
(JC\textsubscript{11}) ([21]) If \( J(F) \in k^* \), then the derivations \( D_1, \ldots, D_n \) are locally finite.

We recall from [21] that a \( k \)-derivation \( d \) of \( k[x] \) is called locally nilpotent if for each \( r \in k[x] \) there exists a natural number \( s \) such that \( d^s(r) = 0 \), and is called locally finite if for any \( r \in k[x] \) there exists a finite generated \( k \)-module \( M \subseteq k[x] \) such that \( r \in M \) and \( d(M) \subseteq M \).

Let \( \text{Der}_k(k[x]) \) be the \( k[x] \)-module of all \( k \)-derivations of \( k[x] \). It is well-known ([5]) that \( \text{Der}_k(k[x]) \) is a free \( k[x] \)-module on the basis \( \partial/\partial x_1, \ldots, \partial/\partial x_n \). Let \( D = \{ d_1, \ldots, d_n \} \) be a basis of \( \text{Der}_k(k[x]) \). We say that \( D \) is locally nilpotent (resp. locally finite) if every derivation \( d_i \) \((i=1, \ldots, n)\) is locally nilpotent (resp. locally finite).

Moreover, we say that \( D \) is commutative if \( d_i d_j = d_j d_i \), for all \( i, j \).

We may prove the next two equivalent formulations of (JC\textsubscript{1})

(JC\textsubscript{12}) ([22]) Every commutative basis of \( \text{Der}_k(k[x]) \) is locally nilpotent.

(JC\textsubscript{13}) ([22]) Every commutative basis of \( \text{Der}_k(k[x]) \) is locally finite.

A list of another equivalent formulations of the Jacobian Conjecture we may find in [30], [31], [33] and [4].

2. Effective results for arbitrary \( n \). Denote by \( M(F) \) the number \( \max(\deg(f_1), \ldots, \deg(f_n)) \). Using a clever argument, Jagićev 1980 [9], Bass, Connell and Wright 1982 [4] reduced the problem to the case \( M(F) \leq 3 \) at
the cost of introducing extra variables. In 1980 Wang [30] has shown
that the Conjecture holds if \( M(F) \leq 2 \). (Another proofs of the Wang's
result we may find in [13], [27]). In 1983 Drużkowski [7] (see also [29])
has proved that the Conjecture reduces, at the cost of increasing \( n \),
to the case where each \( f_i \) has the following form:

\[
f_i = x_i + (a_{i1}x_1 + \ldots + a_{in})^3,
\]

where \( a_{ij} \in k \). In addition, he has obtained positive answers when the
rank of the matrix \( (a_{ij}) \) is equal to 0,1,2 or \( n-1 \) (it is easy to show
that \( \text{rank}(a_{ij}) < n \)).

Denote by \( P(F) \) the projective dimension of \( k[x] \) over \( k[F] \). The
assertion (JC_3) says that if \( J(F) \in k^* \), then \( P(F) = 0 \). In 1980 Wang [30]
has shown that if \( J(F) \in k^* \), then \( P(F) \leq 1 \).

3. The case \( n=2 \). If \( n=2 \) then we shall denote \( f=f_1, g=f_2, x=x_1 \) and
\( y=x_2 \). The Jacobian of \((f,g)\) we shall denote by \([f,g]\), that is,

\[
[f,g] = f_y g_x - f_x g_y.
\]

Moreover, if \( f, g \in k[x,y] \) then \( f \sim g \) means that \( f = \alpha g \), for some \( \alpha \in k^* \).

In this situation (JC_1) has the following form:

"If \( [f,g] \sim 1 \), then \( k[f,g] = k[x,y] \)."

