RADICAL IRREGULARITY OF SOME POLYNOMIAL RINGS

BY

ANDRZEJ NOWICKI (TORUŃ)

1. Introduction. In [1] Gorman gave an example of a non-trivial differential ring of polynomials over a Ritt algebra, with an infinite set of variables, which is a radically regular ring.

In the present paper we prove at first that there is no non-trivial differential ring of polynomials over a differential ring being an integral domain with characteristics $p > 0$, which is a radically regular ring. Then we show that there is also no non-trivial differential ring of polynomials in finite number of variables over a Ritt algebra being an integral domain with finite Krull dimension, which is a radically regular ring.

I wish to thank Professor S. Balcerzyk for valuable remarks and encouragement during the preparation of this paper.

2. Preliminary notions. A differential ring is a pair $(R, d)$, where $R$ is a commutative ring with unit and $d: R \to R$ is a mapping, called derivation, satisfying the conditions

$$d(r + s) = d(r) + d(s) \quad \text{and} \quad d(rs) = rd(s) + s d(r)$$

for arbitrary $r, s \in R$.

Let $(R, d)$ be a differential ring. An ideal $U$ in $R$ is called differential if $d(U) \subseteq U$. For an arbitrary $T \subseteq R$, we denote by $(T)$, $[T]$ and $\{T\}$ the smallest ideal, the smallest differential ideal and the smallest radical differential ideal containing the set $T$, respectively. For an arbitrary Noetherian differential ring, any radical differential ideal is an intersection of a finite number of prime differential ideals (see [3]). $(R, d)$ is called a Ritt algebra if $R$ contains the field $\mathbb{Q}$ of rational numbers. In Ritt algebras any maximal differential ideal is prime (see [2]). $(R, d)$ is called a radically regular ring if, for any $r \in R$, $\{r\} = R$ implies $(r) = R$ (see [1] and [2]). In particular (see [2], Lemma 4), a Ritt algebra is a radically regular ring if and only if, for any $r \in R$, $[r] = R$ implies $(r) = R$. Let $S = R[x_t : t \in T]$ be a polynomial ring over $(R, d)$. The derivation $d$ can be extended to $S$ by setting, for $d(x_t)$, some polynomial $f_t \in S$ ($t \in T$). We say that $(S, d)$
is a non-trivial differential ring if \( d(x_0) \neq 0 \) for some \( t_0 \in T \). For any polynomial \( g \in \mathbf{Q}[x] \), the \( n \)-th derivative of \( g \) will be denoted by \( g^{(n)} \).

3. Polynomial rings over rings with characteristic \( p > 0 \).

**Proposition 1.** Let \((R, d)\) be a differential ring being an integral domain with characteristic \( p > 0 \) and let \( S = R[x_1; t \in T] \) be a non-trivial differential ring of polynomials over \((R, d)\). Then \((S, d)\) is not a radically regular ring.

**Proof.** There exists a variable \( x_0 \) such that \( d(x_0) \neq 0 \). Put \( x_0 = x \) and \( d(x) = f \) and consider the element \( xf^p + 1 \). Obviously, the ideal \((xf^p + 1)\) is distinct from \( R \). Since the element \( f^{p+1} = d(xf^p + 1) \) belongs to the radical ideal \((xf^p + 1)\), so does \( f \); hence \((xf^p + 1) = R \).

4. Polynomial rings over a Ritt algebra. Let \((R, d)\) be a Ritt algebra being an integral domain with finite Krull dimension (as a commutative ring) and let \( S = R[x_1, \ldots, x_m] \) \((m \geq 1)\) be a non-trivial differential ring of polynomials in finite number of variables over \((R, d)\). Assume that \( x_1 = x \) and \( d(x) = f \neq 0 \). We define the sequence \( u_0, u_1, \ldots \) of elements from \( S \) as follows:

\[
\begin{align*}
\quad u_0 & = -1, \\
u_n & = fd(u_{n-1}) - (2n-1)u_{n-1}d(f) \quad \text{for } n \geq 1.
\end{align*}
\]

**Lemma 1.** Let \( g \) be a polynomial from \( \mathbf{Q}[x] \), \( U \) a differential ideal in \((S, d)\), and \( n \) a natural number. If \( u_n + gf^{2n+1} \) belongs to \( U \), then so does \( u_{n+1} + g^{(1)}f^{2n+3} \).

**Proof.** Since \( fd(u_n + gf^{2n+1}) = fd(u_n) + (2n+1)gf^{2n+1}d(f) + g^{(1)}f^{2n+3} \) belongs to \( U \), so does the element \( u_{n+1} + g^{(1)}f^{2n+3} = fd(u_n) - (2n+1)u_n d(f) + g^{(1)}f^{2n+3} \).

**Lemma 2.** For any natural number \( n \) and any polynomial \( g \in \mathbf{Q}[x] \) the element \( u_n + g^{(n)}f^{2n+1} \) belongs to the ideal \([gf-1]\).

**Proof.** For \( n = 0 \), \( u_0 + g^{(0)}f^{2.0+1} = -1 + gf \) is clearly in \([gf-1]\). Assume that \( u_n + g^{(n)}f^{2n+1} \) belongs to \([gf-1]\); then, by Lemma 1, \( u_{n+1} + g^{(n+1)}f^{2n+3} \) is in \([gf-1]\).

**Lemma 3.** Let \( w \) be a polynomial in \( \mathbf{Q}[x] \), \( n \) any fixed natural number, and \( T \) an infinite subset of \( \mathbf{Q} \). Moreover, for any \( t \in T \) let \( u_n + (w + t)f^{2n+1} \) belong to one of non-zero prime differential ideals \( P_t \) in \( S \), and let \( f \notin P_t \). Then there exists a prime differential ideal \( P \) which contains \( u_{n+1} + w^{(1)}f^{2n+3} \) and is essentially contained in some ideal \( P_t \).

