INNER DERIVATIONS OF HIGHER ORDERS

By

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Summary. We define inner derivations of higher order of a ring \( R \) and we prove that they correspond to the inner automorphisms of a suitable ring. Moreover, we prove that any higher derivation of \( R \) is inner if and only if any usual derivation of \( R \) is inner.

I.

Let \( R \) be a ring with identity and let \( S \) be a segment of \( N=\{0, 1, 2, \ldots \} \), that is, \( S=N \) or \( S=\{0, 1, \ldots, s\} \) for some \( s\geq 0 \).

A family \( d=(d_n)_{n\in S} \) of mappings \( d_n: R \to R \) is called a derivation of order \( s \) of \( R \) (where \( s=\sup S \leq \infty \)) if the following properties are satisfied:

1. \( d_n(a+b)=d_n(a)+d_n(b) \),
2. \( d_n(ab)=\sum_{i+j=n} d_i(a)d_j(b) \),
3. \( d_0=\text{id}_R \).

The set of derivations of order \( s \) of \( R \), denoted by \( D_s(R) \), is the group under the multiplication \( * \) defined by the formula

\[
(d * d')_n = \sum_{i+j=n} d_i d'_j ,
\]

where \( d, d' \in D_s(R) \) and \( n \in S \) (\([1], [5], [7]\)).

It is easy to prove the following two lemmas.

**Lemma 1.1.** Let \( a \in R \), \( d_e=id_e \), and

\[
d_n(x)=a^nx-a^{n-1}xa=a^{n-1}(ax-xa)
\]

for \( n\geq 1 \), \( x \in R \). Then \( d=(d_n)_{n\in S} \) belongs to \( D_s(R) \).

**Lemma 1.2.** Let \( d \in D_s(R) \), \( k \in S \setminus \{0\} \) and let \( \delta=(\delta_n)_{n\in S} \) be the family of mappings from \( R \) to \( R \) defined by

\[
\delta_n = \begin{cases} 
0, & \text{if } k \nmid n, \\
\delta_r, & \text{if } n=rk.
\end{cases}
\]
Then $\delta \in D_s(R)$.

The derivation $d$ from Lemma 1.1 will be denoted by $[a, 1]$ and the derivation $\delta$ from Lemma 1.2, for $d=[a, 1]$, will be denoted by $[a, k]$. Therefore, for $a \in R$, $k \in \mathbb{S}\setminus\{0\}$, $x \in R$, $n \in \mathbb{S}$:

$$
[a, k]_n(x) =
\begin{cases}
x & \text{if } n=0, \\
0 & \text{if } k \nmid n, \\
ar^n - ar^{-1}xa & \text{if } n \neq 0 \text{ and } n=kr.
\end{cases}
$$

Let $a=(a_n)_{n \in \mathbb{S}}$ be a sequence in $R$. Denote by $\Delta(a)$ the element in $D_s(R)$ defined by

$$
\Delta(a)_n = ([a_1, 1] \ast [a_2, 2] \ast \cdots \ast [a_n, n])_n.
$$

For example

$$
\Delta(a)_1(x) = a_1x - xa_1 \\
\Delta(a)_2(x) = a_1^2x - a_1xa_1 + a_2x - xa_2 \\
\Delta(a)_3(x) = a_1^3x - a_1^2xa_1 + a_1a_2x + xa_2a_1 - a_1xa_1 - a_2xa_1 + a_3x - xa_3 \\
\Delta(a)_4(x) = a_1^4x - a_1^3xa_1 + a_1^2a_3x + a_2a_3x - a_1^2xa_1 + a_3^2x - xa_3 - a_4x + xa_4,
$$

$$
+ a_1xa_4a_1 + a_1a_3x - a_1xa_1 + xa_3a_1 + a_4x - xa_4.
$$

**Definition 1.3.** Let $d \in D_s(R)$. If there exists a sequence $a=(a_n)_{n \in \mathbb{S}}$ of elements of $R$ such that $d = \Delta(a)$ then $d$ is called an inner derivation of order $s$ of $R$.

**II.**

Denote by $T$ the additive group of the product of $s+1$ copies of $R$. The element $(a_n)_{n \in \mathbb{S}}$ will be always denoted by $a$. We define a multiplication on $T$ as follows:

$$ab = c, \text{ where } c_n = \sum_{i+j=n} a_ib_j.
$$

$T$ is a ring with identity $(1, 0, 0, \ldots)$ ([7], [8]). Notice that an element $a$ is invertible in $T$ iff $a_0$ is invertible in $R$.

