Some Remarks on d-MP Rings

by

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Summary. In this paper we study d-MP rings, i.e. differential commutative rings with the condition that the radical of any differential ideal is again a differential ideal. We give a characterization of d-MP rings of characteristic \( n > 0 \) and a characterization of noetherian d-MP rings. We give also examples and some properties of d-MP rings.

1. Preliminaries. Throughout this paper all rings are commutative with identity. For any ring \( R \) and for any ideal \( A \) of \( R \), \( r(A) \) will denote the radical of \( A \). The term \( d\text{-ring} \) will refer to a ring \( R \) together with a specified derivation \( d: R \to R \). The \( d\text{-ring} \) \( R \) will be called an integral domain \( d\text{-ring} \), a noetherian \( d\text{-ring} \), or a Dedekind \( d\text{-ring} \) iff \( R \) is an integral domain, a noetherian ring, or Dedekind domain, respectively.

Let \( R \) be a \( d\text{-ring} \). An ideal \( A \) is called \( d\text{-ideal} \) if \( d(A) \subseteq A \). For an arbitrary subset \( T \) of \( R \) by \( [T] \) we denote the smallest \( d\text{-ideal} \) containing \( T \) and we set \( T_\# = \{ x \in R | d^n(x) \in T \text{ for all } n \geq 0 \} \).

A \( d\text{-ideal} \) \( A \) is \( d\text{-maximal} \) iff it is not contained in any proper, larger \( d\text{-ideal} \) of \( R \). Zorn's arguments imply that any \( d\text{-ideal} \) is contained in a \( d\text{-maximal} \) ideal and, since \( 0 \) is a \( d\text{-ideal} \), \( d\text{-maximal} \) ideals always exist.

If \( S = R (x_i : i \in I) \) is a ring of polynomials over \( R \) then the derivation \( d \) of \( R \) may be extended to \( S \) by setting for \( d (x_i) \) some element \( f_i \) of \( S \), for all \( i \in I \) (see [2]).

If \( S = R [x_1, \ldots, x_n] \) is a formal power series ring over \( R \) then the derivation \( d \) of \( R \) may be extended to \( S \) by setting for \( d (x_i) \) some element \( f_i \) of \( S \), for \( i = 1, \ldots, n \) (see [2]).

A \( d\text{-ring} \) \( R \) is called \( Ritt algebra \) iff \( R \) contains the field of rational numbers.

Throughout the rest of the paper \( R \) is a \( d\text{-ring} \).

2. A characterization of \( d\text{-MP rings} \). A \( d\text{-ideal} \) \( P \) in \( R \) will be called quasi-prime iff there is a multiplicative subset \( S \) of \( R \) such that \( P \) is maximal among \( d\text{-ideals} \) disjoint from \( S \) ([7, 10]).
Theorem 2.1. The following conditions are equivalent:
(1) Every quasi-prime ideal in \( R \) is prime.
(2) Every quasi-prime ideal in \( R \) is radical.
(3) Every prime ideal minimal over a \( d \)-ideal is a \( d \)-ideal.
(4) The radical of an arbitrary \( d \)-ideal is a \( d \)-ideal.
(5) For every prime ideal \( P \) in \( R \) the ideal \( P_\# \) is prime.

Proof. The equivalence of conditions (1), (3), (4) is given in [4]. The equivalence of conditions (1) and (2) follows immediately from the fact that every quasi-prime ideal is primary (see [8]). The equivalence of (1) and (5) is in [7].

A \( d \)-ring \( R \) is called a \( d \)-MP ring ([3], [4]) or a special \( d \)-ring (see [7]) if \( R \) satisfies the equivalent conditions listed in Theorem 2.1. Some of the properties of the \( d \)-MP rings are given in [3, 4, 7, 9]. A Ritt algebra is a \( d \)-MP ring (see [6]). Any \( d \)-field and every ring with the derivation equal to zero are \( d \)-MP rings.

