HIGHER $R$-DERIVATIONS OF SPECIAL SUBRINGS
OF MATRIX RINGS

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1. Introduction.

Let $R$ be a ring with identity and $P$ be a special subring of $M_n(R)$ ([7]), i.e. $P$ is of the form

$$P = \{A \in M_n(R) ; A_{ij} = 0 \text{ for } (i, j) \in \rho \},$$

where $\rho$ is a (reflexive and transitive) relation on the set $\{1, 2, \ldots, n\}$, and $M_n(R)$ is the ring of $n \times n$ matrices over $R$.

In this paper we study the group $D^r_\rho(P)$ of all $R$-derivations of order $s$ ([5], [8]—[11]) of $P$. We prove (Theorem 5.3) that every element $d \in D^r_\rho(P)$ has a unique representation of the form $d = d'^{(s)} * d'^{(w)}$, where $d'^{(s)}$ is an inner derivation in $D^r_\rho(P)$ ([8]), and $d'^{(w)}$ is an element of a certain abelian subgroup of $D^r_\rho(P)$ whose simple description is given in Section 3 (by * we denote the multiplication in the group $D^r_\rho(P)$). This theorem plays a basic role in our further considerations.

Moreover, in Section 4, we give some necessary and sufficient conditions for a ring $P$ to have all $R$-derivations (all derivations) of order $s$ of $P$ to be inner.

In Sections 7, 8, 9 we investigate $s'$-integrable $R$-derivations of order $s$ (where $s < s'$) i.e. such $R$-derivations of order $s$ which can be extended to $R$-derivations of order $s'$ (comp. [4]). We show in Example 7.4 that, in general, there are non-integrable $R$-derivations of $P$. We prove (Theorem 9.6) that if the homology group $H_s(\Gamma)$ of the simplicial complex $\Gamma$ of the relation $\rho$ (Section 2) is free abelian, then every usual $R$-derivation is $3$-integrable, and if, in addition, $H_s(\Gamma) = 0$ then every $R$-derivation of order $s$ is $s'$-integrable for any $s < s'$ (Theorem 8.6).

At the end of this paper, we formulate three open problems.

2. Preliminaries.

Throughout this paper $R$ is a ring with identity, $n$ is a fixed natural number and $\rho$ is a reflexive and transitive relation on the set $I_n = \{1, 2, \ldots, n\}$.

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We denote by $M_n(R)$ the ring of $n \times n$ matrices over $R$ and by $Z(R)$ the center of $R$.

Moreover, we use the following conventions:

- $S$ = a segment of $N = \{0, 1, \ldots\}$, that is, $S = N$ or $S = \{0, 1, \ldots, k\}$ for some integer $k \geq 0$.
- $s = \sup(S) \leq \infty$.
- $A_{ij} = ij$-coefficient of a matrix $A$.
- $E^r$ = the element of the standard basis of $M_n(R)$.
- $F = $ the diagonal matrix whose all coefficients on the diagonal are equal to $r \in R$.
- $M_n(R)_p$ = the set $\{A \in M_n(R) ; A_{ij} = 0 \text{ for } (i, j) \in p\}$.

It is clear, that $M_n(R)_p$ is a subring of $M_n(R)$. (Conversely, if $\sigma$ is a reflexive relation on $I_n$ and $M_n(R)_p$ is a subring of $M_n(R)$, then $\sigma$ is transitive). We say that the subring $P = M_n(R)_p$ of $M_n(R)$ is special with the relation $p$.

Let $P$ be an arbitrary ring with identity. A sequence $d = (d_m)_{m \in S}$ of mappings $d_m : P \to P$ is called a derivation of order $s$ of $P$ (see [5], [8], [9], [10], [11]) if the following properties are satisfied:

1. $d_m(a + b) = d_m(a) + d_m(b)$,
2. $d_m(ab) = \sum_{i+j=m} d_i(a)d_j(b)$,
3. $d_0(a) = a$,

for all $m \in S$ and $a, b \in P$.

The set $D_s(P)$ of all derivations of order $s$ of $P$ is a group under the multiplication $\ast$ defined by the formula

$$(d \ast d')_m = \sum_{i+j=m} d_i \ast d'_j,$$

where $d, d' \in D_s(P)$ and $m \in S$ ([9], [10], [4]).

If $a \in P$ and $k \in S \setminus \{0\}$ then by $[a, k]$ we denote the element of $D_s(P)$ defined by

$$[a, k]_m(x) = \begin{cases} x, & \text{if } m = 0, \\ 0, & \text{if } k \nmid m, \\ a^s x - a^{-1} x a, & \text{if } m = kr > 0, \end{cases}$$

for $m \in S$, $x \in P$ ([8]).

If $a = (a_m)_{m \in S \setminus \{0\}}$ is a sequence of elements of $P$ then by $\Delta(a)$ we denote the inner derivation of order $s$ of $P$ with respect to $a$ ([8]), i.e., $\Delta(a)$ is a derivation of order $s$ of $P$ such that
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$$\Delta(a) = (a_1, 1) \cdots (a_m, m)$$
for all $m \in S$. The set of inner derivations of order $s$ of $P$, denoted by $ID_s(P)$, is a normal subgroup of $D_r(P)$ ([8] Corollary 3.3).

Recall that the usual derivation of $P$ is an additive mapping $\delta : P \to P$ such that $\delta(ab) = \delta(a) b + a \delta(b)$, for all $a, b \in P$.

The set of usual derivations of $P$ corresponds bijectively to the set $D_r(P)$, namely if $d \in D_r(P)$ then $d_i$ is an usual derivation of $P$.

We now assume that $P$ is a special subring of $M_n(R)$ with the relation $\rho$.

Observe that we can extend every derivation of order $s$ of $R$ to a derivation of order $s$ of $P$.

Indeed, if $\delta \in D_s(R)$ then the sequence $d = (d_m)_{m \in S}$ of mappings $d_m : P \to P$ defined by $d_m(A)_{ij} = \delta_m(A_{ij})$ (for $A \in P, m \in S$) is a derivation of order $s$ of $P$ such that $d_m(r) = \delta_m(r)$ for any $r \in R, m \in S$.

Look also on a generalization of the above fact.

**Example 2.1.** Let $\bar{p}$ be the smallest equivalence relation on $I_\rho$ containing $\rho$, $T$ a fixed set of representatives of equivalence classes of $\bar{p}$, and $v : I_\rho \to T$ the mapping defined by:

$$v(p) = t \text{ iff } p \bar{p} t.$$  

Moreover, let $(d^{(i)})_{i \in I}$ be a sequence of elements of $D_s(R)$. Consider the sequence $\Theta(d) = (d_m)_{m \in S}$ of mappings from $P$ to $P$ defined as follows

$$d_m(A)_{ij} = d^{(i)}_m(A_{ij})$$

for all $m \in S, A \in P$.

It is easy to verify that $\Theta(d)$ belongs to $D_r(P)$.

If a derivation $d \in D_r(P)$ satisfies following equivalent two conditions:

1. $d_m(rA) = rd_m(A)$ for all $m \in S, r \in R, A \in P$,
2. $d_m(r) = 0$ for all $m \in S \setminus \{0\}, r \in R$,

then $d$ is called $R$-derivation of order $s$ of $P$, and the set of all such derivations is denoted by $D_r^R(P)$.

We define similarly an usual $R$-derivation, an inner $R$-derivation and the set $ID_r^R(P)$. It is clear, that $D_r^R(P)$ is a subgroup of $D_r(P)$, and (by [8] Corollary 3.3) $ID_r^R(P)$ is a normal subgroup of $D_r^R(P)$. An inner derivation $\Delta(\Delta)$, where $\Delta = (A^{(m)})_{m \in S \cup \{0\}}$ is a sequence of matrices of $P$, belongs to $ID_r^R(P)$ iff $A^{(m)} \in M_n(Z(R))$ for any $m$. 


LEMMA 2.2. If \( d \in D^p(P) \) then \( d_m(E^p)_{ij} \in Z(R) \) for any \( m \in S \) and all \( i, j, p, q \in I_n \) such that \( pq \).

PROOF. Let \( r \in R. \) Since \( rE^p = E^p_r = 0 \) then

\[
0 = d_m(rE^p) - d_m(E^p)_{ij} = \sum_{u \in m} (d_u(r)E^p - d_u(E^p)r)_{ij} = (d_m(E^p) - d_m(E^p)r)_{ij} = (rE^p)_{ij} - d_m(E^p)_{ij}r.
\]

Usual derivations and usual \( R \)-derivations of \( P \) are investigated in [6], [1], [2], [7]. In this paper (Section 5) we give a description of the group \( D^p_r(P) \).