It is easy to verify that the brackets \([,\) are Lie brackets, so
\((k[x,y], [,),] \) is a Lie-algebra over \( k \). The Jacobian Conjecture is
equivalent to the following conjecture:

\[(JC_{14}) \text{ Every } k\text{-endomorphism of } k[x,y] \text{ which is a Lie map is an auto-}\]

\[\text{morphism.}\]

\[\text{Proof. } (JC_1) \Rightarrow (JC_{14}). \text{ Let } A \text{ be a } k\text{-endomorphism of } k[x,y] \text{ which is a Lie map. Then } 1 = A(1) = A([x,y]) = [A(x), A(y)] \text{ and, by } (JC_1),\]

\[k[x,y] = k[A(x), A(y)] = A(k[x,y]), \text{ that is, } A \text{ is an automorphism.}\]

\[\Rightarrow (JC_1). \text{ Let } f, g \in k[x,y] \text{ with } [f, g] = a \in k^*. \text{ Consider } k\text{-endo-}\]

\[\text{morphism } A \text{ of } k[x,y] \text{ such that } A(x) = a^{-1}f, A(y) = g. \text{ Then } A \text{ is a Lie}\]

\[\text{map so, by } (JC_{14}), k[f, g] = k[A(x), A(y)] = A(k[x,y]) = k[x,y]. \Box\]

\[\text{In the case } n=2 \text{ we know some partial results and several equi-}\]

\[\text{valent formulations of the Jacobian Conjecture. In this case it is known ([17]) that the problem has positive answers when } \text{Max} (\text{deg}(f), \text{deg}(g)) \leq 100. \text{ In this case it is also known the structure of}\]

\[\text{Aut}_k(k[x,y]), \text{ the group of all } k\text{-automorphisms of the ring } k[x,y].\]

\[\text{More precisely, it is well-known ([10], [12], [16], [18], [6], [32]) the following}\]

\[\text{Theorem 3.1. Every } k\text{-automorphism of } k[x,y] \text{ is a composite of } k\text{-auto-}\]

\[\text{morphisms of the following form:}\]

\[a) \quad (x,y) \mapsto (a_1x + b_1y + c_1, a_2x + b_2y + c_2),\]

\[\text{where } a_1, a_2, b_1, b_2, c_1, c_2 \in k \text{ and } a_1b_2 - a_2b_1 \neq 0,\]

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b) \((x, y) \longrightarrow (x, y + u(x))\), where \(u(x) \in k[x]\).

Note the following Abhankar-Moh's Lemma [1] (see also [2], [16]):

**Lemma 3.2.** Let \(u, v \in k[t]\). If \(k[u, v] = k[t]\) then \(\text{deg}(u) \mid \text{deg}(v)\) or \(\text{deg}(v) \mid \text{deg}(u)\).

In the papers [1], [2] and [16] we may find easy proofs of Theorem 3.1 which show that this theorem is a consequence of Lemma 3.2.

Using Lemma 3.2, Abhyankar[2] has shown that if \(A\) is a \(K\)-automorphism of \(k[x, y]\) then \(\text{deg}(A(x)) \mid \text{deg}(A(y))\) or \(\text{deg}(A(y)) \mid \text{deg}(A(x))\). This fact we may prove also without this lemma (see [23]). As a simple consequence of the above fact we may obtain the following equivalent formulation of the Jacobian Conjecture ([2], [23])

\[(JC_{15}) \quad \text{If } [f, g] \sim _1, \text{ then } \text{deg}(f) \mid \text{deg}(g) \quad \text{or} \quad \text{deg}(g) \mid \text{deg}(f).\]

Observe that \((JC_{15})\) is equivalent to the following

\[(JC_{16}) \quad \text{Let } [f, g] \sim _1, \text{ deg}(f) = m, \text{ deg}(g) = n \text{ and let } d \text{ be a natural number. If } \gcd(m, n) = d, \text{ then } \min(m, n) = d.\]

Today we know that the assertion \((JC_{16})\) is true in the following three cases:

1) (1955 Magnus [14]) \(d=1\),

2) (1977 Nakai and Baba [20]) \(d=2\),

3) (1985 Appelgate and Onishi [3]) \(d\) is prime.
Denote by $P$ the set of all prime numbers.

From the above facts it is easy to obtain the following three partial results:

Let $[f, g] \sim 1$, $\deg(f) = m$, $\deg(g) = n$, where

1) $([14], [20])$ $m$ or $n \in P$, or

2) $([20])$ $m$ or $n$ is of the form $2p$, where $p \in P$, or

3) $([3])$ $m$ or $n$ is of the form $pq$, where $p, q \in P$.

Then $k[f, g] = k[x, y]$.

Therefore we have

Theorem 3.3 ([3]) Let $[f, g] \sim 1$, $\deg(f) = m$, $\deg(g) = n$. If $m$ or $n$ is of the form $pq$, where $p, q \in P \cup \{1\}$, then $k[x, y] = k[f, g]$.