Example 2.2. For a field \( K \) let \( R = K \langle x_i : i \in I \rangle \) be a polynomial ring over \( K \) and \( d : R \to R \) be a derivation of \( R \) such that \( d(K) = 0 \), \( d(x_i) = x_i \) for all \( i \in I \). Moreover, let \( A \) be an ideal in \( R \) generated by all variables of \( R \). Thus \( A^2 \) is a \( d \)-ideal and \( d \)-ring \( R/A^2 \) is a \( d \)-MP ring.

3. The \( d \)-MP rings of characteristic \( n > 0 \).

Theorem 3.1. If \( R \) is a ring of characteristic \( n > 0 \) then the following conditions are equivalent:
(1) \( R \) is a \( d \)-MP ring.
(2) Every prime ideal in \( R \) is a \( d \)-ideal.
(3) Every radical ideal in \( R \) is a \( d \)-ideal.

Proof. The equivalence (2) \( \iff \) (3) and the implication (3) \( \Rightarrow \) (1) are obvious. We prove implication (1) \( \Rightarrow \) (3). If \( a \) is an arbitrary element in \( R \) then the ideal \( (a^\alpha) \) is a \( d \)-ideal and from (1) it follows that \( r(a) = r(a^\alpha) \) is a \( d \)-ideal. Now let \( A \) be an arbitrary radical ideal in \( R \). If \( a \in A \) then \( r(a) \subseteq A \), thus \( d(a) \in r(a) \subseteq A \), i.e. \( d(A) \subseteq A \).

Corollary 3.2. If \( R \) is a unique factorization domain with characteristic \( n > 0 \) then the following conditions are equivalent:
(1) \( R \) is a \( d \)-MP ring.
(2) Every ideal in \( R \) is a \( d \)-ideal.

Proof. The implication (2) \( \Rightarrow \) (1) is obvious. Now, since every principal ideal of \( R \) is a product of prime ideals then from Theorem 3.1 it follows that every principal ideal of \( R \) is a \( d \)-ideal and thus every ideal is a \( d \)-ideal.

Corollary 3.3. Let \( R = K \langle x_i : i \in I \rangle \) be a polynomial ring over a field \( K \) and \( d : R \to R \) a nonzero derivation such that \( d(K) = 0 \). The following conditions are equivalent:
(1) \( R \) is a \( d \)-MP ring.
(2) \( K \) is of characteristic zero.
Proof. If $K$ is of characteristic zero then $R$ is a $d$-MP ring, since in this case $R$ is a Ritt algebra. Assume now that $R$ is a $d$-MP ring of characteristic $p > 0$. Since $d \neq 0$ then there is a variable $x_i$ such that $d \left( x_i \right) \neq 0$. If $d \left( x_i \right) = k \in K$ then $(x_i)$ is not a $d$-ideal and by Corollary 3.2 the ring $R$ is not a $d$-MP ring. Now suppose $d \left( x_i \right) = f$, where the polynomial $f$ has a variable $x_i$ such that the highest exponent of $x_i$ is $r > 0$. Consider the ideal $(x_i^p + x_i)$ and assume that $(x_i^p + x_i)$ is a $d$-ideal in $R$. Then $f = d \left( x_i^{rp} + x_i \right)$ belongs to $(x_i^p + x_i)$ and we have $rp = \deg_{x_i} \left( x_i^{rp} + x_i \right) \leq \deg_{x_i} f = r$, i.e. $rp \leq r$. Therefore $(x_i^{rp} + x_i)$ is not a $d$-ideal and from Corollary 3.2 the ring $R$ is not a $d$-MP ring.

Corollary 3.4. Let $R = K \left[ [x_1, \ldots, x_n] \right]$ be the formal power series ring over a field $K$ of characteristic $p$ and $d: R \to R$ a nonzero derivation of $R$ such that $d \left( K \right) = 0$. The following conditions are equivalent:

1. $R$ is a $d$-MP ring.
2. Either $p = 0$ or $p > 0$, $n = 1$, and $d \left( x_i \right) \in \left( x_i \right)$.