Let \( s < \infty \), and \( S' \) be a segment of \( N \) such that \( S \equiv S' \). We say (comp. [4]) that an \( R \)-derivation \( d \in D^p_r(P) \) is \( s' \)-integrable (where \( s' = \sup(S') \leq \infty \)) if there exists an \( R \)-derivation \( d' \in D^p_r(P) \) such that \( d_m = d_{m'} \) for all \( m \in S \). We will study such derivations in Sections 7, 8, 9.

Now we will define the graph \( \Gamma \) of the relation \( \rho \). Let \( \sim \) be the equivalence relation on \( I_n \) defined by:

\[
x \sim y \text{ iff } xpy \text{ and } ypx.
\]

Denote by \([x] \) the equivalence class of \( x \in I_n \) with respect to \( \sim \), and let \( I_n' \) be the set of all equivalence classes. We define a relation \( \rho' \) of partial order on \( I_n \) as follows:

\[
[x] \rho' [y] \text{ iff } xpy.
\]

We will denote the pair \((I_n, \rho')\) by \( \Gamma, \) (or \( \Gamma(\rho) \)) and call it the graph of \( \rho \). Elements of \( I_n \) we call vertices of \( \Gamma \) and pairs \((a, b)\), where \( a \rho' b \) and \( a \neq b \), arrows of \( \Gamma \).

Let us imbed the set of the vertices of \( \Gamma \) in an Euclidean space of a sufficiently high dimension so that the vertices will be linearly independent.

If \( a_0, a_1, \ldots, a_k \) are elements of \( I_n \) such that \( a_i \rho' a_{i+1} \) for \( i = 0, 1, \ldots, k-1 \), then by \((a_0, a_1, \ldots, a_k)\) we denote the \( k \)-dimensional simplex with vertices \( a_0, \ldots, a_k \) ([3]). The union of all 0, 1, 2 or 3-dimensional such simplicies we will denote also by \( \Gamma \). Therefore, \( \Gamma \) is a simplicial complex of dimension \( \leq 3 \).

Let \( C_k(\Gamma) \), for \( k = 0, 1, 2, 3 \), be the free abelian group whose free generators are \( k \)-dimensional simplicies of the complex \( \Gamma \). We have the following standard complex of abelian groups:
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\[ 0 \rightarrow C_0(\Gamma) \xrightarrow{\partial_0} C_1(\Gamma) \xrightarrow{\partial_1} C_1(\Gamma) \xrightarrow{\partial_1} C_0(\Gamma) \rightarrow 0, \]

where
\[
\partial_0(a, b) = (b) - (a), \\
\partial_1(a, b, c) = (b, c) - (a, b) + (a, b), \\
\partial_2(a, b, c, d) = (b, c, d) - (a, c, d) + (a, b, d) - (a, b, c).
\]

Then $H_0(\Gamma) = \text{Ker} \partial_0 / \text{Im} \partial_0$, $H_1(\Gamma) = \text{Ker} \partial_1 / \text{Im} \partial_1$ and (by the Künneth formulas)
\[ H^1(\Gamma, G) = \text{Hom}(H_1(\Gamma), G) \]

for an arbitrary abelian group $G$ (see [3]).

In the sequel $P$ denotes a special subring of $M_n(R)$ with the relation $\rho$.

3. Transitive mappings.

Recall from [7] that a mapping $\varphi: \rho \rightarrow Z(R)$ is called transitive if $\varphi(p, r) = \varphi(p, q) + \varphi(q, r)$ for $ppq$, $qrp$. In this paper such mappings will be called usual transitive mappings from $\rho$ to $R$.

**Definition 3.1.** A sequence $f = (f_m)_{m \in S}$ of mappings $f_m: \rho \rightarrow Z(R)$ is called a transitive mapping of order $s$ from $\rho$ to $R$ if the following properties are satisfied:

(a) $f_s(p, q) = 1$ for all $ppq$,

(b) $f_m(p, r) = \sum_{i+j=m} f_i(p, q)f_j(q, r)$ for all $m \in S$ and $ppqpr$.

We denote by $TM_s(\rho, R)$ the set of transitive mappings of order $s$ from $\rho$ to $R$.

By the above definition it follows that if $f \in TM_s(\rho, R)$ then
\[ f_s(p, r) - f_s(p, q) - f_s(q, r) = 0, \]

i.e. $f_s$ is an usual transitive mapping from $\rho$ to $R$, and
\[ f_s(p, r) - f_s(p, q) - f_s(q, r) = f_s(p, q)f_s(q, r), \]
\[ f_s(p, r) - f_s(p, q) - f_s(q, r) = f_s(p, q)f_s(q, r) + f_s(p, q)f_s(q, r) \]

for all $ppqpr$.

It is easy to prove

**Lemma 3.2.** (1) $f_m(p, p) = 0$, for all $p \in I_m, m \in S \setminus \{0\}$.

(2) If $ppq$ and $qrp$, and $f_3(p, q) = \ldots = f_m(p, q) = 0$ for some $m \geq 2$, then
Example 3.3. If $Q \subseteq R$ and $\varphi: \rho \rightarrow Z(R)$ is an usual transitive mapping then the sequence $(f_m)_{m \in S}$ where $f_m(p, q) = (m!)^{-1} \varphi(p, q)^m$, is a transitive mapping of order $s$ from $\rho$ to $R$.

Example 3.4. Let

\[
\rho = \begin{array}{c}
1 \\
\downarrow \\
3 \\
\downarrow \\
2
\end{array}
\]

Put $f_m(1, 2) = f_m(1, 3) = 1$ and $f_m(2, 3) = 0$ for all $m \in S \setminus \{0\}$. Then $f = (f_m)_{m \in S}$ belongs to $TM_s(\rho, R)$.

Example 3.5. Let

\[
\rho = \begin{array}{c}
1 \\
\downarrow \\
4 \\
\downarrow \\
2
\end{array}
\]

If $f_m$, for any $m \in S \setminus \{0\}$, is an arbitrary mapping from $\rho$ to $Z(R)$ then $(f_m)_{m \in S}$ is a transitive mapping of order $s$ from $\rho$ to $R$.

Let $f, g \in TM_s(\rho, R)$. Denote by $f \ast g$ the sequence $(h_m)_{m \in S}$ of mappings from $\rho$ to $Z(R)$ defined by

\[
h_m(p, q) = \sum_{e \in m} f(e, p, q)g(e, p, q)
\]

for all $m \in S$ and $ppq$.

Then $f \ast g$ belongs to $TM_s(\rho, R)$ and it is easy to check that the set $TM_s(\rho, R)$, under the multiplication $\ast$, is an abelian group.

For every $f \in TM_s(\rho, R)$ we will denote by $\Delta'$ the sequence $(\Delta'_m)_{m \in S}$ of mappings $\Delta'_m : P \rightarrow P$ defined by the following formula

\[
\Delta'_m(A)_{pq} = f_m(p, q)A_{pq},
\]

for all $A \in P$ and $ppq$.

Then we have

Lemma 3.6. The sequence $\Delta'$ is an $R$-derivation of order $s$ of $P$.

Proof. Every $\Delta'_m$ is obviously an $R$-additive mapping. Let $A, B \in P$ and
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Then

\[
\left( \sum_{k=0}^{s} \Delta^k(A)\Delta^k(B) \right)_{pq} = \sum_{k=1}^{s} \sum_{i=0}^{s} \Delta^k(A)_{pi} \Delta^k(B)_{iq} \\
= \sum_{i=1}^{s} \sum_{q=1}^{s} f_m(p, i) f_m(i, q) A_{pi} B_{iq} \\
= \sum_{i=1}^{s} f_m(p, q) A_{pi} B_{iq} \\
= f_m(p, q)(AB)_{pq} \\
= \Delta^m(AB)_{pq}.
\]

Therefore

\[
\Delta^m(AB) = \sum_{i=1}^{s} \Delta^i(A) \Delta^i(B),
\]

for all $m \in S$ and $A, B \in P$.

**Proposition 3.7.** The mapping $f \mapsto \Delta f$ is a group monomorphism from $TM_s(\rho, R)$ to $D_s^f(P)$.

**Proof.** The condition $\Delta f = \Delta f \ast \Delta f$ follows from definition of multiplications. Suppose now that $\Delta f = \Delta f$ for some $f, g \in TM_s(\rho, R)$. Then, for $ppq$ and $m \in S$, we have

\[
f_m(p, q) = \Delta^m(E^{pq})_{pq} = \Delta^m(E^{pq})_{pq} = g_m(p, q),
\]

i.e. $f = g$.

