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Optimal jury design for homogeneous juries with correlated votes

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Abstract In a homogeneous jury, in which each vote is correct with the same probability, and each pair of votes correlates with the same correlation coefficient, there exists a correlation-robust voting quota, such that the probability of a correct verdict is independent of the correlation coefficient. For positive correlation, an increase in the correlation coefficient decreases the probability of a correct verdict for any voting rule below the correlation-robust quota, and increases that probability for any above the correlation-robust quota. The jury may be less competent under the correlation-robust rule than under simple majority rule and less competent under simple majority rule than a single juror alone. The jury is always less competent than a single juror under unanimity rule.

Keywords Dichotomous choice · Condorcet's Jury Theorem · Correlated votes

1 Introduction

Let us suppose that an executive contemplates hiring a group of experts. The executive wants to maximize the probability of the group collectively making the correct decision. How large should the group be and how should the expert opinions be pooled into a collective judgment?

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Condorcet's Jury Theorem answers the first question, and indirectly, also the second one. The experts are jurors who share the goal of convicting the guilty and acquitting the innocent, provided that the following five assumptions apply: (1) the jury must choose between two alternatives (one of which is correct); (2) the jury reaches its verdict by simple majority vote; (3) each juror is competent (i.e., is more likely than not to vote correctly); (4) all jurors are equally competent (have equal probabilities); and (5) each juror decides independently of all other jurors, then

1. any jury comprising an odd number greater than one of jurors is more likely to select the correct alternative than any single juror
2. this likelihood tends to a certainty as the number of jurors tends to infinity.

A proof of the theorem is obtained by showing that this likelihood (Condorcet's probability) increases with the size of the jury (Young 1988; Boland 1989; Ladha 1992). Since adding another juror improves the collective competence, the jury should be as large as possible. The answer to the second question is only implicit. The probability of the event "$k + 1$ jurors vote correctly" cannot exceed that of the event "k jurors vote correctly," as the occurrence of the former implies the latter. Consequently, the probability of the jury being correct cannot increase in the number of votes required to reach a verdict; it is the highest for simple majority and lowest for unanimity. It remains to be shown that a simple majority jury beats a single juror, and the first part of the theorem establishes that.

Note that the definition of collective competence as the probability of the jury collectively reaching the correct decision can be made without reference to the probability of the defendant being guilty. Let p be the probability of the correct decision, which is to convict the guilty or acquit the innocent, and let g be the probability of the defendant being guilty. Since the jury can either acquit or convict the defendant, who can be either guilty or innocent,

$$\underbrace{P(convict|guilty)}_{p} + \underbrace{P(acquit|guilty)}_{1-p} = 1,$$
$$\underbrace{P(acquit|innocent)}_{p} + \underbrace{P(convict|innocent)}_{1-p} = 1.$$

The probability of the correct decision $pg + p(1-g) = p$ is independent of g. This is because the events *guilty* and *innocent* are mutually exclusive, and $P(guilty) + P(innocent) = 1$.

Condorcet's Jury Theorem rationalizes entrusting important decisions to a group rather than an individual and a simple majority as the best decision rule for the group, but it does so under restrictive assumptions. The independence assumption is especially unrealistic. Jurors are influenced by many individual and contextual factors such as differences or similarities in education, experience or ideologies and common information conveyed by the court evidence. The independence assumption impinges on the assumption of individual competence because, as Lindley (1985) succinctly puts it: "The most important source of correlation is the knowledge held in common" (p. 383). However, individual competence is crucial to the validity of the theorem, as the assumption of individual incompetence reverses its conclusion.

We show that the blueprint for an optimal jury is more nuanced when the votes are correlated. First, enlarging the jury under simple majority rule will not necessarily improve the collective competence. Secondly, the optimal jury size may comprise a single juror even if all the jurors are equally competent. The existing literature suggests that this can only occur if some jurors are more competent than others, or when the most competent juror will outperform a jury comprising his/her less competent colleagues (e.g., Nitzan and Paroush 1984; Ben-Yashar and Paroush 2000). Third and most important, the effect of correlation on the jury's competence crucially depends on the voting rule. For example, positive correlation decreases the competence of the jury under simple majority, but increases it under unanimity. This switch in the effect of correlation hints at the existence of an intermediate voting quota for which the probability of the jury being correct is not affected by correlation.