We see that positive results connected with the Jacobian Conjecture (in the case $n=2$) had the beginning from Magnus' Theorem (if $\gcd(m, n) = 1$, then $\min(m, n) = 1$). In 1977 Nakai and Baba [20] have given an elegant second proof of Magnus' Theorem. They have used weighted gradings on $k[x, y]$ and the "method of rotation of lines around the points". In the proof of Theorem 3.3, Appelgate and Onishi have used also Nakai-Baba's methods. Note that similar methods, also in 1977, has used Abhyankar in [2] and he obtained a number of interesting facts about the Jacobian Conjecture. A few Abhyankar results we
may find (with another proofs) in [3] and [23].

Now we shall present several Abhyankar’s and Appelgate-Onishi’s results.

Observe that the Jacobian Conjecture (in the case n=2) is trivially easy if \( \min(\deg(f), \deg(g)) = 1 \). So we may assume that \( \min(\deg(f), \deg(g)) > 1 \). We shall say that \((f, g)\) is a basic pair if \([f, g] \sim 1\) and \(\min(\deg(f), \deg(g)) > 1\).

If \(h \in k[x, y]\) then \(S_h\) denotes the support of \(h\), that is, \(S_h\) is the set of integers points \((i, j)\) such that the monomial \(x^i y^j\) appears in \(h\) with a non-zero coefficient. For example, if \(h = x + 3xy + x^3 y^6 + 2x^9\), then \(S_h = \{(1, 0), (1, 1), (3, 6), (9, 0)\}\).

By a direction we mean a pair \((p, q)\) of integers such that \(\gcd(p, q) = 1\) and \(p > 0\) or \(q > 0\).

Let \((p, q)\) be a direction. We say that a non-zero polynomial \(h\) in \(k[x, y]\) is a \((p, q)\)-form of degree \(n\) if \(h\) is of the following form

\[
h = \sum_{pi+qj=n} a_{ij} x^i y^j,
\]

where \(a_{ij} \in k\); that is, \(h \neq 0\) is a \((p, q)\)-form of degree \(n\) if and only if the support \(S_h\) is contained in the line defined by the equation \(px + qy = n\). The degree of a \((p, q)\)-form \(h\) is denoted by \(d_{pq}(h)\).

We see that every polynomial \(f \in k[x, y]\) has the \((p, q)\)-decomposition
Theorem 3.6([2], [3], [15]). If \((f,g)\) is a basic pair then

\[ h_{11} \sim (ax + by)^s(cx + dy)^t, \]

where \(a, b, c, d \in k, ad - bc \neq 0, s \geq 0, t \geq 0\) and \(s + t = \gcd(\deg(f), \deg(g))\).

Let \(f\) be an element of \(k[x,y]\) and \(r\) a natural number. We say that \(f\) has \(r\) points at infinity if \(f^{11} \sim h_1^{i_1} \cdots h_r^{i_r}\), where \(i_1, \ldots, i_r\) are positive and \(h_1, \ldots, h_r\) are coprime linear forms (that is \((1,1)\)-forms of degree 1). By Theorems 3.5 and 3.6 we see that if \((f,g)\) is a basic pair then \(f\) and \(g\) have at most two points at infinity. Abhyankar [2] has proved that the Jacobian Conjecture is equivalent to the following (JC_17) If \((f,g)\) is a basic pair then \(f\) and \(g\) have only one point at infinity.

An another proof of this equivalence there is in [23].

Therefore, for a proof of the Jacobian Conjecture we must show, by (JC_17), that in Theorem 3.6 \(s = 0\) or \(t = 0\).

If \(f \in k[x,y]\) then by \(W_f\) we shall denote the Newton polygon of \(f\), that is, the convex hull (in the real space \(R^2\)) of the set \(S_f \cup \{(0,0)\}\).