Proof. The implication (2) $\Rightarrow$ (1) is obvious. To prove (1) $\Rightarrow$ (2) suppose that $R$ is a $d$-MP ring, $p > 0$ and $n \geq 2$. Since $d \neq 0$, there is a variable $x_i$ such that $d \left( x_i \right) \neq 0$. From Corollary 3.2 it follows that for any natural number $n$ we have $f = d \left( x_i + x_i^p \right) \in \left( x_i + x_i^p \right)$. It is easy to see that if $n = m$ then the elements $x_i + x_i^p$, $x_i + x_i^{mp}$ have no common divisors. Therefore the element $f$ has an infinite number of common prime divisors. This contradicts the fact that $f \neq 0$. Consequently, if $p > 0$ then $n = 1$. Now, if $d \left( x_i \right) \notin \left( x_i \right)$ then $(x_i)$ is not a $d$-ideal and by Corollary 3.2 it follows that $R$ is not a $d$-MP ring.

If $R$ is a $d$-MP ring then every $d$-maximal $d$-ideal in $R$ is prime but the converse is not necessarily true.

Example 3.5. Let $R = K \left[ [x, y] \right]$ be a formal power series ring over the field $K$ of characteristic $p > 0$ and let $d: R \to R$ be a derivation of $R$ such that $d \left( K \right) = 0$, $d \left( x \right) = y$, $d \left( y \right) = x$. Since the ideal $(x, y)$ is a $d$-maximal $d$-ideal thus every $d$-maximal $d$-ideal in $R$ is prime. By Corollary 3.4 it follows that $R$ is not a $d$-MP ring.

4. Noetherian $d$-MP rings. If $R$ is a $d$-ring then a $d$-ideal $P$ is called $d$-prime iff $P \neq R$ and if for $d$-ideals $A, B$ of $R$ the relation $AB \subseteq P$ implies either $A \subseteq P$ or $B \subseteq P$.

Proposition 4.1. If every $d$-prime $d$-ideal of $d$-ring $R$ is prime then $R$ is a $d$-MP ring.

Proof. Every quasi-prime ideal is $d$-prime ([10] Theorem 4.1). Therefore every quasi-prime ideal in $R$ is prime.

The converse is not necessarily true.

Example 4.2. Let $T = Z_2 \left[ X_1, X_2, \ldots, Y_1, Y_2, \ldots \right]$ be a ring of polynomials over the field $Z_2$ and let $A$ be an ideal in $T$ generated by the squares
of all variables. Moreover, put \( R = T/A \) and let \( x_n = X_n + A, \ y_n = Y_n + A \) for all \( n \in N \). The ring \( R \) is local with the unique maximal ideal \( M = (x_1, x_2, \ldots, y_1, y_2, \ldots) \). Let \( d \) be a derivation of \( R \) such that \( d(x_n) = x_{n+1}, \ d(y_n) = y_{n+1} \) for every natural \( n \). \( R \) is therefore a \( d \)-ring and \( M \) is a \( d \)-ideal of \( R \). Since every quasi-prime ideal in \( R \) is equal to \( M \) ([10] Lemma 6.1) \( R \) is a \( d \)-MP ring. Here \( P = (y_1, y_2, \ldots) \) is a \( d \)-prime \( d \)-ideal in \( R \) ([10] Theorem 6.3) but it is not prime in \( R \).

Let \( A \) be a proper ideal in a ring \( R \) and let \( P \) be a prime ideal in \( R \). \( P \) is called a prime ideal associated with \( A \) iff there is \( x \in R \) such that \( P = (A:x) = \{ r \in R | rx \in A \} \) (see [13]).

**Theorem 4.3.** Let \( R \) be a noetherian \( d \)-ring. The following conditions are equivalent:

1. \( R \) is a \( d \)-MP ring.
2. Any prime ideal in \( R \) associated with a \( d \)-ideal is \( d \)-ideal.
3. Every \( d \)-prime \( d \)-ideal in \( R \) is prime.