A voting rule is defined by the number of votes $k \in \mathbb{N}$ required to pass a decision in a jury of size $n \in \mathbb{N}$, an integer ranging from $\frac{n+1}{2}$ in the case of simple majority rule to n in the case of unanimity. Expressed as a share of total votes, the voting rule implies a voting quota, a real number between 0.5 and 1. We prove the existence of a quota that makes the jury immune to the effect of correlation. Being a general real number, the correlation-robust quota may not define a feasible voting rule. Among the two feasible voting rules which are least sensitive to correlation, we favor the one which leads to higher collective competence.

This article thus contributes to a large body of research on the consequences of relaxing the independence assumption for the validity of Condorcet's Jury Theorem. Our approach is based on an explicit representation of the joint probability distribution on the set of all conceivable interarrangements of votes obtained by Bahadur (1961). Working with the joint distribution allows us to get more precise results that have been obtained in the literature (Sect. 2). We derive the probability of being correct under an arbitrary quota-base voting rule for a homogeneous jury with correlated votes, and provide an alternative representation of this probability in terms of the regularized beta function (Sect. 3). We then state the conditions for which a jury operating under the two most frequently studied voting rules: simple majority and unanimity outperforms a single juror (Sect. 4). The last section offers concluding remarks.

2 The model

We model juror i's vote as a realization v_i of a Bernoulli random variable V_i, such that: $P(V_i = 1) = p_i$ and $P(V_i = 0) = 1 - p_i$. Juror i is correct if $v_i = 1$, and incorrect if $v_i = 0$. Juror i's individual competence is measured by the probability of being correct p_i.

In a jury of n jurors, the n-tuple of votes $\mathbf{v} = (v_1, \ldots, v_n)$ is called a voting profile. There will be 2^n such voting profiles. Let $\mathbf{v}$ occur with the probability $\pi_{\mathbf{v}}$. For example, in a jury of three jurors with $p_1 = 0.75$ and $p_2 = p_3 = 0.6$, the eight voting profiles may occur with the probabilities listed in Table 1.

The competence of a jury is measured by the probability of it collectively reaching the correct decision under a given voting rule. In the example on the left, this probability equals 0.720 when the jury reaches a decision by simple majority vote and 0.270

Table 1 Two examples of joint probability distributions of votes ($n = 3$, $p_1 = 0.75$, $p_2 = p_3 = 0.6$)

v_1	v_2	v_3	$\bar{\pi}_{\mathbf{v}}$	$\pi_{\mathbf{v}}$
1	1	1	0.270	0.357
1	1	0	0.180	0.136
1	0	1	0.180	0.136
1	0	0	0.120	0.122
0	1	1	0.090	0.051
0	1	0	0.060	0.056
0	0	1	0.060	0.056
0	0	0	0.040	0.086

when a unanimous vote is required. In the second example these probabilities equal respectively 0.680 and 0.357.

Computing the probability of a correct verdict requires the probabilities of all voting profiles. In the first example, the votes are independent, and the probabilities were obtained as

$$\bar{\pi}_{\mathbf{v}} = \prod_{i=1}^{n} p_i^{v_i} (1 - p_i)^{1-v_i}.$$

In the second example, the votes are correlated with $c_{1,2} = c_{1,3} = c_{2,3} = 0.2$. Although zero correlation does not imply independence in general, the definition of the Pearson product–moment correlation coefficient for two Bernoulli random variables V_i, V_j with $E(V_i) = p_i$, $E(V_j) = p_j$:

$$c_{i,j} = \frac{P\{V_i = 1, V_j = 1\} - p_i p_j}{\sqrt{p_i(1 - p_i)p_j(1 - p_j)}},$$

shows that two uncorrelated Bernoulli random variables are indeed independent.

2.1 Bahadur's theorem

Computing the probability of a correct verdict in general requires a joint probability distribution on the set of voting profiles. Bahadur (1961) obtained a closed-form expression for the joint probability distribution of n correlated Bernoulli random variables. He established that the probability of a voting profile in the case of correlated votes can be factorized into its probability in the case of independent votes and an adjustment factor that accounts for correlations. Bahadur's result is the starting point of our analysis.