4. Inner derivations.

Recall from [7] that if $f$ is an usual transitive mapping from $\rho$ to $R$ then $f$ is called trivial iff there exists a mapping $\sigma : I_s \rightarrow Z(R)$ such that $f(p, q) = \sigma(p) - \sigma(q)$ for all $ppq$. We say that the relation $\rho$ is regular over $R$ iff every usual transitive mapping from $\rho$ to $R$ is trivial.

Combining [8] Theorem 4.2 with results of the paper [7] we obtain the following two theorems

**Theorem 4.1.** Let $P$ be a special subring of $M_n(R)$ with the relation $\rho$. The following conditions are equivalent:

1. Every $R$-derivation of order $s$ of $P$ is inner,
2. Every usual $R$-derivation of $P$ is inner,
3. The relation $\rho$ is regular over $Z(R)$,
4. The relation $\rho'$ is regular over $Z(R)$,
THEOREM 4.2. Let $P$ be a special subring of $M_*(R)$ with the relation $\rho$. Denote by $w$, $w_*$, $u$, $u'$ the following sentences:

$w$ = "Every usual derivation of $R$ is inner",
$w_*$ = "Every derivation of order $s$ of $R$ is inner",
$u$ = "The relation $\rho$ is regular over $Z(R)$",
$u'$ = "The relation $\rho'$ is regular over $Z(R)$".

Then the following conditions are equivalent:

1. Every derivation of order $s$ of $P$ is inner,
2. Every usual derivation of $P$ is inner,
3. $w$ and $u$,
4. $w_*$ and $u$,
5. $w$ and $u'$,
6. $w_*$ and $u'$,
7. $w$ and $H^1(\Gamma(\rho), Z(R)) = 0$,
8. $w_*$ and $H^1(\Gamma(\rho), Z(R)) = 0$.

EXAMPLE 4.3. If $P = M_*(R)_\rho$ where

a) $n \leq 3$, or
b) the graph $\Gamma(\rho)$ is a tree, or
c) the graph $\Gamma(\rho)$ is a cone (i.e. there exists $b \in I_s$ such that $bpa$ or $apb$ for any $a \in I_s$) in particular $P = M_*(R)$ or $P$ is the ring of triangular $n \times n$ matrices over $R$, or
d) the graph $\Gamma(\rho)$ is of the form

![Diagram]

then every $R$-derivation (or every derivation, if every usual derivation of $R$ is inner) of order $s$ of $P$ is inner (see [7]).

5. The group $D^\rho_1(P)$.

In this section we give a description of the group $D^\rho_1(P)$. ....
We start from the following two lemmas.

**Lemma 5.1.** Let $d \in D^k(P)$, $m \in S \setminus \{0\}$. Assume that $d_k(E^m)_{pq} = 0$ for $k = 1, 2, \ldots, m$ and all $p \neq q$. Then

(i) $d_k(E^{pp})_{pq} = 0$ for $k = 1, 2, \ldots, m$ and any $p \in I_m$,

and

(ii) $d_k(E^{ip})_{pq} = 0$ for $k = 1, 2, \ldots, m$ and all $ipj$, $ppq$ such that $(p, q) \neq (i, j)$.

**Proof.** (by induction with respect to $m$). If $m = 1$ then this lemma follows from [7] Lemma 3.1. Let $m > 1$ and suppose that the conditions (i) and (ii) hold for any $k < m$. We show that then

1. $d_m(E^{ij})_{pq} = 0$ for $i \neq p$, $j \neq q$,
2. $d_m(E^{pp})_{pp} = 0$ for any $p \in I_m$,
3. $d_m(E^{pp})_{pq} = 0$ for $p \neq q$,
4. $d_m(E^{pp})_{iq} = 0$ for $p \neq i$,
5. $d_m(E^{pp})_{pq} = 0$ for $q \neq j$.

For example we verify (1) and (2). The proofs of the conditions (3)-(5) are similar.

1. Let $i \neq p$, $j \neq q$, and $ppq$, $ipj$. Then

$$d_m(E^{ij})_{pq} = \sum_{s, r \in m} (d_s(E^{ij}) d_r(E^{ij})_{pq})$$

$$= \sum_{s, r \in m} \sum_{r} d_s(E^{ij}) d_r(E^{ij})_{rq}.$$

Hence, by induction, we have

$$d_m(E^{ij})_{pq} = \sum_{r} (d_s(E^{ij})_{pr} d_m(E^{ij})_{rq} + d_m(E^{ij})_{pr} d_s(E^{ij})_{rq})$$

$$= \sum_{r} (0 d_m(E^{ij})_{rq} + d_m(E^{ij})_{pr} 0) = 0.$$  

2. Let $p \in I_m$. Then

$$d_m(E^{pp})_{pp} = d(E^{pp} E^{pp})_{pp}$$

$$= \sum_{s, r \in m} (d_s(E^{pp}) d_r(E^{pp})_{pp})$$

$$= \sum_{s, r \in m} \sum_{r} d_s(E^{pp}) d_r(E^{pp})_{rp}$$

$$= \sum_{r} (d_s(E^{pp})_{pr} d_m(E^{pp})_{rp} + d_m(E^{pp})_{pr} d_s(E^{pp})_{rp})$$

$$= d_m(E^{pp})_{pp} + d_m(E^{pp})_{pp}.$$
Hence \( d_m(E^{\rho})_{pp} = 0 \).

**Lemma 5.2.** Let \( d \in D_R^2(P) \). Assume that \( d_m(E^{\rho})_{pq} = 0 \) for all \( m \in S \setminus \{0\} \) and all \( ppq \). Then the sequence \( f = (f_m)_{m \in S} \) of mappings from \( \rho \) to \( R \) defined by \( f_m(p, q) = d_m(E^{\rho})_{pq} \) for \( ppq \) is a transitive mapping of order \( s \) from \( \rho \) to \( R \).

**Proof.** Lemma 2.2 implies that \( f_m(p, q) \in Z(R) \) for all \( ppq \). Now let \( ppqpr \), \( m \in S \). By Lemma 5.1 we have

\[
\begin{align*}
  f_m(p, r) &= d_m(E^{\rho})_{pr} = d_m(E^{\rho}E^{\rho})_{pr} \\
  &= \left( \sum_{i \neq j} d_i(E^{\rho})d_j(E^{\rho}) \right)_{pr} \\
  &= \sum_{i \neq j} d_i(E^{\rho})_{pi}d_j(E^{\rho})_{ir} \\
  &= \sum_{i \neq j} d_i(E^{\rho})_{pq}d_j(E^{\rho})_{qr} \\
  &= \sum_{i \neq j} f_i(p, q)f_j(q, r),
\end{align*}
\]

i.e. \( f \in TM_s(\rho, R) \).

Now we can prove the following

**Theorem 5.3.** Let \( P \) be a special subring of \( M_n(R) \) with the relation \( \rho \). Every \( R \)-derivation \( d \) of order \( s \) of \( P \) has a unique representation:

(0) \( d = \Delta(\Delta^{*})^{\Delta} \),

where

1. \( \Delta = (A^{(m)})_{m \in S} \) is a sequence of matrices \( A^{(m)} \in \mathcal{P} \setminus M_n(Z(R)) \) such that \( A^{(m)}_{pq} = 0 \) for \( i = 1, 2, \ldots, n \),

2. \( f \) is a transitive mapping of order \( s \) from \( \rho \) to \( R \).

**Proof.** (1). Let \( d \in D_R^2(P) \). We define matrices \( A^{(0)}, A^{(1)}, \ldots \) inductively as follows:

\[
A^{(m)}_{pq} = d_m(E^{\rho})_{pq},
\]

and

\[
A^{(m+1)}_{pq} = d^{(m+1)}_{m+1}(E^{\rho})_{pq} \quad \text{for } 1 \leq m < s,
\]

where

\[
d^{(m)} = ([A^{(0)}, 1]^{*} \cdot \ldots \cdot [A^{(m)}, m])^{-1} d.
\]

Put \( \delta = (\delta_m)_{m \in S} \), where \( \delta_i = id_P \) and \( \delta_m = d^{(m)}_{m+1} \) for \( m \geq 1 \). Let \( \Delta = (A^{(m)})_{m \in S} \) and let \( f = (f_m)_{m \in S} \) be the sequence of mappings from \( \rho \) to \( R \) defined by

\[
f_m(p, q) = \delta_m(E^{\rho})_{pq}
\]

for all \( m \in S \), \( ppq \).

