ANDRZEJ NOWICKI (Toruń)

The primary decomposition of differential modules

1. Introduction. In [8] A. Seidenberg proved the following theorem: for any noetherian Ritt algebra each differential ideal $A$ has an irredundant primary decomposition $A = A_1 \cap \ldots \cap A_s$, where $A_1, \ldots, A_s$ are differential ideals.

A more general case is presented in [1]. In [7] the above theorem was proved by S. Sato for arbitrary noetherian differential rings.

In this paper, using methods similar to those of S. Sato, we prove that: if $R$ is a noetherian differential ring and $M$ is a differential $R$-module finitely generated over $R$, then any differential submodule $N$ of $M$ has an irredundant primary decomposition $N = N_1 \cap \ldots \cap N_s$, where all $N_i$ are differential submodules.

From this fact a number of interesting conclusions follow concerning differential modules over a noetherian $d-MP$-ring.

In the last section we show an example of a differential ring for which the Differential Nakayama Lemma does not hold and a particular version of this lemma is given.

The author wishes to thank Professor S. Balcerzyk for many valuable discussions and criticism which helped to improve the text.

2. Preliminary notions. A differential ring (shortly: a d-ring) is a pair $(R, d)$, where $R$ is a commutative ring with unit and $d: R \to R$ is a mapping, called derivation, which satisfies the conditions:

$$d(r+s) = d(r) + d(s), \quad d(rs) = rd(s) + sd(r) \quad \text{for arbitrary } r, s \in R.$$ 

A differential module (shortly: a d-module) over a d-ring $(R, d)$ is a pair $(M, \delta)$, where $M$ is a $R$-module and $\delta: M \to M$ is a mapping which satisfies the conditions: $\delta(m + n) = \delta(m) + \delta(n), \quad \delta(mn) = r\delta(m) + d(r)m$ for arbitrary $m, n \in M, r \in R$.

Let $(R, d)$ be a d-ring and $(M, \delta)$ a d-module over $(R, d)$. An ideal $A$ in $R$ is called a d-ideal if $d(A) \subseteq A$. Similarly a submodule $N$ of $M$ is called a d-submodule if $\delta(N) \subseteq N$. 
If $A$ is a $d$-ideal in $R$, then $AM$ is a $d$-submodule of $M$. If $N$ and $P$ are $d$-submodules of $M$, then $(N: P) = \{ r \in R; rP \subset N \}$ is a $d$-ideal in $R$. Similarly, if $A$ is a $d$-ideal and $N$ a $d$-submodule, then $(N: A) = \{ m \in M; Am \subset N \}$ is a $d$-submodule.

For an arbitrary subset $T$ of $R(M)$ by $[T]$ we denote the smallest $d$-ideal (or sub-module) containing $T$.

We say that a $d$-module $M$ is $d$-finitely generated if there is a finite number of elements $m_1, \ldots, m_n \in M$ such that $M = [m_1, \ldots, m_n]$. The $d$-ring $(R, d)$ is called a $d$-MP ring if a radical of an arbitrary $d$-ideal in $R$ is a $d$-ideal. Equivalent definitions of a $d$-MP ring may be found in [3]. If the $d$-ring $(R, d)$ contains the field of rational numbers $Q$, then we call it a Ritt algebra. Every Ritt algebra is a $d$-MP ring. A $d$-ideal $A$ is $d$-maximal if it is maximal among all $d$-ideals in $R$ different from $R$. If $R$ is a $d$-MP ring, then $d$-maximal ideals are prime (see [3]).

With every $d$-ring $(R, d)$ we associate some ring (non-commutative in general) $D = D(R, d)$ (see [4], [5]) which is a left free $R$-module having basis $\{ 1, t, t^2, \ldots \}$, with the multiplication defined by: $r \cdot t = rt$, $t^n \cdot t^m = t^{n+m}$, $t \cdot r = d(r) + rt$. If $(M, \delta)$ is a $d$-module over $(R, d)$, then $M$ together with the multiplication $(r_n t^n + \ldots + r_0 m = r_n \delta^n(m) + \ldots + r_0 m$ is a left $D$-module. If $M$ is a $D$-module, then the mapping $\delta: M \to M$, $\delta(m) = tm$, makes $(M, \delta)$ a $d$-module over $(R, d)$. Any $d$-module over $(R, d)$ is $d$-finitely generated if it is finitely generated as $D(R, d)$-module.