The Pearson product–moment correlation coefficient is a pairwise, or second-order, measure of dependence. Ladha (1992) and Berend and Sapir (2007) discuss examples of a jury comprising three jurors, in which the votes are pairwise uncorrelated, and yet not independent. This is possible because the marginal probabilities and second-order correlation coefficients do not uniquely define the joint probability distribution unless all higher-order correlation coefficients are equal to zero. Higher order correlation coefficients measure dependence between the general tuples of votes. In a jury of size n, there will be $\sum_{i=2}^{n} C_n^i = 2^n - n - 1$ correlation coefficients of all orders, which together with n marginal probabilities (expected values) uniquely define the joint probability distribution of n correlated Bernoulli random variables.[1] Let $Z_i = (V_i - p_i)/\sqrt{p_i(1-p_i)}$ for all $i = 1, 2, \ldots, n$, and

$$\begin{aligned} c_{i,j} &= E(Z_i Z_j) \quad \text{for all } 1 \le i < j \le n; \\ c_{i,j,k} &= E(Z_i Z_j Z_k) \quad \text{for all } 1 \le i < j < k \le n; \\ &\ldots \\ c_{1,2,\ldots,n} &= E(Z_1 Z_2 \ldots Z_n). \end{aligned}$$

The joint probability distribution of n correlated Bernoulli random variables is

$$\pi_{\mathbf{v}} = \bar{\pi}_{\mathbf{v}} \left(1 + \sum_{i<j} c_{i,j} z_i z_j + \sum_{i<j<k} c_{i,j,k} z_i z_j z_k + \cdots + c_{1,2,\ldots,n} z_1 z_2 \ldots z_n \right).$$

Here, $z_i = (v_i - p_i)/\sqrt{p_i(1-p_i)}$ denotes a realization of the random variable Z_i.

The large number of parameters required to define a distribution severely limits the application of Bahadur's result. Foreseeing this difficulty, he proposed truncating the distribution to second-order correlation coefficients:

$$\pi_{\mathbf{v}} = \bar{\pi}_{\mathbf{v}} \left(1 + \sum_{i<j} c_{i,j} z_i z_j \right). \tag{1}$$

The truncated solution is exact if all higher-order correlation coefficients equal to zero. If not, additional constraints on $c_{i,j}$'s are required for the coordinates of the truncated solution to be nonnegative for given n and p_i's.

2.2 Homogeneous jury

This article studies the simplest extension of the Condorcet's Jury Theorem to correlated votes:

Definition 1 In a homogeneous jury each vote has an equal probability of being correct, and each pair of votes correlates with the same correlation coefficient. Condorcet's Jury Theorem assumes a homogeneous jury in which the correlation coefficient equals zero.

The homogeneous jury model is an example of a representative agent model. In a homogeneous jury, the probability of occurrence of any voting profile depends on the total number of correct votes in that profile, but not on the identity of the jurors who cast them. In order to avoid the need for a tie-breaking rule, we assume that

[1] C_n^x denotes the binomial coefficient $C_n^x = n!/[x!(n-x)!]$ for $n, x \in \mathbb{N}$, where $C_n^x = 0$ for $n < x$.

the jury comprises an odd number of jurors $n, n \geq 3$. If $p_i = p \in (0.5, 1)$ for all $i = 1, 2, \ldots, n$ and $c_{i,j} = c \in [0, 1)$ for all $1 \leq i < j \leq n$, then (1) becomes:

$$\pi_{\mathbf{v}} = p^t (1-p)^{n-t} \left\{ 1 + \frac{c}{2p(1-p)} \left[t_{\mathbf{v}}^2 - t_{\mathbf{v}} + p(n-1)(np - 2t_{\mathbf{v}}) \right] \right\}, \quad (2)$$

where $t_{\mathbf{v}} = \sum_{i=1}^{n} v_i$ is the total number of correct votes. Bahadur stated the bounds on c that ensure $\pi_{\mathbf{v}} \in [0, 1]$:

$$-\frac{2(1-p)}{n(n-1)p} \leq c \leq \frac{2p(1-p)}{(n-1)p(1-p) + 0.25 - \gamma},$$
$$\text{where } \gamma = \min_{0 \leq t \leq n} \{[t - (n-1)p - 0.5]^2\} \leq 0.25. \quad (3)$$

Since $c \sim O(n^{-1})$ for $c > 0$, and $|c| \sim O(n^{-2})$ for $c < 0$, the upper bound is tighter for $c < 0$. We focus on nonnegative correlation,[2] so that only the right inequality is used. Effectively, these constraints imply that in a large homogeneous jury the votes may be only weakly dependent. This makes our model better suited for modeling voting committees than general electorates.

The set of admissible model parameters is given by

$$\mathcal{B}_n = \left\{ (p, c) : 0.5 < p < 1,\ 0 \leq c \leq \frac{2p(1-p)}{(n-1)p(1-p) + 0.25 - \gamma} \right\}. \quad (4)$$

In the following, we assume that $(p, c) \in \mathcal{B}_n$. Figure 1 plots the upper bound on $c \geq 0$ for two values of n. In Appendix, we prove that in a homogeneous jury c can be at most $\frac{1}{n-1}$ for $p \approx 1$ and at most $\frac{2}{n-1}$ for $p \approx 0.5$.