**Proof.** Since \( u_n + (w+t)f^{2n+1} \) is in \( P_t \) for any \( t \in T \), Lemma 1 implies that the polynomial

\[
\begin{align*}
u_{n+1} + w^{(1)}f^{2n+3} & = u_{n+1} + (w+t)^{(1)}f^{2n+3}
\end{align*}
\]
belongs to all $P_i$, and so it is in a differential ideal

$$I = \bigcap_{i \in T} P_i.$$ 

$I$ is a radical differential ideal in a Noetherian ring $S$ and, consequently, $I = I_1 \cap \ldots \cap I_k$, where $I_j$ $(1 \leq j \leq k)$ are prime differential ideals. Since $P_t \supseteq I$ for all $t \in T$, each $P_t$ contains some ideal $I_{t_i}$ $(1 \leq i \leq k)$. The set $T$ is infinite, and so there exist two different elements $s, t \in T$ such that $I_{t_s} = I_{t_t}$. Put $P = I_{t_s} = I_{t_t}$; then, clearly $P_s \supseteq P$ and $P_t \supseteq P$. If $P = P_s$ and $P = P_t$, then $u_n + (w + s)f^{2n+1}$ and $u_n + (w + t)f^{2n+1}$ belong to $P_t$. Thus $(s - t)f^{2n+1}$ is in $P_t$, and so $f$ belongs to $P$, contrary to the assumption. Hence $P$ is essentially contained in one of the ideals $P_t$ and $P_s$ and, obviously, $u_{n+1} + w^{(1)}f^{2n+3}$ belongs to $P$.

**Theorem 1.** If $(R, d)$ is a Ritt algebra being an integral domain with finite Krull dimension and $S = R[x_1, \ldots, x_m]$ $(m \geq 1)$ is a non-trivial differential ring $(S, d)$ of polynomials, then the ring $(S, d)$ is not radically regular.

**Proof.** Assume that $(S, d)$ is radically regular.

(A) First we show that, for any polynomial $g \in Q[x]$ of degree greater than zero and for any natural number $n$, elements $u_n + g(n)f^{2n+1}$ are non-zero polynomials. Assume that $u_n + g(n)f^{2n+1} = 0$ for some $n$ and put $h = g + x^n$. Since $(S, d)$ is a radically regular ring and since the polynomial $hf - 1$ is not invertible in $S$, there exists a prime differential ideal $P$ such that $[hf - 1] \subseteq P$. This fact and Lemma 2 imply that the element

$$u_n + h(n)f^{2n+1} = u_n + g(n)f^{2n+1} + (n!)f^{2n+1} = (n!)f^{2n+1}$$

belongs to $P$ and, consequently, $f \in P$. But $hf - 1 \in P$ implies $P = S$, which is impossible.

(B) Since the Krull dimension of $R$ is finite, so is the Krull dimension of $S$. Let $\dim S = u$ and let $v$ be a fixed natural number such that $v \geq u$. Clearly, $v \geq 1$.

We will write any polynomial $g = a_rx^r + \ldots + a_0$, where $a_r \in Q$, $a_r \neq 0$, in the form $g = g(a_r, \ldots, a_0)$ and its $k$-th derivative, for $k < r$, as $g^{(k)} = g_k(a_r, \ldots, a_k)$.

Now consider all polynomials $g \in Q[x]$ of degree $v$. Since for any $a_v$, $a_{v+1}$, $a_0$, where $a_v \neq 0$, the polynomial $g(a_v, \ldots, a_0)f - 1$ is not invertible in $S$, by the assumption that $S$ is radically regular there exist prime differential ideals $P(a_v, \ldots, a_0)$ such that

$$g(a_v, \ldots, a_0)f - 1 \in P(a_v, \ldots, a_0).$$

All these prime ideals do not contain $f$ and, by part (A) of this proof, they are non-zero ideals. Having fixed elements $a_v, \ldots, a_1$ and taking $T = Q$ we see that the assumptions of Lemma 3 are satisfied. Thus there-
exists a prime differential ideal \( P(a_v, \ldots, a_1) \) containing the element 
\( u_1 + g_1(a_v, \ldots, a_1) f^3 \) and essentially contained in some prime ideal 
\( P(a_v, \ldots, a_1, b_0(a_v, \ldots, a_1)) \). Running over sequences \((a_v, \ldots, a_1)\), where 
\( a_v \neq 0 \), we get ideals \( P(a_v, \ldots, a_1) \) for which all assumptions of Lemma 3 
are satisfied. So we get new prime ideals \( P(a_v, \ldots, a_2) \) and the inclusion 
\[ P(a_v, \ldots, a_2) \subseteq P(a_v, \ldots, a_2, b_1(a_v, \ldots, a_2)). \]

Thus 
\[ P(a_v, \ldots, a_2) \subseteq P(a_v, \ldots, a_2, b_1(a_v, \ldots, a_2)) \]
\[ \subseteq P(a_v, \ldots, a_2, b_1(a_v, \ldots, a_2), b_0(a_v, \ldots, a_2, b_1(a_v, \ldots, a_2))). \]

Continuing this process we get the following sequence of \( v+1 \) prime ideals:
\[ 0 \subseteq P(b^0_v) \subseteq P(b^0_v, b^0_{v-1}) \subseteq \cdots \subseteq P(b^0_v, \ldots, b^0_1, b^0_0). \]

This contradicts \( \text{dim} S = u < v+1 \).

**REFERENCES**


INSTITUTE OF MATHEMATICS
N. COPERNICUS UNIVERSITY, TORUŃ

Reçu par la Rédaction le 26. 1. 1977