**Proof.** (1) \( \Rightarrow \) (2). See [9], Corollary 7. (2) \( \Rightarrow \) (1). If prime ideals of \( R \) associated with \( d \)-ideals are prime then, in particular, prime ideals minimal over \( d \)-ideals are \( d \)-ideals and by Theorem 2.1 \( R \) is a \( d \)-MP ring. (3) \( \Rightarrow \) (1) follows from Proposition 4.1. (1) \( \Rightarrow \) (3). Since \( R \) is noetherian then \( d \)-prime \( d \)-ideals are quasi-prime ([10 Theorem 4.2]).

The implication (1) \( \Rightarrow \) (3) for noetherian Ritt algebras was proved by Jordan in [5].

**Corollary 4.4.** Let \( R \) be a noetherian \( d \)-MP ring and let \( x \) be an element of \( R \). If \( d(x) \) is not a zero-divisor in \( R \) then \( x \) is not one either.

**Proof.** In a noetherian ring \( R \) the set of all zero-divisors is the union of all prime ideals, associated with 0. Since 0 is a \( d \)-ideal, thus by Theorem 4.3 all prime ideals, associated with 0 are \( d \)-ideals. Therefore if \( x \) were zero-divisor then \( d(x) \) would be such too.

This Corollary for noetherian Ritt algebras was proved by Vasconcelos in [12]. If \( R \) is not \( d \)-MP ring then this corollary is not necessarily true, e.g. for \( R = Z_2 [X]/(X^2) \) with \( d(x) = 1 \), where \( x = X + (X^2) \), \( d(x) \) is not zero-divisor in \( R \) but \( x \) is such.

**Corollary 4.5.** Let \( R \) be a Dedekind \( d \)-ring. The following conditions are equivalent:

1. \( R \) is a \( d \)-MP ring.
2. Every proper \( d \)-ideal in \( R \) is a product of prime \( d \)-ideals.

**Proof.** (1) \( \Rightarrow \) (2). Let \( A \) be a \( d \)-ideal different from \( R \). Since \( R \) is a Dedekind domain then \( A = P_1^{e_1} \ldots P_n^{e_n} \), where \( P_1, \ldots, P_n \) are different prime ideals in \( R \) (see [13]). Thus we have \( r(A) = P_1 \cap \ldots \cap P_n \), i.e. the ideals \( P_1, \ldots, P_n \) are prime ideals associated with \( d \)-ideal \( r(A) \). By Theorem 4.3 ideals \( P_1, \ldots, P_n \) are \( d \)-ideals.

(2) \( \Rightarrow \) (1). Let \( A \) be a \( d \)-ideal different from \( R \). By (2) we have
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\[ A = P_{1}^{a_1} \ldots P_{n}^{a_n}, \text{where } P_1, \ldots, P_n \text{ are prime } d\text{-ideals in } R. \text{ Thus } r(A) = P_1 \cap \ldots \cap P_n \text{ is a } d\text{-ideal.} \]

**Lemma 4.6.** Let \( A \) be a \( d\)-ideal in a noetherian \( d\)-ring \( R \). For any \( x \in R \) there is a natural number \( k \) such that \((A:x^k)\) is a \( d\)-ideal in \( R \).

**Proof.** See [11] or [9].

**Theorem 4.7.** Let \( R \) be a noetherian \( d\)-MP ring and let \( A \) be a \( d\)-ideal different from \( R \). There are a natural number \( n \) and prime \( d\)-ideals \( P_1, \ldots, P_n \) in \( R \) such that \( P_1 \supseteq A \supseteq P_1 P_2 \ldots P_n \) for \( i = 1, \ldots, n \).

