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PRIME IDEAL STRUCTURE IN ADDITIVE CONSERVATIVE SYSTEMS

In this paper we prove that the set of prime ideals in an additive conservative system is a spectral space and that every morphism of additive conservative systems induces a spectral map.

Throughout this paper all rings are commutative with identity and all ring homomorphisms preserve the identity.

An additive conservative system ([4], [7]) is a pair \((R, M)\), where \(R\) is a ring and \(M\) is a set of ideals of \(R\) satisfying the following conditions:

A1. The intersection of any set of elements of \(M\) is an element of \(M\).

A2. The union of any nonempty set, totally ordered by inclusion, of elements of \(M\) is an element of \(M\).

A3. The sum of any two ideals of \(M\) is an element of \(M\).

Let \((R, M)\) be an additive system. The elements of \(M\) are called \(M\)-ideals. By A1 applied to the empty set of elements of \(M\), \(R\) is an \(M\)-ideal. If \(E\) is a subset of \(R\), then by \([E]\) we denote the smallest \(M\)-ideal containing \(E\). It is clear, that if \(A\) and \(B\) are ideals in \(R\) then \([A+B] = [A] + [B]\).

If \((R, M)\) and \((S, N)\) are additive conservative systems then a ring homomorphism \(f: R \rightarrow S\) will be called a morphism of additive conservative systems \((R, M)\) and \((S, N)\) iff the inverse images of \(N\)-ideals are \(M\)-ideals.

Lemma 1. Let \((R, M)\) be an additive conservative system and let \(E\) be a subset of \(R\). If \(x \in [E]\), then there exists a finite subset \(T \subseteq E\) such that \(x \in [T]\).
Proof. See [7] or [4].

Lemma 2. Let \((R, \mathcal{M})\) be an additive conservative system and let \(A\) be an ideal of \(R\). Then \(A\) is an \(\mathcal{M}\)-ideal if and only if for every \(x \in R\) the condition \(x \in A\) implies \([x] \subset A\).

Proof. The necessity of the condition is obvious. To prove sufficiency we show that \([A] \subset A\). Let \(x \in [A]\), then, by Lemma 1, we have \(x \in [a_1, \ldots, a_n]\), for some \(a_1, \ldots, a_n \in A\). Since \([a_1, \ldots, a_n] = [a_1] + \ldots + [a_n]\) and \([a_i] \subset A\), for \(i = 1, \ldots, n\), thus \(x \in A\).

For any ring \(R\), \(\text{Spec}(R)\) will denote the set of prime ideals in \(R\) with the Zariski topology ([1] p.125). If \(E\) is a subset of \(R\), then by \(V(E)\) we denote the set of prime ideals of \(R\) containing \(E\) and by \(D(E)\) we denote the set \(\text{Spec}(R) \setminus V(E)\).

Let \((R, \mathcal{M})\) be an additive conservative system. The set of prime \(\mathcal{M}\)-ideals in \(R\) will be denoted by \(\text{Spec}(R, \mathcal{M})\) and will be called the prime spectrum of \((R, \mathcal{M})\). As a topological space, it has the subspace topology from \(\text{Spec}(R)\), so that the closed sets in \(\text{Spec}(R, \mathcal{M})\) are of the form \(V(E) \cap \text{Spec}(R, \mathcal{M})\), where \(E\) is a subset of \(R\).

Lemma 3. If \((R, \mathcal{M})\) is an additive conservative system, then

\[
\text{Spec}(R, \mathcal{M}) = \bigcap \{D(r) \cup V([r]), \ r \in R\}.
\]

Proof. Let \(P \in \text{Spec}(R, \mathcal{M}), \ r \in R\). If \(r \notin P\) then \(P \notin D(r)\). If \(r \in P\), then \([r] \subset P\). Therefore, for every \(r \in R\), we have \(\text{Spec}(R, \mathcal{M}) \subset D(r) \cup V([r])\). Conversely, assume that \(P \in D(r) \cup V([r])\), for any \(r \in R\). We prove that \(P \in \mathcal{M}\). Let \(r \in P\). Then \(P \notin D(r)\) and thus \(P \in V([r])\). Therefore \([r] \subset P\) and, by Lemma 2, we have \(P \in \mathcal{M}\).

Let \(f: (R, \mathcal{M}) \rightarrow (S, \mathcal{N})\) be a morphism of additive conservative systems. Then \(f\) induces a continuous mapping \(a_f: \text{Spec}(S) \rightarrow \text{Spec}(R)\) via \(a_f(P) = f^{-1}(P)\), where \(P \in \text{Spec}(S)\). Since \(a_f(\text{Spec}(S, \mathcal{N})) \subset \text{Spec}(R, \mathcal{M})\), we have a continuous mapping

\[
b_f : \text{Spec}(S, \mathcal{N}) \rightarrow \text{Spec}(R, \mathcal{M}),
\]

given by \(b_f(P) = a_f(P)\), for any \(P \in \text{Spec}(S, \mathcal{N})\).
Recall from ([5]) that a topological space $X$ is called spectral if it is $T_0$ and quasi-compact, the quasi-compact open subsets are closed under finite intersection and form an open basis, and every nonempty irreducible closed subset has a generic point. Moreover, a continuous map of spectral spaces is called spectral if the inverse image of quasi-compact open sets are quasi-compact. It is well known, that for every ring $R$, $\text{Spec}(R)$ is a spectral space, and, for every ring homomorphism $f$, $a_f$ is a spectral map.