In general \(W_f\) is a point or a line segment or a convex polygon. But it is easy to show that if \((f,g)\) is a basic pair then \(W_f\) and \(W_g\) are convex polygons (that is, they are not points or line segments). Moreover we may show that if \((f,g)\) is a basic pair then the polygons \(W_f\) and \(W_g\)
\[ f = \sum_{n} f_n \] into \((p,q)\)-components \(f_n\) of degree \(n\). We denote by \(f^*_{pq}\) the \((p,q)\)-component of \(f\) of the highest degree. By \((p,q)\)-degree of a polynomial \(f\) we mean the number \(d_{pq}(f) = d_{pq}(f^*)\). Observe that \(d_{11}(f) = \deg(f)\). For example, let \(f = x + 3xy + x^3y^6 + 2x^9\).

a) If \(p=1\) and \(q=1\) then \(f^*_{pq} = x^3y^6 + 2x^9\) and \(d_{pq}(f) = 9\).

b) If \(p=2\) and \(q=1\) then \(f^*_{pq} = 2x^9\) and \(d_{pq}(f) = 18\).

c) If \(p=-5\) and \(q=2\) then \(f^*_{pq} = 3xy + x^3y^6\) and \(d_{pq}(f) = -3\).

We see that in the example c) the \((p,q)\)-degree of \(f\) is negative. But we may prove (\([25]\)), the following:

**Lemma 3.4.** If \((f,g)\) is a basic pair then the numbers \(d_{pq}(f)\) and \(d_{pq}(g)\) are positive for any direction \((p,q)\).

Moreover we may prove

**Theorem 3.5.** (\([3]\), \([25]\)). Let \((f,g)\) be a basic pair with \(\deg(f) = \deg(g) = 1\). Then, for every direction \((p,q)\), there exists a \((p,q)\)-form \(h = h_{pq}\) of positive degree such that

\[ f^*_{pq} \sim h^m \quad \text{and} \quad g^*_{pq} \sim h^n. \]

A similar theorem there is in \([2]\) or \([20]\), but in these papers are only that \(f^*_{pq} \sim h^{m'}\) and \(g^*_{pq} \sim h^{n'}\), for some \(m', n'\).

In \([2]\) and \([3]\) there is a description of the \((p,q)\)-form \(h_{pq}\) from Theorem 3.5. For example, in the case \(p=q=1\) we have the following result.
are similar. More precisely, using Nakai-Baba’s methods, we may obtain the following

**Theorem 3.7** ([3], [25]). Let \( (f, g) \) be a basic pair with \( \text{deg}(f) = dm, \text{deg}(g) = dn, \) where \( \gcd(m, n) = 1. \) There exists a convex polygon \( W \) with vertices in \( \mathbb{Z} \times \mathbb{Z} \) such that \( W_f = mW \) and \( W_g = nW. \)

Abhyankar [2] (see also [23]) has proved that the Jacobian Conjecture is equivalent to

\[(JC_{18}) \quad \text{If } (f, g) \text{ is a basic pair then the polygons } W_f \text{ and } W_g \text{ are triangles.}\]

Denote by \( t_x(f) \) (resp. \( t_y(f) \)), where \( f \in k[x, y] \), the greatest integer \( s \) such that the monomial \( x^s \) (resp. \( y^s \)) appears in \( f \) with a non-zero coefficient, that is, \( t_x(f) \) (resp. \( t_y(f) \)) is the greatest integer \( s \) such that the point \( (s, 0) \) (resp. \( (0, s) \)) belongs to the support of \( f. \)

Using the above facts, it is easy to prove (see [23]) that the Jacobian Conjecture is equivalent to

\[(JC_{19}) \quad \text{If } (f, g) \text{ is a basic pair, then } t_x(f) \mid t_x(g) \text{ or } t_x(g) \mid t_x(f).\]

Moreover, by symmetry, the Jacobian Conjecture is equivalent to

\[(JC_{20}) \quad \text{If } (f, g) \text{ is a basic pair, then } t_y(f) \mid t_y(g) \text{ or } t_y(g) \mid t_y(f).\]
4. Remarks and questions.

(4.1) The polygon $W$ from Theorem 3.7 is called the basic web of $(f, g)$.

By Appelgate-Onishi's results [3] we can deduce that if the Jacobian Conjecture is not true, then there exists a basic pair $(f, g)$ with

de$(f) = dm$, de$(g) = dn$, $gcd(m, n) = 1$, such that its basic web $W$ is contained in a rectangle with vertices $(0, 0), (a, 0), (0, b)$ and $(a, b)$,

where

1) $a > 0, b > 0$,

2) $a \neq b$,

3) $a + b = d$, and

4) $gcd(a, b) \geq 1$.