For correlated votes, the asymptotic part of Condorcet's Jury Theorem follows directly from inequality (3), as $n \to \infty \Longrightarrow c \to 0 \Longrightarrow M_n(p, c) \to M_n(p, 0)$, and we know from the original theorem that $M_n(p, 0) \to 1$ as $n \to \infty$.

We use distribution (2) to obtain Condorcet's probability under an arbitrary voting rule. Formally, this is the probability of at least k successes in n correlated Bernoulli trials:

$$P_n^k(p, c) = \sum_{t=k}^{n} \sum_{\mathbf{v}: t_{\mathbf{v}} = t} \pi_{\mathbf{v}}, \quad \text{where } \frac{n+1}{2} \leq k \leq n. \quad (5)$$

For simple majority and unanimity rules, we use simpler notations: $P_n^{\frac{n+1}{2}}(p, c) = M_n(p, c)$ and $P_n^n(p, c) = U_n(p, c)$, where $U_n(p, c) \leq P_n^k(p, c) \leq M_n(p, c)$. A

[2] It is nonnegative correlation that we typically find in voting data. For example, in the U.S. Supreme Court (Kaniovski and Leech 2009), the Supreme Court of Canada (Heard and Swartz 1998), the non-judicial voting bodies such as the European Union Council of Ministers (Hayes-Renshaw et al. 2006) and the institutions of the United Nations (Newcombe et al. 1970).

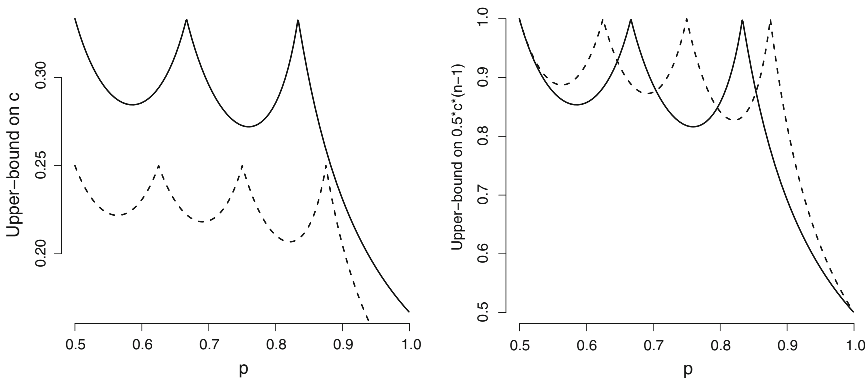

Fig. 1 The upper bound on $c \geq 0$. The *left panel* shows the upper bound on a positive second-order correlation coefficient c for $n = 7$ (*solid line*) and $n = 9$ (*dashed line*) implied in inequality (3) on the existence of a joint probability distribution of n Bernoulli exchangeable correlated random variables. The *right panel* shows that $\max\{0.5c(n-1)\} \in (0.5, 1]$

solution for $P_n^k(p, c)$ allows us to compare different voting rules and find the correlation-robust voting quota. A solution for $M_n(p, c)$ allows us to study its sensitivity to c and its monotonicity with respect to n.

The versatility of working with the joint probability distribution comes at the cost of retaining the homogeneity assumption. Workable explicit solutions for the joint probability distribution are available for homogeneous juries only. In this, our approach is less general than the heterogeneous jury model by Ladha (1992), and the sequential voting models by Boland (1989) and Berg (1993). In sequential voting models, there are both heterogeneity, as a juror's probability of being correct may depend on his/her position in the voting sequence, and dependence, although correlations may decay with the time lag between the votes.

In heterogeneous jury models, the joint probability distribution and exact conditions for its existence are typically unknown. Nevertheless such conditions clearly exist. For example, the probability of jurors i and j both being correct cannot exceed $\min\{p_i, p_j\}$ and therefore

$$c_{i,j} \leq \min\left\{\sqrt{\frac{p_i(1-p_j)}{p_j(1-p_i)}}, \sqrt{\frac{p_j(1-p_i)}{p_i(1-p_j)}}\right\}.$$

By working with the joint distribution, we are able to differentiate between the conditions required for the distribution to exist and those required for Condorcet's theorem to hold.