For any $k \in \mathbb{S}$, let $\pi_k$ denote the $k$-th projection from $T$ to $R$. If $a \in R$ then $j_k(a)$, $p_k(a)$ and $q_k(a)$ (where $k \in \mathbb{S}$, $l \in \mathbb{S}\setminus\{0\}$) denote the elements of $T$ defined by the following conditions:

$$
\pi_n j_k(a) =
\begin{cases}
0, & \text{for } n \neq k, \\
a, & \text{for } n = k,
\end{cases}
$$

$$
\pi_n p_k(a) =
\begin{cases}
0, & \text{if } l \nmid n, \\
a^*, & \text{if } n = rl,
\end{cases}
$$

$$
\pi_n q_k(a) =
\begin{cases}
0, & \text{if } l \nmid n, \\
a, & \text{if } n = rl.
\end{cases}
$$
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\[ \pi_n q_l(a) = \begin{cases} 
1, & \text{for } n=0, \\
0, & \text{for } n \geq 1, n \neq l, \\
a_l, & \text{for } n = l.
\end{cases} \]

Let \( T_k \) (for \( k \in \mathbb{S} \setminus \{0\} \)) denote the set of elements \( a \) in \( T \) such that \( a_0 = 1 \) and \( a_i = 0 \) for \( i = 1, 2, \ldots, k \), and let \( T_k \) be the set of elements \( a \) in \( T \) such that \( a_0 = 1 \).

Observe that \( q_k(a) = 1 + q_k(a) \), and every element in \( T_k \) is of the form \( 1 + f_{n+l}(1)a \), for some \( a \in T \).

It is easy to verify the following

**Lemma 2.1.** Let \( k \in \mathbb{S}, a \in R \).

1. If \( a, b \in T_k \) then \( ab, a^{-1} \in T_k \).
2. \( p_k(a)^{-1} = q_k(a) \).
3. If \( b \in T_k \) then \( bp_k(-b) = a \), where \( a_n = b_n \) for \( n = 0, 1, \ldots, k-1 \), and \( a_k = 0 \).

Now we prove two lemmas.

**Lemma 2.2.** Let \( b \in T_k \). Then there exists an element \( a \) in \( T_k \) such that \( bp_k(a) \in T_k \), for any \( k \in \mathbb{S} \setminus \{0\} \).

**Proof.** Let \( a_i = -b_i \). Then, by Lemma 2.1(3), we have \( bp_k(a_i) \in T_i \). Suppose that elements \( a_1, \ldots, a_n \) satisfy the condition

\[ v^{(k)} = bp_k(a_i) \cdots p_k(a_k) \in T_k \]

for \( k = 1, 2, \ldots, n \).

Put \( a_{n+1} = -\pi_{n+1}(v^{(n)}) \). Then

\[ v^{(n+1)} = v^{(n)} p_{n+1}(a_{n+1}) \]

\[ -bp_k(a_i) \cdots p_{n+1}(a_{n+1}) \in T_{n+1} \]

by Lemma 2.1(3).

**Lemma 2.3.** Let \( a \in T_k \). Then there exists \( b \in T_k \) such that

\[ p_k(a) p_k(a) \cdots p_k(a) b \in T_k \]

for any \( k \in \mathbb{S} \setminus \{0\} \).

**Proof.** Put \( b_n = 1 \) and \( b_n = \pi_n(u_{(m)} \), for \( n \geq 1 \), where \( u_{(m)} = q_n(-a_n) \cdots q_1(-a_1) \).

Then \( b_n = \pi_n(u^{(n)}) \) for any \( n \in \mathbb{S} \setminus \{0\} \) and \( n \geq n \). In fact, if \( k \geq 2 \) then
\[ \pi_n(u^{(k+1)}) = \pi_n(u^{(k)}) + j_{n+1}(-a_{k+1})u^{(k)} \]
\[ = \pi_n(u^{(k)}) + \pi_n(j_{n+1}(-a_{k+1})u^{(k)} \]
\[ = \pi_n(u^{(k)}). \]

Therefore, if \( b = (b_n)_{n \in S} \) then \( \pi_n(b - u^{(k)}) = 0 \) for \( i = 0, 1, \ldots, k \). So \( b = u^{(k)} + j_{n+1}(1)v^{(k)} \), for some \( v^{(k)} \in T \), and, by Lemma 2.1, we have
\[ p_1(a_1)p_2(a_2) \ldots p_{n}(a_n)b = p_1(a_1) \ldots p_{n}(a_n)q_1(-a_1) \ldots q_{n}(-a_{n}) + j_{n+1}(1)v^{(k)} \]
\[ = 1 + j_{n+1}(1)c, \]
for some \( c \in T \). This completes the proof.