**Lemma 4.** Let $Y$ be a subspace of a spectral space $X$. Every quasi-compact open set in $Y$ is of the form $U \cap Y$, where $U$ is a quasi-compact open set in $X$.

**Proof.** Let $V$ be a open set in $X$ such that $V \cap Y$ is a quasi-compact set in $Y$, and let $V = \bigcup U_i$, where $U_i$ are quasi-compact open sets in $X$. Since $V \cap Y$ is a quasi-compact set, we have $V \cap Y = (U_1 \cup \ldots \cup U_n) \cap Y$. The set $U_1 \cup \ldots \cup U_n$ is quasi-compact open in $X$, because, for every topological space, the quasi-compact open subsets are closed under finite union.

**Theorem.** If $(R, M)$ is an additive conservative system then $\text{Spec}(R, M)$ is a spectral space. If $f : (R, M) \to (S, N)$ is a morphism of additive conservative systems then $b_f$ is a spectral map.

**Proof.** Lemma 3 implies that $\text{Spec}(R, M)$ is a closed set in the patch topology on $\text{Spec}(R)$ (see [5] p.45). It follows that $\text{Spec}(R, M)$ is a spectral subobject of $\text{Spec}(R)$ (see [5] p.45), and therefore $\text{Spec}(R, M)$ is a spectral space.

Let $f : (R, M) \to (S, N)$ be a morphism of additive conservative systems and let $G$ be a quasi-compact open set in $\text{Spec}(S, N)$. By Lemma 4, there exists a quasi-compact open set $U$ in $\text{Spec}(S)$, such that $G = U \cap \text{Spec}(S, N)$. Since $b_f^{-1}(G) = (a_f^{-1})_1(U) \cap \text{Spec}(R, M)$ and $a_f^{-1}(U)$ is a quasi-compact open set in $\text{Spec}(R)$, and since $\text{Spec}(R, M)$ is a spectral subobject of $\text{Spec}(R)$, then $b_f^{-1}(G)$ is a quasi-compact open set in $\text{Spec}(R, M)$.
Corollary. Let \( f: (R, M) \rightarrow (S, N) \) be a morphism of additive conservative systems. Then there exist rings \( R' \) and \( S' \) and a ring homomorphism \( f': R' \rightarrow S' \) such that \( \text{Spec}(R, M) \) is homeomorphic to \( \text{Spec}(R') \), \( \text{Spec}(S, N) \) is homeomorphic to \( \text{Spec}(S') \), and \( b_f \) is equivalent to \( a_f \) i.e. \( b_f = ga_f p \), where \( p: \text{Spec}(S, N) \rightarrow \text{Spec}(S') \), \( q: \text{Spec}(R') \rightarrow \text{Spec}(R, M) \) are homeomorphisms.

Proof. This follows from Theorem and from [5] Th.6. p.51.

Examples: 1. Let \( R \) be a ring, \( F \) a set of endomorphisms of the additive group of \( R \), and let \( M \) be the set of ideals \( A \) in \( R \) such that \( f(A) \subseteq A \) for any \( f \in F \). Then \( (R, M) \) is an additive conservative system.

2. Let \( D \) be a set of derivations of \( R \) and let \( M \) be the set of differential ideals of \( R \) ([7]). Then \( (R, M) \) is an additive conservative system. Every differential homomorphism is a morphism of additive conservative systems. The theorem, for special differential rings, was proved by W.F. Keigher in [6], and, for arbitrary differential rings, was proved by G. Carra in [3].

3. Let \( D \) be a set of higher derivations of \( R \) and let \( M \) be a set of \( D \)-ideals ([2], [4]). Then \( (R, M) \) is an additive conservative system and every \( D \)-homomorphism is a morphism of additive conservative systems.

4. Let \( R \) be a graded ring and let \( M \) be the set of homogeneous ideals of \( R \). Then \( (R, M) \) is an additive conservative system and every homomorphism of graded rings is a morphism of additive conservative systems.

Remarks 1. Lemma 1 is also true without the assumption that \( M \) satisfies A3 (see [7], [4]).

2. If \( M \) is a set of ideals of \( R \) satisfying only A1 and A2 then Lemma 2 is not necessarily true e.g. Let \( R = \mathbb{Z}_2[X, Y]/A \) where \( A \) is an ideal in \( \mathbb{Z}_2[X, Y] \) generated by \( X^2, XY, Y^2 \). Let \( x = X + A \), \( y = Y + A \) and \( M = \{0, (x), (y), (x+y), R\} \). Then the ideal \( (x, y) \) satisfies the condition of Lemma 2, and it is not \( M \)-ideal.
REFERENCES


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