In sequential voting models, the correlation between the votes is induced by group dynamics. The probability of a vote being correct may change every time a vote is cast. There is a state dependence in the process of reaching a decision, with the possibility of a lock-in on the incorrect alternative (Page 2006). In spite of sequential voting models being potentially very useful in modeling certain types of group dynamics, strictly speaking, they do not apply to the baseline case of simultaneous and anonymous voting, or when the expertise of several experts whose opinions have been expressed individually is pooled into a collective judgment. This is the original setting of Condorcet's Jury Theorem.

In our analysis, we abstract from the effect of asymmetric information on jurors' behavior and assume sincere voting. In this, our approach follows the tradition of the classic Condorcet's Jury Theorem. A conceptually more general approach is that of strategic voting. Each juror receives private information—a signal—concerning the guilt of the defendant. Whether or not the juror acts according to his/her private signal (i.e., sincerely) depends on the probability of his/her being pivotal conditioned on the behavior of other jurors.

The literature on strategic voting shows that sincere voting can be irrational in the presence of informational asymmetries among jurors (Austen-Smith and Banks 1996; Myerson 1998). Feddersen and Pesendorfer Feddersen and Pesendorfer (1996, 1997) show the inefficiency of unanimity rule in aggregating private information, while non-unanimous voting rules are asymptotically efficient.

The strategic voting framework has been extended in several ways, all of which highlight the importance of taking account of communication between the jurors when designing voting procedures. Extensions include, for example, the role of incentives to acquire costly information (Persico 2004; Gerardi and Yariv 2008) and deliberation prior to voting (Austen-Smith and Feddersen 2006; Gerardi and Yariv 2007).

3 The correlation-robust voting rule

In this section, we prove the existence of a voting quota α for which the probability of the jury being correct is independent of the correlation coefficient c. In this case, the jury's competence cannot be impaired or improved by independence, as $P_n^\alpha(p,c) = P_n^\alpha(p,0)$. We call α the correlation-robust quota. Its existence follows from the following theorem:

Theorem 1 *If $(p,c) \in \mathcal{B}_n$, then Condorcet's probability under a k-voting rule, where $\frac{n+1}{2} \le k \le n$, is given by:*

$$P_n^k(p,c) = P_n^k(p,0) + 0.5cn(n-1)\left(\frac{k-1}{n-1} - p\right) C_{n-1}^{k-1} p^{k-1}(1-p)^{n-k}. \quad (6)$$

In particular, under simple majority and unanimity rules, we have

$$M_n(p,c) = M_n(p,0) + cn(n-1)(0.5-p) C_{n-2}^{\frac{n-1}{2}} (p-p^2)^{\frac{n-1}{2}} \quad (7)$$

$$U_n(p,c) = p^n \left(1 + 0.5cn(n-1)\frac{1-p}{p}\right). \quad (8)$$

Proof $P_n^k(p,0)$ is the probability of at least k successes in n independent Bernoulli trials:

$$P_n^k(p,0) = \sum_{t=k}^{n} C_n^t p^t (1-p)^{n-t}.$$

The above probability leads to two recurrence relations:

$$P_n^k(p,0) - P_{n-1}^k(p,0) = C_{n-1}^{k-1} p^k (1-p)^{n-k}; \tag{9}$$

$$P_n^k(p,0) - P_{n-2}^k(p,0) = C_{n-1}^{k-1} p^k (1-p)^{n-k} \left(1 + \frac{n-k}{(1-p)(n-1)}\right). \tag{10}$$

By substituting (2) in (5) we obtain:

$$\begin{aligned} P_n^k(p,c) = \ & P_n^k(p,0) + \frac{c}{2p(1-p)} \\ & \times \left[\sum_{t=k}^{n} \sum_{\mathbf{v}:t_{\mathbf{v}}=t} t^2 p^t (1-p)^{n-t} - \sum_{t=k}^{n} \sum_{\mathbf{v}:t_{\mathbf{v}}=t} t p^t (1-p)^{n-t} \right. \\ & \left. + p(n-1)\left(np - 2\sum_{t=k}^{n} \sum_{\mathbf{v}:t_{\mathbf{v}}=t} t p^t (1-p)^{n-t}\right)\right]. \end{aligned}$$