### III.

If \( d \in D_s(R) \) then \( \exp(d) \) will denote the ring automorphism of \( T \) defined as follows:
\[ \exp(d)(a) = b, \text{ where } b_n = \sum_{i+j=n} d_i(a_j) \quad ([5], [7], [8]). \]

In [7] Ribenboim showed that the mapping \( \exp \) is a group isomorphism from \( D_s(R) \) to the group \( B_k(R) \) of such automorphisms \( h : T \to T \) that \( h(j_1(1)) = j_1(1) \), \( \pi_n h = id_R \). If \( h \in B_k(R) \) then the derivation \( d = (d_n)_{n \in S} \), where \( d_n(x) = \pi_n h j_n(x) \) for \( x \in R \), satisfies the condition \( h = \exp(d) \) ([7]).

For any \( a \in T_s \) denote by \( \langle a \rangle \) the inner automorphism of \( T \) defined by
\[ \langle a \rangle(x) = a^{-1}xa. \]
Observe that \( \langle a \rangle \) belongs to \( B_k(R) \).

**Lemma 3.1.**

1. If \( a \in R, \ k \in S \setminus \{0\} \) then \( \exp([a, k]) = \langle q_k(-a) \rangle \).
2. Let \( a \in T_s \). If \( d = (d_n)_{n \in S} \) is an element of \( D_s(R) \) such that \( \exp(d) = \langle a \rangle \), then \( d_1 = d_2 = \cdots = d_s = 0 \).

**Proof.**

1. If \( d \in D_s(R) \) satisfies \( \exp(d) = \langle q_k(-a) \rangle \) then
\[ d_n(x) = \pi_n q_k(-a) j_n(x) \]
\[ = \pi_n q_k(-a) j_n(x) q_k(-a) \]
\[ = \pi_n p_n(a) j_n(x)(1 + j_k(-a)), \quad \text{for } n \in S. \]

Hence \( d_n(x) = 0 \) if \( k \not\in n \), and \( d_n(x) = a^n x - a^{n-1} x a \) if \( n = kr \). Therefore \( d = [a, k] \).

2. It follows from Lemma 2.1 since \( d_n = \pi_n \langle a \rangle j_n \).

Now we are ready to prove the following

**Theorem 3.2.** Let \( d \in D_s(R) \). Then \( d \) is inner iff there exists \( b \in T_0 \) such
that $\exp(d) = \langle b \rangle$.

**Proof.** Let $d = \Delta(a)$, where $a \in T_a$ and let $b$ be as in Lemma 2.3. Moreover, let $\delta = \langle d \rangle_{\text{ess}}$ be the unique derivation satisfying $\exp(\delta) = \langle b \rangle$. We show that $\delta = d$.

Let $n \in S \setminus \{0\}$. It follows from Lemmas 2.3, 2.1 that

$$b = q_a \langle -a_a \rangle \cdots q_1 \langle -a_1 \rangle v^{(n)},$$

where $v^{(n)}$ is an element of $T_a$.

Therefore, if $F = \exp^{-1}$ then

$$\delta = F \langle b \rangle = F \langle v^{(n)} \rangle \ast F \langle q_1 \langle -a_1 \rangle \rangle \ast \cdots \ast F \langle q_a \langle -a_a \rangle \rangle,$$

and, by Lemma 3.1,

$$\delta_a = [[a_a, 1] \ast \cdots \ast [a_a, n]]_a = d_a.$$ 

Conversely, let $b \in T_a$, $d = \exp^{-1}(\langle b \rangle)$ and let $a$ be such as in Lemma 2.2. We show that $d = \Delta(a)$.