We show that \( \Delta \) and \( f \) satisfy conditions (0), (1) and (2) of this theorem. Observe first that
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a) $d^{(m)}_k = d^{(k)}_k$ for any $k \leq m$,

b) $\delta$ is an $R$-derivation of order $s$ of $P$,

c) $d = \Delta(A) \ast \delta$.

Now we prove that

d) $\delta_{m}(E^{(q)})_{pq} = 0$ for $m \in S \setminus \{0\}$ and $p \neq q$.

In fact, for $m = 1$ we have

$$\delta_1(E^{(q)})_{pq} = d^{(1)}_1(E^{(q)})_{pq}$$

$$= (\Delta^{(1)}, 1) \ast d_1(E^{(q)})_{pq}$$

$$= -[A^{(1)}, 1]_1 (E^{(q)})_{pq} + d_1(E^{(q)})_{pq}$$

$$= -(A^{(1)} E^{(q)} - E^{(q)} A^{(1)})_{pq} + A^{(1)}_{pq}$$

$$= -A^{(1)}_{pq} + A^{(1)}_{pq} = 0$$

and, if $m > 1$ then

$$\delta_m(E^{(q)})_{pq} = d^{(m)}_m(E^{(q)})_{pq}$$

$$= (\Delta^{(m)}, m) \ast d^{(m-1)}_m(E^{(q)})_{pq}$$

$$= (\sum_{i \neq m} [A^{(m)}, m] \ast d^{(m-1)}_i(E^{(q)})_{pq}$$

$$= -[A^{(m)}, m]_{mq} (E^{(q)})_{pq} + (\sum_{i \neq m} O d^{(m-1)}_i(E^{(q)})_{pq}$$

$$= -(A^{(m)} E^{(q)} - E^{(q)} A^{(m)})_{pq} + A^{(m)}_{pq}$$

$$= -A^{(m)}_{pq} + A^{(m)}_{pq} = 0.$$
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\[
\sum_{i,j} X_{ij} \delta_m (E^{ij})_{pq} = \sum_{i,j} X_{ij} \delta_m (E^{ij})_{pq} = X_{pq} \delta_m (E^{pq})_{pq} \quad \text{(by d) and Lemma 5.1)}
\]

\[
= X_{pq} \delta_m (E^{pq})_{pq}, \quad \text{i.e.} \quad \delta = \Delta'.
\]

(II) Suppose that

\[
\Delta(A) \ast \Delta' = \Delta(B) \ast \Delta^s,
\]

where \(A, f\) and \(B, g\) satisfy conditions (1) and (2).

Then, for \(p \neq q\),

\[
A^{(1)}_{pq} = (\Delta(A) \ast \Delta')_{pq} = (\Delta(B) \ast \Delta^s)_{pq} = B^{(s)}_{pq}.
\]

So \(A^{(1)} = B^{(1)}\).

Suppose that \(A^{(1)} = B^{(1)}, \ldots, A^{(m)} = B^{(m)}\) for some \(m < s\). Then

\[
\Delta(0, \ldots, 0, A^{(m+1)}, A^{(m+2)}, \ldots) \ast \Delta' = ([A^{(1)}, 1] \ast \ldots \ast [A^{(m)}, m])^{-1} \ast \Delta(A) \ast \Delta'
\]

\[
= ([B^{(1)}, 1] \ast \ldots \ast [B^{(m)}, m])^{-1} \ast \Delta(B) \ast \Delta^s
\]

\[
= \Delta(0, \ldots, 0, B^{(m+1)}, B^{(m+2)}, \ldots) \ast \Delta^s,
\]

hence

\[
A^{(m+1)}_{pq} = (\Delta(0, \ldots, 0, A^{(m+1)}, A^{(m+2)}, \ldots) \ast \Delta')_{m+1} (E^{pq})_{pq}
\]

\[
= (\Delta(0, \ldots, 0, B^{(m+1)}, B^{(m+2)}, \ldots) \ast \Delta^s)_{m+1} (E^{pq})_{pq}
\]

\[
= B^{(m+1)}_{pq} \quad \text{for} \quad p \neq q,
\]

and hence

\[
A^{(m+1)} = B^{(m+1)}.
\]

Therefore, by induction, \(A = B\).

Further we have

\[
\Delta' = (\Delta(A) \ast (\Delta(A) \ast \Delta'))^{-1} \ast (\Delta(A) \ast \Delta')
\]

\[
= (\Delta(B) \ast (\Delta(B) \ast \Delta^s))^{-1} \ast (\Delta(B) \ast \Delta^s) = \Delta^s
\]

hence, by Proposition 3.7, we obtain that \(f = g\). This completes the proof.

6. Corollaries to Theorem 5.3.

Let \(S'\) be a segment of \(N\) such that \(S \subseteq S'\) and let \(s' = \sup(S') \leq \infty\). We say that a transitive mapping \(f \in TM_1(\rho, R)\) is \(s'\)-integrable if there exists a transitive mapping \(f' \in TM_1(\rho, R)\) such that \(f'_m = f_m\) for all \(m \in S\).

As an immediate consequence of Theorem 5.3 we have
COROLLARY 6.1. The following conditions are equivalent:

(1) Every $R$-derivation of order $s$ of $P$ is $s'$-integrable,
(2) Every transitive mapping of order $s$ from $P$ to $R$ is $s'$-integrable.

If $U$ is an ideal in $P$, then $U=\{U_{ij}\}$, where $U_{ij}$ are ideals of $R$ for any $i, j$ (see [7] Lemma 2.1). Therefore from Theorem 5.3 we get

COROLLARY 6.2. If $d\in D_1^s(P)$ and $U$ is an ideal in $P$ then $d_m(U)\subseteq U$ for all $m\in S$.

Observe also that from Theorem 5.3 follows

COROLLARY 6.3. If $d\in D_1^s(P)$ and $C$ is the center of $P$, then $d_m(C)=0$ for all $m\in S\setminus\{0\}$.

Denote by $I(P)$ the set of all matrices $A\in P$ such that $A_{\rho,\rho}=0$ for all $\rho\in I_n$. It is easy to verify the following two lemmas.

LEMMA 6.4. The following conditions are equivalent:

(1) $I(P)$ is an ideal in $P$,
(2) $I(P)$ is a left-ideal in $P$,
(3) $I(P)$ is a right-ideal in $P$,
(4) $AB\in I(P)$ for all $A, B\in I(P)$,
(5) $AB-BA\in I(P)$ for all $A, B\in I(P)$,
(6) $AB-BA\in I(P)$ for all $A\in I(P), B\in P$,
(7) The relation $\rho$ is a partial order.

LEMMA 6.5 The following two conditions are equivalent:

(1) $AB=0$ for all $A, B\in I(P)$,
(2) There do not exist three different elements $a, b, c\in I_n$ such that $apbpc$.

Combining Lemma 6.4 with Theorem 5.3 and Lemma 3.2(1) we obtain

COROLLARY 6.6. Let $d\in D_1^s(P)$. If the relation $\rho$ is a partial order then $d_m(P)\subseteq I(P)$ for all $m\in S\setminus\{0\}$.

We end this section with

COROLLARY 6.7. Assume that there do not exist three different elements $a, b, c\in I_n$ such that $apbpc$. Let $d=(d_m)_{m\in S}$ be a sequence of mappings from $P$ to
$P$ such that $d_{n} = id_{P}$.

Then $d$ is an $R$-derivation of order $s$ of $P$ if and only if every mapping $d_{m}$ (for $m \in S \setminus \{0\}$) is a usual $R$-derivation of $P$.

**Proof.** If $d \in D_{r}^{s}(P)$ then, by Corollary 6.6 and Lemma 6.5, $d_{i}(A)d_{j}(B) = 0$ for $i > 0$ or $j > 0$ and any $A, B \in P$. Therefore $d_{m}(AB) = Ad_{m}(B) + d_{m}(A)B$, for any $m \in S \setminus \{0\}$ and $A, B \in P$. Conversely, if any $d_{m}$ is an usual $R$-derivation of $P$ then, by Corollary 6.6, $d_{m}(A) \leq l(P)$ for any $A \in P$, hence, by Lemma 6.5, $d_{i}(A)d_{j}(B) = 0$ for any $A, B \in P$ and $i > 0$ or $j > 0$. Therefore

$$d_{m}(AB) = Ad_{m}(B) + d_{m}(A)B$$

$$= \sum_{i=j=m}^{\infty} d_{i}(A)d_{j}(B), \quad \text{i.e.} \quad d \in D_{r}^{s}(P).$$

7. **Integrable $R$-derivations.**

Let $S'$ be a segment of $N$ such that $S \subseteq S'$ and let $s' = sup(S') \leq \infty$.