We evaluate the double sums as follows:

$$\begin{aligned} \sum_{t=k}^{n} \sum_{\mathbf{v}:t_{\mathbf{v}}=t} t p^t (1-p)^{n-t} &= \sum_{t=k}^{n} C_n^t t p^t (1-p)^{n-t} = n \sum_{t=k}^{n} C_{n-1}^{t-1} p^t (1-p)^{n-t}; \\ \sum_{t=k}^{n} \sum_{\mathbf{v}:t_{\mathbf{v}}=t} t^2 p^t (1-p)^{n-t} &= \sum_{t=k}^{n} C_n^t t^2 p^t (1-p)^{n-t} \\ &= n(n-1) \sum_{t=k}^{n} C_{n-2}^{t-2} p^t (1-p)^{n-t} \\ &\quad + n \sum_{t=k}^{n} C_{n-1}^{t-1} p^t (1-p)^{n-t}, \end{aligned}$$

and, consequently,

$$\begin{aligned} P_n^k(p,c) = P_n^k(p,0) + \frac{cn(n-1)}{2p(1-p)} & \left(\sum_{t=k}^{n} C_{n-2}^{t-2} p^t (1-p)^{n-t} \right. \\ & \left. -2p \sum_{t=k}^{n} C_{n-1}^{t-1} p^t (1-p)^{n-t} + p^2 P_n^k(p,0)\right). \end{aligned} \tag{11}$$

Next we express the sums in (11) in terms of $P_n^k(p, 0)$ by using Eqs. (9)–(10) and identities $xC_n^x = nC_{n-1}^{x-1}$ and $x^2C_n^x = n(n-1)C_{n-2}^{x-2} + nC_{n-1}^{x-1}$. Let $i = t - 1$. Then,

$$\begin{aligned}\sum_{t=k}^{n} C_{n-1}^{t-1} p^t (1-p)^{n-t} &= \sum_{i=k-1}^{n-1} C_{n-1}^{i} p^{i+1} (1-p)^{n-i-1}\\ &= p(P_{n-1}^k(p,0) + C_{n-1}^{k-1} p^{k-1}(1-p)^{n-k})\\ &= pP_n^k(p,0) + (1-p)C_{n-1}^{k-1} p^k (1-p)^{n-k}.\end{aligned}$$

Similarly, let $j = t-2$. In view of identities $C_{n-2}^{x-1} = \frac{n-x}{n-1}C_{n-1}^{x-1}$ and $C_{n-2}^{x-2} = \frac{x-1}{n-1}C_{n-1}^{x-1}$,

$$\begin{aligned}\sum_{t=k}^{n} C_{n-2}^{t-2} p^t (1-p)^{n-t} &= \sum_{j=k-2}^{n-2} C_{n-2}^{j} p^{j+2} (1-p)^{n-j-2}\\ &= p^2 \Big(P_{n-2}^k(p,0) + C_{n-2}^{k-2} p^{k-2}(1-p)^{n-k}\\ &\quad + C_{n-2}^{k-1} p^{k-1}(1-p)^{n-k-1}\Big)\\ &= p^2 P_n^k(p,0)\\ &\quad + C_{n-1}^{k-1} p^k (1-p)^{n-k} \left[\frac{(k-1)+p(n-k)}{n-1} - p^2\right].\end{aligned}$$

A substitution of the above sums in (11) leads to

$$\begin{aligned}P_n^k(p,c) = P_n^k(p,0) + \frac{cn}{2p(1-p)}\,[(k-1)\\ + p(n-k) + p(p-2)(n-1)]\, C_{n-1}^{k-1} p^{k-1}(1-p)^{n-k},\end{aligned}$$

which simplifies to (6). Equation (7) follows from substituting $k = \frac{n+1}{2}$ in (6), while equation (8) from substituting $k = n$ in (6). □

Theorem 1 establishes that for a given size and individual competence, the effect of positive correlation on the jury's competence depends on the voting rule.

Corollary 1 *Let* $(p, c) \in \mathcal{B}_n$. *Given* n *and* p, *the probability* $P_n^k(p, c)$ *decreases in* c *for* $k < p(n-1) + 1$, *and increases in* c *for* $k > p(n-1) + 1$. *In particular, an increase in* c *increases the jury's competence under unanimity but decreases it under simple majority.*

Proof The piecewise monotonicity follows immediately from (6). Under simple majority, $k = \frac{n+1}{2} \implies \frac{k-1}{n-1} - p = 0.5 - p < 0$, while under unanimity $k = n \implies \frac{k-1}{n-1} - p = 1 - p > 0$. □

The above corollary corroborates the finding by Ladha (1992) and Berg (1993) who found negative correlation to improve the collective competence under simple majority, and positive correlation to have the opposite effect. We add that under simple majority, whereas correlation has no effect on the competence of a randomizing jury in which $p = 0.5$. The effect of correlation is the opposite under unanimity: negative correlation is detrimental, whereas positive correlation is beneficial.