Let $n \in S \setminus \{0\}$. It follows from Lemmas 2.2, 2.1 that

$$b = v^{(n)} q_a \langle -a_a \rangle \cdots q_1 \langle -a_1 \rangle,$$

where $v^{(n)} \in T_a$, and hence

$$d = F \langle b \rangle = F \langle q_1 \langle -a_1 \rangle \rangle \ast \cdots \ast F \langle q_a \langle -a_a \rangle \rangle \ast F \langle v^{(n)} \rangle,$$

where $F = \exp^{-1}$.

Therefore, by Lemma 3.1, we have

$$d_a = [[a_a, 1] \ast \cdots \ast [a_a, n]]_a \quad \text{i.e.} \quad d = \Delta(a).$$

**Corollary 3.3.** The set of inner derivations of order $s$ of $R$ is a normal subgroup of $D_s(R)$.

**IV.**

Recall that the usual (classical) derivation of $R$ is the additive mapping $\delta : R \to R$ such that $\delta(ab) = \delta(a)b + a\delta(b)$, for all elements $a, b \in R$. The set of usual derivations of $R$ corresponds bijectively, in the natural way, to the set $D_s(R)$. Evidently a usual derivation is inner iff there exists an element $a \in R$ such that $\delta(x) = ax - xa$ for any $x \in R$.

It is easy to see that

**Lemma 4.1.** Let $d, d' \in D_s(R)$. If $d_i = d'_i$ for $i = 0, 1, \ldots, n < s$ then $d_{s+1} - d'_{s+1}$
is a usual derivation.

Now we can prove

**Theorem 4.2.** If every classical derivation of \( R \) is inner then so is every derivation of order \( s \) of \( R \).

**Proof.** Let \( d \in D_s(R) \). We must construct an element \( a \in T \) such that \( d = \Delta(a) \).

Since \( d \) is a classical derivation then there exists \( a \in R \) such that \( d(x) = a_x - xa \), for any \( x \in R \). So we have \( d = [a, 1] \).

Let \( d'_3 = [a, 1]_3 \). Then \( (1_R, a, d'_3) \) and \( (1_R, d'_3, d'_3) \) are derivations of order 2 and hence, by Lemma 4.1, there exists \( a_2 \in R \) such that \( d'_3(x) = d'_3(x) + a_2 x - xa \), for any \( x \in R \). Therefore,

\[
d'_3 = d'_3 + [a, 2]_3 = [a, 1]_3 + [a, 2]_3, \\
= ([a, 1] + [a, 2])_3.
\]

Next let \( d'_3 = ([a, 1] + [a, 2])_3 \). Since \( (1_R, d'_3, d'_3) \), \( (1_R, d'_3, d'_3) \), \( d'_3, d'_3, d'_3, \) \( d'_3 \) are derivations of order 3 then, by Lemma 4.1, \( d'_3(x) = d'_3(x) + a_3 x - xa \), for some \( a_3 \in R \). So we have

\[
d'_3 = d'_3 + [a_3, 3]_3, \\
= ([a, 1] + [a, 2] + [a, 3])_3, \\
= ([a, 1] * [a, 2] * [a, 3])_3,
\]

and so on.

The assumption of the above theorem is satisfied for a large class of rings (see for example [3], [4], [2]).

**V.**

We end this paper with the following three remarks.

**Remark 5.1.** Let \( a \in R \). If \( d = [a, 1]^{-1} \) then \( d_n(x) = xa^n - axa^{n-1} \), for \( n \geq 1 \), \( x \in R \).

**Remark 5.2.** Let \( a \in R \). Let \( d = (d_n)_{n \in \mathbb{N}} \) be the family of mappings from \( R \) to \( R \) defined by

\[
d_n(x) = x \\
d_1(x) = ax - xa
\]
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\[ d_n(x) = a^n x + x(-a)^n + \sum_{k=1}^{n-1} a^{n-k} x(-a)^k, \quad \text{for } n \geq 2. \]

Then \(d \in D_s(R)\) (in general) but \(\delta = (2d_n)_{n \geq s}\) is an inner derivation of order \(s\) of \(R\). Namely, \(\delta = [a, 1]^s [-a, 1]^{-1}\).

**Remark 5.3.** Let \(d \in D_s(R)\). Suppose that there exists an element \(a \in R\) such that \(d_n = a^{n-1} d_1\) for any \(n \in S \setminus \{0\}\). If the set \(d_s(R)\) contains a regular element then \(d = [a, 1]\).

**References**


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