For a $R$-module $M$ by $\text{Ass}_R(M)$ we denote the set of all prime ideals in $R$ associated with $M$ (see [5]).

3. Primary decomposition. Let $(R, d)$ be a noetherian $d$-ring, $(M, \delta)$ a $d$-module finitely generated over $R$, and $N$ a $d$-submodule of $M$.

**Lemma 1.** For any $x \in R$ there is a natural number $k$ such that $(N: x^k)$ is a $d$-submodule of $M$ and $(N: x^n) = (N: x^k)$ for any $n \geq k$.

**Proof.** For any $m \in U = \bigcup_{s=0}^{\infty} (N: x^s)$ we have $x^s m \in N$ for some $s$ and then the element $\delta(x^s m) = x^s \delta(m) + sx^{s-1} d(x) m$ is in $N$, thus $x^{s+1} \delta(m) \in N$, i.e. $\delta(m) \in U$. It means that $U$ is a $d$-submodule of $M$. It suffices now to consider the sequence $(N: x^1) \subset (N: x^2) \subset \ldots$

**Definition 2.** A $d$-submodule $N$ of $M$ is $d$-primary if for any $d$-ideal $A$ and any $d$-submodule $P$ of $M$, from $AP \subset N$ it follows that either $P \subset N$ or $A^n \subset (N: M)$ for some natural number $n$.

**Definition 3.** A $d$-submodule $N$ of $M$ is $d$-irreducible if it is not an intersection of two $d$-submodules different from $N$.

**Lemma 4.** If $N$ is a $d$-primary $d$-submodule, then it is a primary submodule.

**Proof.** Let for given $r \in R, m \in M$, the element $rm$ be in $N$. We must
show that either \( m \in N \) or \( r \in \sqrt{(N: M)} \). By Lemma 1 there is a natural number \( k \) such that \((N: r^k)\) is a \( d \)-submodule of \( M \). \( rm \in N \) implies \( m \in (N: r^k) \) since \( r \in (N: r^k) \) and hence \([m] \subseteq (N: r^k)\) and therefore \( r^k \in (N: [m]) \).

Now, \((N: [m])\) is a \( d \)-ideal in \( R \), thus \([r^k] \subseteq (N: [m])\), i.e. \([r^k][m] \subseteq N\). Since \( N \) is \( d \)-primary, we have either \([m] \subseteq N\) or \([r^k]^n \subseteq (N: M)\), i.e. \( m \in N \) or \( r \in \sqrt{(N: M)} \).

**Lemma 5.** If \( N \) is \( d \)-irreducible \( d \)-submodule of \( M \), then \( N \) is a \( d \)-primary \( d \)-submodule.

**Proof.** Assume that for a \( d \)-ideal \( A \) and a \( d \)-submodule \( P \) we have \( AP \subseteq N \) and \( A \nsubseteq \sqrt{(N: M)} \). Let \( N = N_1 \cap \ldots \cap N_k \) be a primary decomposition of \( N \). Since \( A \nsubseteq \sqrt{(N_i: M)} \) for some \( i \), we have \( A \nsubseteq \sqrt{(N_i: M)} \) for \( i = 1, 2, \ldots, s \) and \( A \subseteq \sqrt{(N_j: M)} \) for \( j = s+1, \ldots, k \).

If \( s = k \), then, for any \( i = 1, 2, \ldots, k \), \( (N_i: A) = N_i \) and therefore \( P \subseteq (N: A) = \bigcap \limits_{i=1}^{k} (N_i: A) = \bigcap \limits_{i=1}^{k} N_i = N \). Assume that \( s < k \). Since \( R \) is noetherian, there is a natural number \( n \) such that \( A^n \subseteq (N_j: M) \) for \( j = s+1, \ldots, k \). In this case \((N_i: A^n) = N_i \) for \( i = 1, 2, \ldots, s \) and \((N_j: A^n) = M \) for \( j = s+1, \ldots, k \). Thus we have

\[
N \subseteq (N: A^n) \cap (N + A^n M) \subseteq \bigcap \limits_{i=1}^{s} N_i \cap \bigcap \limits_{j=s+1}^{k} N_j = N,
\]
i.e. \( N = (N: A^n) \cap (N + A^n M) \).

Since \( AP \subseteq N \), we have \( A^n P \subseteq N \) and \( N + P \subseteq (N: A^n) \). Therefore

\[
N \subseteq (N + P) \cap (N + A^n M) \subseteq (N: A^n) \cap (N + A^n M) = N,
\]
i.e. \( N = (N + P) \cap (N + A^n M) \).