In theory, choosing $\alpha = p(n-1)+1$ as the voting rule would make the jury immune to correlation. In practice, α may not be an integer between $\frac{n+1}{2}$ and n, and thus may not define a valid voting rule. Two feasible correlation-robust voting rules can be defined using the greatest integer smaller or equal to α ($\lfloor\alpha\rfloor$), and the smallest integer larger or equal to α ($\lceil\alpha\rceil$). The respective probabilities $P_n^{\lfloor\alpha\rfloor}(p,c)$ and $P_n^{\lceil\alpha\rceil}(p,c)$ will have different sensitivities with respect to c. The sensitivity of a k-voting rule to correlation can be measured by

$$\left|\frac{\partial P_n^k(p,c)}{\partial c}\right| = 0.5n(n-1)\left|\frac{k-1}{n-1} - p\right| C_{n-1}^{k-1} p^{k-1}(1-p)^{n-k}.$$

If $\alpha \in \mathbb{N}$, then $\frac{\partial P_n^\alpha(p,c)}{\partial c} = 0$ and $P_n^\alpha(p,c) = P_n^\alpha(p,0)$. In this case, α is a valid voting rule that makes the jury immune to correlation. If $\alpha \notin \mathbb{N}$, then using $\lfloor\alpha\rfloor = \lceil\alpha\rceil - 1$ and $xC_{n-1}^x = (n-x)C_{n-1}^{x-1}$ one can show that

$$\left|\frac{\partial P_n^{\lfloor\alpha\rfloor}(p,c)}{\partial c}\right| < \left|\frac{\partial P_n^{\lceil\alpha\rceil}(p,c)}{\partial c}\right| \iff (1-p)\frac{\alpha-\lfloor\alpha\rfloor}{n-\lfloor\alpha\rfloor} < p\frac{\lfloor\alpha\rfloor-\alpha+1}{\lfloor\alpha\rfloor}. \quad (12)$$

Corollary 2 *If* $(p,c) \in \mathcal{B}_n$, *then inequality* (12) *holds for* $p \approx 0.5$ *and* $p \approx 1$.

Proof We show first that (12) holds for $p \geq \frac{n-2}{n-1}$. In this case, $\lfloor\alpha\rfloor = n-1$ and

$$(1-p)\frac{\alpha-\lfloor\alpha\rfloor}{n-\lfloor\alpha\rfloor} < p\frac{\lfloor\alpha\rfloor-\alpha+1}{\lfloor\alpha\rfloor} \iff$$
$$(1-p)(\alpha-n+1) < \frac{p(n-\alpha)}{n-1} \iff (n-1)(1-p) > 1-p.$$

Next we prove that (12) holds for $0 < 2p-1 < \frac{n-1}{2} - \sqrt{\frac{(n-1)^2}{4} - 1}$. In this case, $\lfloor\alpha\rfloor = \frac{n+1}{2}$ since $\alpha = p(n-1)+1 < \frac{n+3}{2} \iff 2p-1 < \frac{2}{n-1}$ and $\frac{n-1}{2} - \sqrt{\frac{(n-1)^2}{4} - 1} < \frac{2}{n-1}$. We have

$$\begin{aligned}
&(1-p)\frac{\alpha-\lfloor\alpha\rfloor}{n-\lfloor\alpha\rfloor} < p\frac{\lfloor\alpha\rfloor-\alpha+1}{\lfloor\alpha\rfloor}\\
&\iff \frac{(1-p)(2\alpha-n-1)}{n-1} < \frac{p(n+3-2\alpha)}{n+1}\\
&\iff (1-p)(2p-1) < \frac{p(n+1-2p(n-1))}{n+1}\\
&\iff (2p-1)^2 - (2p-1)(n-1) + 1 > 0.
\end{aligned}$$

□

Despite the different sensitivities under the two rules, $P_n^{\lfloor\alpha\rfloor}(p,c) > P_n^{\lceil\alpha\rceil}(p,c)$ because $P_n^k(p,c)$ is non-increasing in k. We, therefore, call $\lfloor\alpha\rfloor$ the optimal correlation-robust voting rule. Since $\frac{n+1}{2} \le \lfloor\alpha\rfloor \le n-1$, the optimal correlation-robust voting rule may coincide with simple majority rule but not with unanimity rule.

We emphasize that the optimal correlation-robust voting rule does not necessarily maximize the jury's competence; this is accomplished by simple majority rule, as is also the case when the votes are independent (Fey 2003). However, we shall see that when the votes are correlated, the jury may be less competent than a single juror even under simple majority rule.