In the sequel we shall study $s'$-integrable $R$-derivations of order $s$ of $P$.

In this section, we give some examples of such $R$-derivations and we show that in general there are non-integrable $R$-derivations.

Notice first that, by Corollary 6.1, we may reduce our investigations and to study only $s'$-integrable transitive mappings of order $s$ from $\rho$ to $R$.

Observe also, that it suffices to consider the case where $\rho$ is a partial order.

It follows from the following

**Lemma 7.1.** The following conditions are equivalent:

1. Every transitive mapping of order $s$ from $\rho$ to $R$ is $s'$-integrable,
2. Every transitive mapping of order $s$ from $\rho'$ to $R$ is $s'$-integrable.

**Proof.** Denote by $W$ some fixed set of representatives of the cosets with respect to $\sim$.

1) $\Rightarrow$ 2). Let $g \in TM_{s}(\rho', R)$. Consider the sequence $f = \{f_{m}\}_{m \in S}$ of mappings from $\rho$ to $Z(R)$ defined by $f_{m}(x, y) = g_{m}([x], [y])$ for all $m \in S$ and $x \rho y$. If $x \rho y \rho z$ then $[x] \rho' [y] \rho' [z]$ and we have

$$f_{m}(x, z) = g_{m}([x], [z])$$

$$= \sum_{i, j = m}^{\infty} g_{i}([x], [y])g_{j}([y], [z])$$

$$= \sum_{i, j = m}^{\infty} f_{i}(x, y)f_{j}(y, z)$$

for all $m \in S$. Therefore $f \in TM_{s}(\rho, R)$, and, by (1), there exists $f' \in TM_{s'}(\rho, R)$
such that \( f'_m = f_m \) for all \( m \in S \).

Put \( g'_k([a], [b]) = f'_k(a, b) \) for \( k \in S' \) and \( a, b \in W \).

Then \( g' = (g'_k)_{k \in S'} \) is a transitive mapping of order \( s' \) from \( \rho' \) to \( R \). Indeed, if \([a] \rho'([b]) \rho'([c])\), then \( a b p c \) and we have

\[
g_k([a], [c]) = f'_k(a, c) = \sum_{p \in S'} f'_p(a, b)f'_p(b, c) = \sum_{p \in S'} g'_p([a], [b])g'_p([b], [c]) \quad \text{for all} \quad k \in S'.
\]

Moreover, if \( m \in S \), \([a] \rho'([b])\) then

\[
g'_m([a], [b]) = f'_m(a, b) = f_m(a, b) = g_m([a], [b]),
\]

i.e. \( g'_m = g_m \) for all \( m \in S \).

(2) \Rightarrow (1). Let \( f' \in TM_1(\rho', R) \). We define the element \( g' \in TM_1(\rho', R) \) by

\[
g'_k([a], [b]) = f'_m(a, b),
\]

where \( m \in S \) and \( a, b \in W \).

Let \( g' \) be such an element in \( TM_1(\rho', R) \) that \( g'_m = g_m \) for all \( m \in S \). We shall construct (by induction) a sequence \( f' \in TM_1(\rho', R) \) such that

(i) \( f'_m = f_m \) for all \( m \in S \),

and

(ii) \( f'_k(a, b) = g'_k([a], [b]) \) for all \( a, b \in W \) and \( k \in S' \).

If \( t \leq s \) then we put \( f'_t = f_t \).

Now let \( s \leq t < s' \) and assume that \( (f'_s, f'_t, \ldots, f'_i) \in TM_1(\rho', R) \) and the mappings \( f'_s, f'_t, \ldots, f'_i \) satisfy the condition (ii). If \( xpy \) then we put

\[
f'_{i+1}(x, y) = g'_{i+1}([a], [b])
\]

\[
= \sum_{c \in S'} f'(x, a) f'_{i+1}(a, y) - \sum_{c \in S'} f'(y, b) f'_{i+1}(b, y) + \sum_{c \in S'} f'(a, b) f'_{i+1}(b, y),
\]

where \( a, b \) are elements of \( W \) such that \( x \sim a, y \sim b \). Lemma 3.2 implies that \( f'_{i+1}(a, b) = g'_{i+1}([a], [b]) \) for \( a, b \in W \).

It remains to show that
\[ f'_{i+1}(x, z) - f'_{i+1}(x, y) - f'_{i+1}(y, z) = \sum_{i=1}^{\infty} f'(x, y) f'_{i+1}(y, z) \]

for \( x \neq y \).

For this purpose we introduce the following notation:

\[ (x_1, x_2, x_3) = \sum_{i=1}^{\infty} f'(x_1, x_2) f'_{i+1}(x_2, x_3) \quad \text{for} \quad x_1 \neq x_2 \neq x_3, \]

\[ A(x_1, x_2, x_3) = (x_2, x_3, x_1) - (x_1, x_2, x_3) \quad \text{for} \quad x_1 \neq x_2 \neq x_3 \neq x_4. \]

Observe that

(iii) \[ A(x_1, x_2, x_3) = 0. \]

In fact,

\[ A(x_1, x_2, x_3) = -\sum_{i=1}^{\infty} f'(x_1, x_2) f'(x_2, x_3) f'_{i+1}(x_3, x_4) \]

\[ + \sum_{i=1}^{\infty} f'(x_1, x_2) f'_{i+1}(x_2, x_3) f'(x_3, x_4) \]

\[ = -\sum_{i=1}^{\infty} f'(x_1, x_2) f'_{i+1}(x_3, x_4) \]

\[ - \sum_{i=1}^{\infty} f'(x_1, x_2) f'(x_2, x_3) f'(x_3, x_4) \]

\[ + \sum_{i=1}^{\infty} f'(x_1, x_2) f'_{i+1}(x_3, x_4) \]

\[ + \sum_{i=1}^{\infty} f'(x_1, x_2) f'(x_2, x_3) f'(x_3, x_4) \]

\[ = 0. \]

Observe also that if \( a, b, c \) are such elements of \( W \) that \( a \neq b \neq c \) then, by (ii), we have

(iv) \[ g'_{i+1}([a], [c]) - g'_{i+1}([a], [b]) - g'_{i+1}([b], [c]) = (a, b, c). \]

In fact, since \( g' \in TM_p(\rho', R) \) we have

\[ g'_{i+1}([a], [c]) - g'_{i+1}([a], [b]) - g'_{i+1}([b], [c]) \]

\[ = \sum_{i=1}^{\infty} g([a], [b]) g'_{i+1}([b], [c]) \]

\[ = \sum_{i=1}^{\infty} f(a, b) f'_{i+1}(b, c) \]

\[ = (a, b, c). \]

Now, let \( x \neq y \neq z \) and let \( a, b, c \) be such elements of \( W \) that \( a \sim x, b \sim y, c \sim z \).

Then, by (iii), (iv) and by the fact that \( y, y, z = 0 \) (Lemma 3.2) we obtain
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\[ f'_{i+1}(x, z) - f'_{i+1}(x, y) + f'_{i+1}(y, z) \]
\[ = (a, b, c) \]
\[ + (x, a, z) - (x, c, z) + (a, c, z) \]
\[ - (x, a, y) + (y, b, y) - (a, b, y) \]
\[ - (y, b, z) + (x, c, z) - (b, c, z) \]
\[ = (a, y, z) - (x, y, z) + (x, a, z) - (x, a, y) \]
\[ - ((b, c, z) - (a, c, z) + (a, b, z) - (a, b, c)) \]
\[ + ((b, y, z) - (a, y, z) + (a, b, z) - (a, b, y)) \]
\[ - ((b, y, z) - (y, y, z) + (y, b, z) - (y, b, y)) \]
\[ + (x, y, z) - (y, y, z) \]
\[ = A(x, a, y, z) - A(a, b, c, z) + A(a, b, y, z) - A(y, b, y, z) \]
\[ + (x, y, z) - (y, y, z) \]
\[ = (x, y, z) - (y, y, z) \]
\[ = (x, y, z). \]

This completes the proof.

**Example 7.2.** Let $P$ be such as in Example 4.3. Since $D^R(P) = ID^R(P)$ then every $R$-derivation of order $s$ of $P$ is $s'$-integrable (for any $s'$).