By \( d \)-irreducibility of \( N \neq N + A^n M \) we have that \( N = N + P \), i.e. \( P \subseteq N \).

**Theorem 6.** Let \((R, d)\) be a noetherian \( d \)-ring and \((M, \delta)\) a \( d \)-module finitely generated over \( R \). Then any \( d \)-submodule \( N \) of \( M \) has an irredundant primary decomposition \( N = N_1 \cap \ldots \cap N_n \) such that \( N_i \) are \( d \)-submodules of \( M \).

**Proof.** Using Lemmas 4 and 5 the argument is standard.

**4. Conclusions from Theorem 6 for noetherian \( d \)-\( MP \) rings.** We assume now that \( R \) is a noetherian \( d \)-\( MP \) ring and \( M \) is a \( d \)-module finitely generated over \( R \).

From Theorem 6 we have an immediate

**Corollary 7.** Any prime ideal associated with a \( d \)-module \( M \) is a \( d \)-ideal.

**Lemma 8.** For any \( m \in M \) if \((\alpha: m)\) is a \( d \)-ideal, then \((\alpha: m) = (\alpha: [m])\).
Proof. See [2], Lemma 2.

**Lemma 9.** If $M \neq 0$, then there exists a $d$-submodule $N \neq 0$ and a prime $d$-ideal $P$ such that $N$ is a torsion-free $d$-submodule over $R/P$.

**Proof.** Let $P$ be a maximal ideal in the family $\{(o; m); o \neq m \in M\}$. It is known that $(o; x) = P$ is a prime ideal. By Corollary 7, $P$ is a $d$-ideal.

Put $N = [x]$. Clearly, $N$ is a non-zero $d$-submodule and, by Lemma 8, $P = (o; x) = (o; [x]) = (O; N)$, thus $PN = 0$, i.e. $N$ is a $d$-module over the $d$-ring $R/P$.

Now assume that $rn = o$, $r \in R \setminus P$, $o \neq n \in N$. Then $r \in (o; n)$, $r \notin (o; x) = P$, which gives $(o; x) \not\subset (o; n)$, contrary to the maximality of $(o; x)$.

**Corollary 10.** If $M \neq 0$, then there exist a sequence of $d$-submodules $0 = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_k = M$ and a sequence of prime $d$-ideals $P_1, \ldots, P_k$ in $R$ such that $M_i/M_{i-1}$ is a torsion-free $d$-module over the $d$-ring $R/P_i$, $i = 1, \ldots, k$.

**Proof.** Let $N$ and $P$ be as in Lemma 9. We put $M_1 = N$ and $P_1 = P$. Then $M_1/M_0 = N$ is a torsion-free $d$-module over $R/P_1$. If $M_1 = M$, then there is nothing more to do. If $M_1 \not\subsetneq M$, then we apply Lemma 9 to the $d$-module $M/M_1 \neq 0$. Thus there exist a $d$-submodule $N_1 \neq 0$ of $M/M_1$ and a prime $d$-ideal $P_2$ such that $N_1$ is a torsion-free $d$-module over $R/P_2$. We take $M_2 = \varphi^{-1}(N_1)$, where $\varphi: M \to M/M_1$ is canonical. So we have $0 \subsetneq M_1 \subsetneq M_2$ and $M_2/M_1 = N_1$ is torsion-free $d$-module over $R/P_2$. Since $M$ is noetherian, this procedure ends.

**Corollary 11.** Assume that $M \neq 0$ is a $d$-simple $d$-module (i.e. $M$ is without any proper $d$-submodels). Then

1. $(O; M)$ is a prime $d$-ideal,
2. $M$ is a torsion-free $d$-module over $R/(O; M),$
3. for any $o \neq m \in M$, we have $(o; m) = (O; M)$.

**Proof.** (1) Since $M \neq 0$, the set $\text{Ass}_R(M)$ is non-empty. Let $P = (o; m) \in \text{Ass}_R(M)$. By Corollary 7, $P$ is a $d$-ideal. Thus Lemma 8 implies that $(O; M) = (O; [m]) = (o; m) = P$ is a prime $d$-ideal;

(2) Follows from (1) and from the proof of Lemma 9;

(3) For such $m$, since $[m] = M$, we have $(O; M) = (O; [m]) \subset (o; m)$. Assume that $(O; M) \subsetneq (o; m)$ and take $x \in (o; m)$ such that $x \notin (O; M)$. Since $M$ is $d$-simple, $O$ is a $d$-primary $d$-submodule, and by Lemma 4 it is a primary submodule of $M$. But $xm = o$; hence $m = o$ or $x \in \sqrt{(O; M)} = (O; M)$, a contradiction.