This means that, if we prove that the above situation is impossible, then we shall have a proof of the Jacobian Conjecture in the case $n=2$.

(4.2) If $k[x_1, \ldots, x_n] = k[f_1, \ldots, f_n]$ then, for any $a_1, \ldots, a_n \in k$, the polynomials $f_1 + a_1, \ldots, f_n + a_n$ are irreducible.

How may we prove that if $f_1, \ldots, f_n$ are polynomials with $J(f_1, \ldots, f_n) \in k^*$ then $f_1, \ldots, f_n$ are irreducible?

Observe that, in the case $n=2$, by Lemma 3.4, we may only deduce that if $(f, g)$ is a basic pair then $f$ and $g$ have not linear divisors. We know, by Theorem 1.1, that if $J(f_1, \ldots, f_n) \in k^*$ then the polynomials $f_1, \ldots, f_n$ are irreducible in $k[[x_1, \ldots, x_n]]$. 

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Note also the following

Theorem 4.2.1 ([28]). Let \((f, g)\) be a basic pair. Assume that there is an element \(a \in k\) such that the polynomial \(f + ag + b\) is irreducible for all \(b \in k\). If there is a polynomial \(h \in k[x, y]\) such that \(k[f, g, h] = k[x, y]\) then \(k[f, g] = k[x, y]\).

(4.3) The next problem is about a generalization of Theorem 3.7 in the case \(n > 2\). If \(f \in k[x_1, \ldots, x_n]\) then denote by \(S_f\) the support of \(f\) and let \(W_f\) be the convex hull (in the space \(R^n\)) of the set \(S_f \cup \{0, \ldots, 0\}\).

In the proof of Theorem 3.7 we often use (see [25]) the following simple rule:

\[
[x^a y^b, x^c y^d] = (ad - bc)x^{a+c-1} y^{b+d-1}.
\]

The same rule we have for arbitrary \(n\), that is,

\[
J(x_1, x_2, \ldots, x_n, \ldots, x_1, x_2, \ldots, x_n) = \det(a_{ij}) x_1^{s_1} \ldots x_n^{s_n},
\]

where \(s_i = a_{i1} + a_{i2} + \ldots + a_{in}\), for any \(i = 1, \ldots, n\).

Let, for example, \(n = 3\). Assume that \(f, g, h \in k[x, y, z]\) are polynomials such that \(\min(\deg(f), \deg(g), \deg(h)) > 1\) and \(J(f, g, h) \in k^*\). Then we see, using the above rule and using weighted gradings on \(k[x, y, z]\), that there exist vertices \(A \in W_f, B \in W_g\) and \(C \in W_h\) such that the points \(A, B, C\)
and \(0 = (0,0,0)\) lie on the same plane. However we have not a generalization of Lemma 3.4, so we cannot yet use the Nakai-Baba's methods.

Note also that it is not true the generalization of Theorem 3.1, which give a structure of \(\text{Aut}_k(k[x,y])\) (see Nagata [19]). The Nagata's counter-example is the following: \(n=3,\)

\[
f = x - 2y(zx + y^2) - z(zx + y^2)^2,\]
\[
g = y + z(zx + y^2),\]
\[
h = z.
\]

In this counter-example, \(S_f\) is contained in the plane \(3x+y-z=3,\)
and \(S_g\) is contained in the parallel plane \(3x+y-z=1.\)

(4.4) An important role (in the case \(n=2\)) plays the following

**Lemma 4.4.1([2],[20],[3]).** Let \((p,q)\) be a direction and let \(f\) and \(g\) be \((p,q)\)-forms of positive degrees. If \([f,g] = 0\) then there exists a \((p,q)\)-form \(h\) such that \(f \sim h^s\) and \(g \sim h^t\), for some positive \(s\) and \(t\).

We may prove the following generalization of this fact:

**Proposition 4.4.2([24]).** Let \(f,g \in k[x,y] \smallsetminus k\). If \([f,g] = 0\) then there exists a polynomial \(h \in k[x,y]\) such that \(f = u(h)\) and \(g = v(h)\), for some polynomials \(u(t), v(t) \in k[t]\).
References


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