**Example 7.3.** Let $P = M_n(R)$, where

\[ \rho = \begin{array}{c}
1 \\
\downarrow \\
4
\end{array} \quad \begin{array}{c}
\rightarrow 3 \\
\uparrow \\
2
\end{array} \quad \text{i.e.} \quad P = \begin{bmatrix}
R & 0 & R \\
0 & R & R \\
0 & 0 & R \\
0 & 0 & 0
\end{bmatrix}. \]

There exist $R$-derivations of order $s$ of $P$ which are not inner ([7]). But, by Corollary 6.1 and Example 3.5, every $R$-derivation of order $s$ of $P$ is $s'$-integrable, for any $s' \leq \infty$ (see also Corollary 6.7).

**Example 7.4.** Consider the following relation $\rho$ on the set $I_{17}$
Let $R=\mathbb{Z}_2$ and let $f_1: \rho \rightarrow \mathbb{Z}_2$ be the usual transitive mapping from $\rho$ to $\mathbb{Z}_2$ defined by the numbers at the arrows (for example $f_1(14,1)=1$, $f_1(10,2)=0$).

Let $f_4(a, b)=1$ for all $a \neq b$. Then $f=(f_0, f_1)$ is a transitive mapping of order 1 from $\rho$ to $\mathbb{Z}_2$. We show that $f$ is not 2-integrable. Suppose that there exists $f_4: \rho \rightarrow \mathbb{Z}_2$ such that

$$f_4(a, c) = f_4(a, b) + f_4(b, c) + f_1(a, b)f_3(b, c),$$

for any $a \neq b \neq c$.

Denote $f_4(a, b)$ by $\langle a, b \rangle$. Then we have

$$1 = f_1(14,1)f_1(1,6)$$
$$= (14,6) + (14,1) + (1,6)$$
$$= (14,12) + (10,12) + (10,1) + (1,2) + (3,2) + (3,4) + (5,4) + (5,6)$$
$$+ (1,2) + (3,2) + (3,4) + (5,4) + (5,6) + (1,6) + (10,1) + (10,12) + (14,12)$$
$$= 0.$$

The above example and Corollary 6.1 show that there exist non-integrable $R$-derivations of $P$.

8. A necessary condition for $s'$-integrability.

Let $\Gamma'=\Gamma'_{\rho}=(I'_{\rho}, \rho')$ be the graph of the relation $\rho$ (see Section 2), and $f \in TM_{\rho}(\rho', R)$.

If $a, b, c$ are such elements in $I'_{\rho}$ that $a \rho' b \rho' c$ then by $\iota(a, b, c)$ we denote the element $\langle a, c \rangle - \langle a, b \rangle - \langle b, c \rangle$ of $C_1(\Gamma)$, and by $f_{m*}(a, b, c)$, for $m \in S$, we denote the element

$$\sum_{m \in S} f_1(a, b)f_{m*}(b, c)$$
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of \( Z(R) \).

For example:
\[
\begin{align*}
\bar{f}_1(a, b, c) &= 0, \\
\bar{f}_2(a, b, c) &= \bar{f}_3(a, b)f_3(b, c), \\
\bar{f}_4(a, b, c) &= \bar{f}_4(a, b)f_4(b, c) + f_3(a, b)f_3(b, c).
\end{align*}
\]

Consider the following equality (in the group \( C_5(I') \)):
\[
(*) \quad \sum_{k=1}^{s} z_k f_{s+1}(a_k, b_k, c_k) = 0,
\]
where \( k \in \mathbb{N}, z_1, \ldots, z_s \in Z \) and \( a_i \rho' b_i \rho' c_i \) for \( i = 1, 2, \ldots, k \).

**Definition 8.1.** Let \( s < \infty \). We say that \( I' \) is an \( s \)-graph over \( R \) if for any transitive mapping \( f \) of order \( s \) from \( \rho' \) to \( R \) and for any equality of the form \((*)\) holds
\[
\sum_{k=1}^{s} z_k f_{s+1}(a_k, b_k, c_k) = 0.
\]

For example, \( I' \) is a 1-graph over \( R \) if for every usual transitive mapping \( \varphi : \rho' \to Z(R) \) and for every equality \((*)\) holds
\[
\sum_{k=1}^{s} z_k \varphi(a_k, b_k \varphi(b_k, c_k)) = 0,
\]
and \( I' \) is a 2-graph over \( R \) if for every \( f = (f_e, f_f, f_g) \in TM_2(\rho', R) \) and for every equality \((*)\) holds
\[
\sum_{k=1}^{s} z_k(f_{1}(a_k, b_k)f_{2}(b_k, c_k) + f_{3}(a_k, b_k)f_{4}(b_k, c_k)) = 0.
\]

In Section 9 we prove that every graph \( I' \) is a 1-graph and is a 2-graph over an arbitrary ring \( R \).

**Example 8.2.** Let
\[
\rho = \begin{array}{c}
4 \\
\downarrow \\
\downarrow \\
\downarrow \\
2 \\
\downarrow \\
1 \\
\downarrow \\
3
\end{array}
\]

We show that \( I' = (I_\rho, \rho) \) is an \( s \)-graph over an arbitrary ring \( R \), for any \( s \in \mathbb{N} \).

Observe, that for \( I' \) we have only one equality of the form \((*)\). Namely,
\[
[(1, 4) - (1, 2) - (2, 4)] - [(1, 3) - (1, 2) - (2, 3)]
\]
\[ +[(2, 4) - (2, 3) - (3, 4)] - [(1, 4) - (1, 3) - (3, 4)] = 0, \]

i.e. \[ t(1, 2, 4) - t(1, 2, 3) + t(2, 3, 4) - t(1, 3, 4) = 0. \]

If \( s \in \mathbb{N}, f \in TM_s(\rho, R), \) then we have

\[
\begin{align*}
f_{s+t}(1, 2, 4) - f_{s+t}(1, 2, 3) + f_{s+t}(2, 3, 4) - f_{s+t}(1, 3, 4) \\
= \sum_{k=1}^{s} [f_{s}(1, 2) f_{s+k}(2, 4) - f_{s}(1, 2) f_{s+k}(2, 3) \\
+ f_{s}(2, 3) f_{s+k}(3, 4) - f_{s}(1, 3) f_{s+k}(3, 4)] \\
= \sum_{k=1}^{s} f_{s}(1, 2) (f_{s+k}(3, 4) + \sum_{p \in S_k, q \in S_{s+k}} f_{p}(3, 4) f_{q}(2, 3)) \\
- \sum_{k=1}^{s} f_{s}(1, 2) + \sum_{p \in S_k, q \in S_{s+k}} f_{p}(3, 4) f_{q}(2, 3) f_{s+k}(3, 4) = 0.
\end{align*}
\]

Now we prove a necessary condition for any \( R \)-derivation of order \( s \) of \( P \) to be \((s+1)\)-integrable.

**Proposition 8.3.** Let \( P = M_s(R)_\rho \). If every \( R \)-derivation of order \( s \) of \( P \) is \((s+1)\)-integrable then \( \Gamma = \Gamma(\rho) \) is an \( s \)-graph.

**Proof.** Consider in \( C_1(\Gamma) \) the equality of the form \((\ast)\) and let \( f \in TM_s(\rho', R) \). There exists, by Corollary 6.1 and Lemma 7.1, a transitive mapping \( f' \in TM_{s+t}(\rho', R) \) such that \( f'_m = f_m \) for all \( m = 0, 1, \ldots, s \). Observe that, for \( i = 1, 2, \ldots, k \), we have

\[ f'_{s+t}(a_i, c_i) - f'_{s+t}(a_i, b_i) - f'_{s+t}(b_i, c_i) = f_{s+t}(a_i, b_i, c_i). \]

Let \( \varphi : C_1(\Gamma) \to Z(R) \) be the group homomorphism defined (for free generators) by \( \varphi(a, b) = f'_{s+t}(a, b) \).

Then we have

\[
\begin{align*}
\sum_{i=1}^{k} \varepsilon_i f_{s+t}(a_i, b_i, c_i) & = \sum_{i=1}^{k} \varepsilon_i (f'_{s+t}(a_i, c_i) - f'_{s+t}(a_i, b_i) - f'_{s+t}(b_i, c_i)) \\
& = \sum_{i=1}^{k} \varepsilon_i (\varphi(a_i, b_i) - \varphi(a_i, b_i) - \varphi(b_i, c_i)) \\
& = \varphi(\sum_{i=1}^{k} \varepsilon_i t(a_i, b_i, c_i)) \\
& = \varphi(0) \\
& = 0. \quad \text{This completes the proof.}
\end{align*}
\]

We obtain some examples of \( s \)-graphs by the following
LEMMA 8.4. If $H_s(I)=0$ then $I$ is an s-graph over $R$ for any natural $s$.