**Proof.** (i) If \( A \) is a primary \( d\)-ideal then \( P = r(A) \) is a prime \( d\)-ideal and \( P^k \subseteq A \subseteq P \) for some natural number \( k \). Therefore, for \( n = k \) and \( P_1 = P_2 = \ldots = P_n = P \) we are done. (ii) Assume now that there are \( d\)-ideals such that the theorem is not true. Let \( M \) be the family of all such \( d\)-ideals and let \( B \) be a maximal element of \( M \). By (i) it follows that \( B \) is not primary. Thus there are two elements \( x, y \) of \( R \) such that \( xy \in B \), \( x \notin B \), \( y \notin r(B) \). Consider \( d\)-ideals \( B_1 = B + [x] \), \( B_2 = (B: [x]) \).

If \( B_1 = R \) then \( 1 = b + u \), where \( b \in B \), \( u \in [x] \), and by Lemma 4.6 there is a natural number \( k \) such that \((B:y^k)\) is a \( d\)-ideal. Since \( xy \in B \), thus \( xy^k \in B \), i.e. \( x \in (B:y^k) \). Hence \([x] \subseteq (B:y^k) \), i.e. \( y^k [x] \subseteq B \). So we have \( y^k = y^k \).

If \( B_2 = R \) then \( 1 \in (B: [x]) \) and thus \( x \in B \).

Therefore the \( d\)-ideals \( B_1 \) and \( B_2 \) are different from \( R \). Since \( B_1 \nsubseteq B \), \( B_2 \nsubseteq B \) then \( B_1 \notin M \) and \( B_2 \notin M \). So there are prime \( d\)-ideals \( P_1, \ldots, P_n \), \( Q_1, \ldots, Q_m \) such that \( P_i \supseteq B_1 \supseteq P_1 \ldots P_n \) for \( i = 1, \ldots, n \), \( Q_j \supseteq B_2 \supseteq Q_1 \ldots Q_m \) for \( j = 1, \ldots, m \). Moreover,

\[ B_1 B_2 = (B + [x])(B: [x]) \subseteq B (B: [x]) + [x] (B: [x]) \subseteq B + B = B. \]

Thus we have

\[ P_i \supseteq B \supseteq B_1 B_2 \supseteq P_1 \ldots P_n Q_1 \ldots Q_m \text{ for } i = 1, \ldots, n, \]

\[ Q_j \supseteq B \supseteq B_1 B_2 \supseteq P_1 \ldots P_n Q_1 \ldots Q_m \text{ for } j = 1, \ldots, m. \]

But this contradicts the fact that \( B \) is in \( M \).

5. **Integral extensions.** Let \( R \) and \( S (R \subseteq S) \) be \( d\)-rings such that the derivation of \( R \) is the restriction to \( R \) of the derivation of \( S \). In [3] Gorman proved the following

**Lemma 5.1.** For \( d\)-MP rings \( R \subseteq S \) such that \( S \) is an integral over \( R \), let \( P \) be a prime ideal of \( R \) and let \( Q \subseteq S \) be a prime ideal lying over \( P \). Then \( P \) is a \( d\)-ideal if and only if \( Q \) is.

Now we give some extra information on the integral extensions of \( d\)-MP rings.

**Proposition 5.2.** Let \( R \subseteq S \) be \( d\)-rings with \( S \) integral over \( R \). If \( S \) is a \( d\)-MP ring then \( R \) is such too.
Proof. For a multiplicative subset \( T \) of \( R \), let \( A \) be a \( d \)-ideal in \( R \) disjoint from \( T \). We prove that a \( d \)-ideal \( B \) in \( R \), maximal among \( d \)-ideals of \( R \) containing \( A \) and disjoint from \( T \), is prime. Clearly, \( SB \) is a \( d \)-ideal disjoint from \( T \). Since \( T \) is also a multiplicative subset of \( d \)-MP ring \( S \) then a \( d \)-ideal \( Q \) in \( S \), maximal among \( d \)-ideals containing \( SB \) and disjoint from \( T \), is prime. Let \( P = R \cap Q \). Thus \( P \) is a prime \( d \)-ideal in \( R \) and \( P \cap T = 0 \), \( B \subset P \). Therefore \( B = P \), i.e. \( B \) is a prime ideal.