**Corollary 12.** If for all $d$-maximal $d$-ideals $\frak{M}$ in $R$, $M_{\frak{M}} = 0$, then $M = 0$.

**Proof.** Assume that $M \neq 0$. Then there is a prime $d$-ideal $P$ of the form $P = (o; x)$, for some $x \in M$, $x \neq o$, since $\text{Ass}_R(M) \neq \emptyset$. Let $\frak{M}$ be
a $d$-maximal $d$-ideal containing $P$. Then $M_{dR} = 0$, thus $x/l = o$ in $M_{dR}$. It follows now that, for some $a \in R \setminus M$, $ax = 0$, hence $s \in (o : x) = P \subset M$, a contradiction.

5. The Differential Nakayama Lemma. Let $Jd(R)$ denote the intersection of all $d$-maximal $d$-ideals of the $d$-ring $(R, d)$. We call $Jd(R)$ the Jacobson $d$-radical.

Definition 13. We say that $d$-ring $(R, d)$ satisfies the Differential Nakayama Lemma if for any $d$-ideal $A \subset Jd(R)$ and any $d$-finitely generated $d$-module $M$ the condition $AM = M$ implies $M = 0$.

Now we give an example of a $d$-ring which does not satisfy the Differential Nakayama Lemma.

Example 14. Let $k$ be a field of characteristics zero, $R = k[x]$ a ring of polynomials in one variable $x$ over $k$ and let $d(x) = x$, $d(k) = 0$. Since the only $d$-maximal $d$-ideal in the $d$-ring $(R, d)$ is $(x)$, we have $Jd(R) = (x)$.

Note that for any $w \in D = D(R, d)$ there is $w' \in D$ such that $wx = xw'$. Indeed,

(a) if $w \in R$, then $wx = xw$,

(b) since $tx = d(x) + xt = x + xt = x(1 + t)$, we have $t^n \cdot x = x(1 + t)^n$,

(c) if $w = r_0 t_1 t + \ldots + r_n t^n$ is an arbitrary element of $D$, then

$$wx = \left( \sum_{i=0}^{n} r_i t^i \right) x = \sum_{i=0}^{n} r_i x(1 + t)^i = x\left( \sum_{i=0}^{n} r_i (1 + t)^i \right).$$

Let $f : k[x] \rightarrow k$ be such a homomorphism of rings that $f(x) = 1$ and $f(k) = k$ for any $k \in k$. The homomorphism $f$ induces on $k$ a structure of $R$-module given $wk = f(w)k$.

Put $M = D \otimes_R k$. Since $M$ is a left $D$-module generated by the element $1 \otimes 1$, $M$ is a $d$-module $d$-finitely generated over $(R, d)$. We show now that $(x)M = M$. Take $m \in M$. Then $m = w(1 \otimes 1)$ for some $w \in D$. Thus we have:

$$m = w(1 \otimes 1) = w(1 \otimes x \cdot 1) = w(1 \otimes f(x) \cdot 1) = w(1 \otimes x \cdot 1)$$

$$= w(1 \cdot x \otimes 1) = wx(1 \otimes 1) = xw'(1 \otimes 1), \quad \text{i.e.} \quad m \in (x)M.$$

This proves that the $d$-ring $(R, d)$ does not satisfy the Differential Nakayama Lemma.

With some limitations on $d$-ring $R$ and $d$-module $M$ one may prove the following version of the Differential Nakayama Lemma, different from previous one.

Proposition 15. Let $(R, d)$ be a noetherian $d$-MP ring and $(M, \delta)$ a $d$-module finitely generated over $R$. If $A$ is $d$-ideal such that $A \subset Jd(R)$ and $AM = M$, then $M = 0$. 
Proof. If $\mathfrak{M}$ is an arbitrary $d$-maximal $d$-ideal in $R$, then $A \subset Jd(R) \subset \mathfrak{M}$, $A_{SR} M_{SR} = M_{SR}$ and $A_{SR} \subset \mathfrak{M} R_{SR}$. From the Nakayama Lemma, $M_{SR} = 0$; hence by Corollary 12, $M = 0$.

References


INSTITUTE OF MATHEMATICS, N. COPERNICUS UNIVERSITY, TORUŃ