PROOF. Suppose that in $C_s(I)$ the equality (*) holds, and let $f \in TM_s(\rho', R)$. We must show that $\sum_{i=1}^{l} z_i \bar{f}_{\ast} + (a_i, b_i, c_i) = 0$.

Consider the group homomorphism $\varphi: C_s(I) \to R$ defined for free-generators by $\varphi(a, b, c) = \bar{f}_{\ast} + (a, b, c)$. Since $\sum_{i=1}^{l} z_i (a_i, b_i, c_i) \in \text{Ker } \partial_s$ and $\text{Ker } \partial_s = \text{Im } \partial_s$ (see Section 2) then

$$\sum_{i=1}^{l} z_i (a_i, b_i, c_i) = \frac{1}{\partial_s} \sum_{j=1}^{m} u_j [\bar{f}_{\ast} + (x_j, y_j, t_j) + (x_j, w_j, t_j) - (y_j, w_j, t_j)]$$

for some $u_1, \ldots, u_l \in Z$ and $x_j, y_j, w_j, t_j, j=1, 2, \ldots, l$.

Therefore, by Example 8.2, we have

$$\sum_{i=1}^{l} z_i \bar{f}_{\ast} + (a_i, b_i, c_i) = \varphi \left( \sum_{i=1}^{l} z_i (a_i, b_i, c_i) \right)$$

$$= \frac{1}{\partial_s} \sum_{j=1}^{m} u_j [\bar{f}_{\ast} + (x_j, y_j, t_j) - \bar{f}_{\ast} + (x_j, y_j, t_j)]$$

$$+ \bar{f}_{\ast} + (y_j, w_j, t_j) - \bar{f}_{\ast} + (y_j, w_j, t_j)$$

$$= \frac{1}{\partial_s} u_0 = 0.$$ This completes the proof.

REMARK 8.5. The necessary condition for any $R$-derivation of order $s$ of $P$ to be $(s+1)$-integrable given in Proposition 8.3 is not sufficient. For example, let $I$ be such as in Example 7.4. Then $I$ is one-dimensional triangulation of the projective plane, and therefore $H_s(I)=0$ (see [3]). So, by Lemma 8.4, $I$ is a 1-graph over $Z_s$. But, by Example 7.4, there exists an $R$-derivation $d$ of order 1 of $P=M_s(R)$ (where $R=Z_s$) such that $d$ is not 2-integrable.

THEOREM 8.6. Let $P$ be a special subring of $M_s(R)$ with the relation $\rho$, and let $I=I(\rho)$ and $s < s' \leq \infty$. If $H_s(I)=0$ and $H_s(I)$ is a free abelian group then every $R$-derivation of order $s$ of $P$ is $s'$-integrable.

PROOF. It follows from Corollary 6.1 and Lemma 7.1 that it is sufficient to prove that every transitive mapping of order $s$ from $\rho'$ to $R$ is $(s+1)$-integrable.

Let $f \in TM_s(\rho', R)$ and consider a group homomorphism $\varphi: \text{Im } \partial_s \to Z(R)$ defined (for generators) by $\varphi(\partial_s(a, b, c)) = \bar{f}_{\ast} + (a, b, c)$. Observe that, by Lemma 8.4, $\varphi$ is a well defined mapping. Since $H_s(I)$ is free then $\varphi$ we can extend to a group homomorphism $\varphi': \text{Ker } \partial_s \to Z(R)$. Further, by [7] Lemma 5.5, we can extend $\varphi'$ to a group homomorphism $\varphi^*: C_s(I) \to Z(R)$. Put $f_{\ast} + (a, b) = \varphi^*(a, b)$ for all $apb$. We show that, for any $apb$,
\[ f_{s+1}(a, c) = \sum_{i=0}^{s-1} f_i(a, b) f_{s+i}(b, c) \]
\[ = f_{s+1}(a, b) + f_{s+1}(b, c) + \sum_{i=1}^{s} f_i(a, b) f_{s+i-1}(b, c) \]

In fact
\[ f_{s+1}(a, b) - f_{s+1}(a, b) = f_{s+1}(b, c) \]
\[ = \varphi^*(a, c) - \varphi^*(a, b) - \varphi^*(b, c) \]
\[ = -\varphi^*(\delta_0(a, b, c)) \]
\[ = -\varphi^*(\delta_0(a, b, c)) \]
\[ = f_{s+1}(a, b, c) \]
\[ = \sum_{i=1}^{s} f_i(a, b) f_{s+i-1}(b, c). \]

Therefore \((1, f_s, \ldots, f_n, f_{s+1})\) is a transitive mapping of order \((s+1)\) from \(\rho'\) to \(R\), i.e. \(f\) is \((s+1)\)-integrable. This completes the proof.

9. \(s\)-graphs.

In this section, using some additional properties of \(s\)-graphs, we describe (for fixed \(s<\alpha\)) a new class of special subrings of \(M_{\alpha}(R)\) in which every \(R\)-derivation of order \(s\) is \(s'\)-integrable.

Let \(G=(\mu, \rho')\) be the graph of the relation \(\rho\) and let \(W(G)=Z[X_{(a, b)}; \alpha\rho' b]\) be the ring of polynomials over \(Z\) in commuting indeterminates, one for each pair \((a, b)\), where \(\alpha\rho' b\). Denote by \(T(G)\) the ring \(W(G)/I(G)\), where \(I(G)\) is the ideal in \(W(G)\) generated by all elements of the form
\[ X_{(a, b)} - X_{(a, b)} - X_{(a, c)} \]
for \(\alpha\rho' \beta \rho' c\).

Moreover, denote by \(\langle a, b \rangle\) the coset of the element \(X_{(a, b)}\) in \(T(G)\).

The following lemma plays a basic role in our further considerations.

**Lemma 9.1.** Let \(n\) be a power of a prime number \(p\). If in the group \(G_\alpha(G)\) holds the equality of the form \((*)\), then in the ring \(T_\alpha(G)\) the following equality holds
\[ \sum_{i=1}^{n} z_i \sum_{j=1}^{n} (1/p)^{(n)} (a_i, b_j, c_i)^{*-1} = 0. \]

**Proof.** Observe that the equality \((*)\) is equivalent to an equality of the form
\[ \sum_{i=1}^{n} (a_i, c_i) + \sum_{j=1}^{n} (a_i^*, b_j^*) + (b_j, c_j) \]

\[ \sum_{i=1}^{n} (a_i, b_i, c_i) \]
Higher \( R \)-derivations of special subrings of matrix rings

\[
= \sum_{j=1}^{m} (a^n_j, c^n_j) + \sum_{j=1}^{m} ((a^n_j, b^n_j) + (b^n_j, c^n_j)), 
\]

where \( a^n_j b^n_j c^n_j \) for some integers \( u, v \) and \( i=1, \ldots, u, \ j=1, \ldots, v \).

Hence it suffices to prove that, in the ring \( T(\Gamma) \), we have

\[
\sum_{j=1}^{m} (1/p)(\binom{n}{k}) (a^n_j, b^n_j)^{k} + (b^n_j, c^n_j)^{k} = \sum_{j=1}^{m} (1/p)(\binom{n}{k}) (a^n_j, b^n_j)^{k} + (b^n_j, c^n_j)^{k}.
\]

Let \( \alpha, \beta : C_\Gamma \to W(\Gamma) \) be the group homomorphisms defined, for free generators, as follows:

\[
\alpha(a, b) = X_{(a, b)} \quad \text{and} \quad \beta(a, b) = X_{(a, b)}.
\]

Further we denote \( X_{(a, b)} \) by \( (a, b) \) (for all \( ap^{\rho}b \)).

Applying \( \alpha \) to the equality (***) we obtain the equality (**) in the ring \( W(\Gamma) \).