If a \( d \)-ring \( R \) has no \( d \)-ideals different from \( R \) and \( O \) then \( R \) is called a \( d \)-simple \( d \)-ring. A \( d \)-ideal \( A \) of \( d \)-ring \( R \) is a \( d \)-maximal \( d \)-ideal if and only if the \( d \)-ring \( R/A \) is \( d \)-simple. The intersection of all \( d \)-maximal \( d \)-ideals of \( d \)-ring \( R \), denoted by \( Jd \ (R) \), is called the Jacobson \( d \)-radical (see [3, 4, 9]).

Proposition 5.3. Let \( R \subset S \) be \( d \)-rings with \( S \) integral over \( R \).

1. If \( S \) is \( d \)-simple then \( R \) is such too.
2. If \( S \) is an integral domain and \( R \) is \( d \)-simple then \( S \) is \( d \)-simple.

Proof. (1). Let \( A \) be a \( d \)-ideal different from \( R \). Thus \( SA \) is a \( d \)-ideal in \( S \) different from \( S \). Since \( S \) is \( d \)-simple then \( SA = 0 \), i.e. \( A = 0 \).

(2). Let \( B \) be a \( d \)-ideal in \( S \) different from \( S \). Thus \( A = B \cap R \) is a \( d \)-ideal in \( R \) different from \( R \). Since \( R \) is \( d \)-simple then \( A = 0 \). Let \( Q \) be maximal among ideals in \( S \) containing \( B \) which lie over 0. Thus we have \( O = O \cap R = Q \cap R \), \( O \subset Q \) and \( O, Q \) are prime ideals in \( S \). By [1], Corollary 5.9, \( O = Q \), i.e. \( B = O \).

Corollary 5.4. Let \( R \subset S \) be \( d \)-rings with \( S \) integral over \( R \), let \( Q \) be a \( d \)-ideal in \( S \) and \( P = Q \cap R \).

1. If \( Q \) is a \( d \)-maximal \( d \)-ideals in \( S \) then \( P \) is \( d \)-maximal in \( R \).
2. If \( Q \) is prime and \( P \) is \( d \)-maximal in \( R \) then \( Q \) is \( d \)-maximal in \( S \).

Corollary 5.5. If \( R \subset S \) are \( d \)-rings with \( S \) integral over \( R \) then \( Jd \ (R) = Jd \ (S) \cap R \).

Theorem 5.6. Let \( R \subset S \) be \( d \)-MP rings and let \( S \) be integral over \( R \). For every \( d \)-maximal \( d \)-ideal \( P \) in \( R \) there is a \( d \)-maximal \( d \)-ideal \( Q \) in \( S \) which lies over \( P \).

Proof. Let \( P \) be \( d \)-maximal in \( R \). Since \( R \) is \( d \)-MP ring \( P \) is prime. Since \( S \) is an integral over \( R \) then there is a prime ideal \( Q \) in \( S \) which lies over \( P \). By Lemma 5.1 it follows that \( Q \) is a \( d \)-ideal and by Corollary 5.4 \( Q \) is a \( d \)-maximal \( d \)-ideal.

Corollary 5.7. If \( R \subset S \) are \( d \)-MP rings and \( S \) is integral over \( R \) then \( Jd \ (R) = Jd \ (S) \cap R \).
REFERENCES


А. Новицкий. Некоторые замечания о d-MP кольцах

В этой работе рассматриваются d-MP кольца, т.е. коммутативные дифференциальные кольца, у которых радикал произвольного дифференциального идеала также является дифференциальным идеалом.

Доказывается несколько необходимых и достаточных условий для того, чтобы: 1) дифференциальное кольцо положительной характеристики, 2) дифференциальное неётерово кольцо, являлись d-MP кольцами.

Доказывается также несколько свойств таких колец.