Applying \( \beta \) to the equality (**) we obtain the following equality in \( W(\Gamma) \):

\[
\sum_{j=1}^{m} (a^n_j, c^n_j)^{n} + \sum_{j=1}^{m} ((a^n_j, b^n_j)^{n} + (b^n_j, c^n_j)^{n})
\]

\[
= \sum_{j=1}^{m} (a^n_j, c^n_j)^{n} + \sum_{i=1}^{m} ((a^n_i, b^n_i)^{n} + (b^n_i, c^n_i)^{n}).
\]

Let

\[
A_i = (a^n_i, c^n_i),
\]

\[
B_i = (a^n_i, b^n_i) + (b^n_i, c^n_i) \quad \text{for} \quad i=1, 2, \ldots, u,
\]

and

\[
C_j = (a^n_j, c^n_j),
\]

\[
D_j = (a^n_j, b^n_j) + (b^n_j, c^n_j) \quad \text{for} \quad j=1, 2, \ldots, v.
\]

Rise both sides of the equality (**) in \( W(\Gamma) \) to the \( n \)-th power and apply (1). Then we have

\[
= \sum_{j=1}^{m} \sum_{k=1}^{n} (\binom{n}{k}) (a^n_j, b^n_j)^{k} + (b^n_j, c^n_j)^{k} - \sum_{j=1}^{m} \sum_{k=1}^{n} (\binom{n}{k}) (a^n_j, b^n_j)^{k} + (b^n_j, c^n_j)^{k}.
\]

\[
= \sum_{i=1}^{u} (i_1, \ldots, i_u) [A^{i_1} \cdots A^{u} - B^{i_1} \cdots B^{i_u}]
\]

\[
+ \sum_{j=1}^{v} (j_1, \ldots, j_v) [D^{j_1} \cdots D^{j_v} - C^{j_1} \cdots C^{j_v}]
\]

\[
+ \sum_{k=1}^{u} (\binom{n}{k}) [\sum_{i=1}^{u} (A_i)^k - (\sum_{j=1}^{v} B_j)^k]^{n-k},
\]
where \((i_1, \ldots, i_n)\), \((j_1, \ldots, j_n)\) are Newton symbols, i.e.
\[
(n_1, \ldots, n_k) = \left(\frac{(n_1 + \cdots + n_k)!}{n_1! \cdots n_k!}\right)
\]
for integers \(n_1, \ldots, n_k \geq 0\).

Since \(n\) is a power of a prime number \(p\) then every Newton symbol in the equality (2) is divisible by \(p\), and therefore, since \(W(I')\) is a ring with no \(Z\)-torsion, we can divide both sides of the equality (2) by \(p\). We obtain the new equality in \(W(I')\), we denote it by (3).

Observe, that the right side of the equality (3) is an element of the ideal \(I(I')\). Therefore, in the ring \(T(I')\), we have the equality (**). This completes the proof.

As a consequence of Lemma 9.1 we obtain

**Theorem 9.2.** Every graph \(\Gamma\) is a 1-graph over an arbitrary ring \(R\).

Observe, that this theorem is obvious if \(R\) is a 2-torsion-free ring. In fact. Let \(f_1: \rho' \rightarrow Z(R)\) be an usual transitive mapping and suppose that \(\in C_1(I')\) the equality of the form (**) holds. Consider the group homomorphism \(\varphi:C_1(I') \rightarrow Z(R)\) such that \(\varphi(a, b) = f_1(a, b)^s\), for all \(a \rho' b\). Then we have

\[
2 \sum_{i=1}^{k} z_i f_1(a_i, b_i) f_1(b_i, c_i)
\]

\[
= \sum_{i=1}^{k} z_i [(f_1(a_i, b_i) + f_1(b_i, c_i))^s - f_1(a_i, b_i) - f_1(b_i, c_i)]
\]

\[
= \sum_{i=1}^{k} z_i [\varphi(a_i, c_i) - \varphi(a_i, b_i) - \varphi(b_i, c_i)]
\]

\[
= \varphi\left(\sum_{i=1}^{k} z_i (a_i, b_i, c_i)\right)
\]

\[
= \varphi(0)
\]

\[
= 0.
\]

**Proof of Theorem 9.2.** Let \(f \in TM_1(\rho', R)\) and suppose that in \(C_1(I')\) the equality of the form (**) holds. Let \(h: W(I') \rightarrow Z(R)\) be the ring homomorphism such that \(h(X(a, b)) = f_1(a, b)\) for all \(a \rho' b\). Since \(f_1\) is an usual transitive mapping then \(h\) induces a ring homomorphism \(\tilde{h}: T(I') \rightarrow Z(R)\) such that \(\tilde{h}(a, b) = f_1(a, b)\). From Lemma 9.1, for \(n=2\), we have

\[
\sum_{i=1}^{k} z_i f_1(a_i, b_i) f_1(b_i, c_i) = \tilde{h}\left(\sum_{i=1}^{k} z_i (a_i, b_i, c_i)\right)
\]

\[
= \tilde{h}(0) = 0.
\]

This completes the proof.
Lemma 9.3. If in $C_i(I)$ the equality (*) holds then in the ring $T(I)$ we have

$$\sum_{c_1} z_i \langle a_i, b_i \rangle \langle b_i, c_i \rangle \langle a_i, c_i \rangle = 0.$$ 

Proof. From Lemma 9.1, for $n=3$, we get

$$0 = \sum_{c_1} z_i \langle a_i, b_i \rangle \langle b_i, c_i \rangle + \langle a_i, b_i \rangle \langle b_i, c_i \rangle$$

$$= \sum_{c_1} z_i \langle a_i, b_i \rangle \langle b_i, c_i \rangle \langle a_i, b_i \rangle + \langle b_i, c_i \rangle$$

$$= \sum_{c_1} z_i \langle a_i, b_i \rangle \langle b_i, c_i \rangle \langle a_i, c_i \rangle.$$ 

Theorem 9.4. Every graph $I$ is a 2-graph over an arbitrary ring $R$.

Proof. Let $f \in TM_6(\rho', R)$ and suppose that in $C_i(I)$ holds (*). Consider the group homomorphism $\varphi: C_i(I) \to Z(R)$ such that

$$\varphi(a, b) = f_3(a, b) f_3(a, b)$$

for all $a \rho' b$.

Then we have

$$0 = \varphi(0)$$

$$= \sum_{c_1} z_i \varphi(a_i, c_i) - \varphi(a_i, b_i) - \varphi(b_i, c_i)$$

$$= \sum_{c_1} z_i [(f_3(a_i, b_i) + f_3(b_i, c_i)) f_3(a_i, b_i) + f_3(b_i, c_i)]$$

$$+ f_3(a_i, b_i) f_3(b_i, c_i) - f_3(a_i, b_i) f_3(b_i, c_i)]$$

$$= \sum_{c_1} z_i [ f_3(a_i, b_i) f_3(b_i, c_i) + f_3(a_i, b_i) f_3(b_i, c_i)]$$

$$+ \sum_{c_1} z_i f_3(a_i, b_i) f_3(b_i, c_i) f_3(a_i, c_i).$$

Since, by Lemma 9.3,

$$\sum_{c_1} z_i f_3(a_i, b_i) f_3(b_i, c_i) f_3(a_i, c_i) = 0$$

then

$$\sum_{c_1} z_i [ f_3(a_i, b_i) f_3(b_i, c_i) + f_3(a_i, b_i) f_3(b_i, c_i)] = 0.$$ 

This completes the proof.

Using a similar method we can prove the following
Theorem 9.5. Let $\Gamma$ be a graph and $R$ be a ring.

a) If $R$ is 2-torsion-free then $\Gamma$ is a 3-graph over $R$,
b) $\Gamma$ is a 4-graph over $R$,
c) If $R$ is 6-torsion-free then $\Gamma$ is a 5-graph over $R$,
d) $\Gamma$ is a 6-graph.

Using the above theorems and arguments from the proof of Theorem 8.6 we obtain

Theorem 9.6. Let $P$ be a special subring of $M_n(R)$ with the relation $\rho$.
Assume that the homology group $H(\Gamma(\rho))$ is free abelian. Then

1) Every $R$-derivation of order $s<3$ of $P$ is 3-integrable.
2) If $R$ is 2-torsion-free then every $R$-derivation of order $s<5$ of $P$ is 5-
   integrable.
3) If $R$ is 31-torsion-free then every $R$-derivation of order $s<7$ of $P$ is 7-
   integrable.

We end this paper with the following open problems:

1. Let $\Gamma=(I_n, \rho)$ be a fixed graph (i.e. $\rho$ is a partial ordering relation on
   $I_n$) and let $s<s'$. Suppose that for every $R$ any $R$-derivation of order $s$ of
   $M_n(R)$, is $s'$-integrable. Is $H_1(\Gamma)$ a free group?

2. Find numbers $n, s$, a ring $R$, and a partial order $\rho$ on $I_n$ such that the
   graph $\Gamma=(I_n, \rho)$ is not $s'$-graph over $R$.

3. Is every graph a 3-graph over an arbitrary ring?

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