4 Individual versus collective competence

In this section, we compare the competence of the jury of size n with that of a single juror under the two most common voting rules: simple majority and unanimity, when $(p,c) \in \mathcal{B}_n$.

It is convenient to represent $P_n^k(p,c)$ in terms of the regularized incomplete beta function:

$$I_x(a,b) = \frac{B_x(a,b)}{B(a,b)} \quad \text{for } a, b > 0, \tag{13}$$

where $B(a,b)$ is the complete and $B_x(a,b)$ is the incomplete beta functions:

$$B(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}du \text{ and } B_x(a,b) = \int_0^x u^{a-1}(1-u)^{b-1}du, \quad x \in [0,1].$$

From the definition of $B_x(a,b)$ it is easy to verify that

$$\frac{\partial I_x(a,b)}{\partial x} = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}. \tag{14}$$

Properties of the regularized incomplete beta function (13) and its first derivative (14) are well known due to their being, respectively, the probability distribution and density functions of the Beta(a,b) distribution. We will also use a sum representation:

$$I_x(a,b) = \sum_{i=a}^{a+b-1} C_{a+b-1}^i x^i (1-x)^{a+b-1-i}. \tag{15}$$

We can now formulate

Corollary 3 *If $(p,c) \in \mathcal{B}_n$, then Condorcet's probability under a k-voting rule, where $\frac{n+1}{2} \le k \le n$, can be written as*

$$P_n^k(p,c) = I_p(k, n-k+1) + 0.5c(n-1)\left(\frac{k-1}{n-1} - p\right)\frac{\partial I_p(k,n-k+1)}{\partial p}. \tag{16}$$

In particular, under simple majority rule, we have

$$M_n(p,c) = I_p\left(\frac{n+1}{2}, \frac{n+1}{2}\right) + 0.5c(n-1)(0.5-p)\frac{\partial I_p(\frac{n+1}{2}, \frac{n+1}{2})}{\partial p}. \quad (17)$$

Proof Substituting $x = p$, $a = k$ and $b = n - k + 1$ in (15) and (14), we obtain:

$$I_p\,(k, n-k+1) = \sum_{i=k}^{n} C_n^i p^i (1-p)^{n-i} = P_n^k(p,0), \quad (18)$$

$$\frac{\partial I_p(k, n-k+1)}{\partial p} = nC_{n-1}^{k-1} p^{k-1}(1-p)^{n-k}. \quad (19)$$

Here, we use the formula $nC_{n-1}^{x} = \frac{1}{B(x+1,n-x)}$. A substitution of (18) and (19) in (6) furnishes the corollary. Equation (17) follows from (16) by substituting $k = \frac{n+1}{2}$. □

4.1 Independent votes

Figure 2 shows examples of the functions $M_n(p,0)$ and $U_n(p,0)$. The fact that $M_n(p,0)$ coincides with the probability distribution function of the symmetric Beta distribution provides a simple proof of the non-asymptotic part of Condorcet's Jury

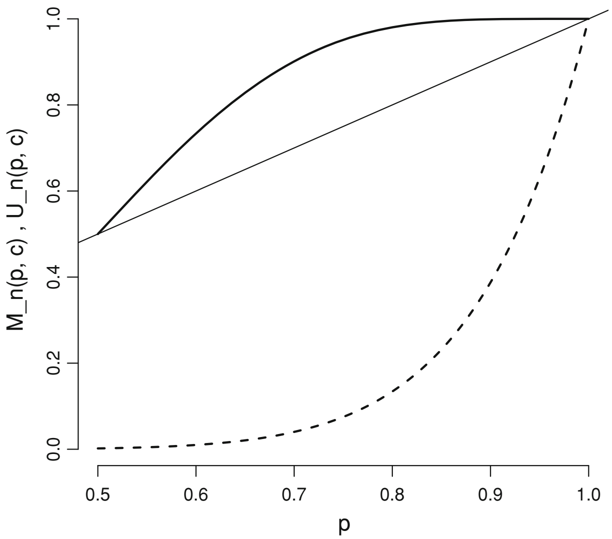

Fig. 2 The functions $M_n(p,0)$ and $U_n(p,0)$. The figure shows the functions $M_n(p,0)$ (*solid line*) and $U_n(p,0)$ (*dashed line*) for $n = 9$. In the interval $(0.5, 1)$, $M_n(p,0)$ lies above the 45°, while $U_n(p,0)$ lies below. The jury is more competent than a single juror under simple majority rule, but a single juror is more competent than the jury under unanimity rule

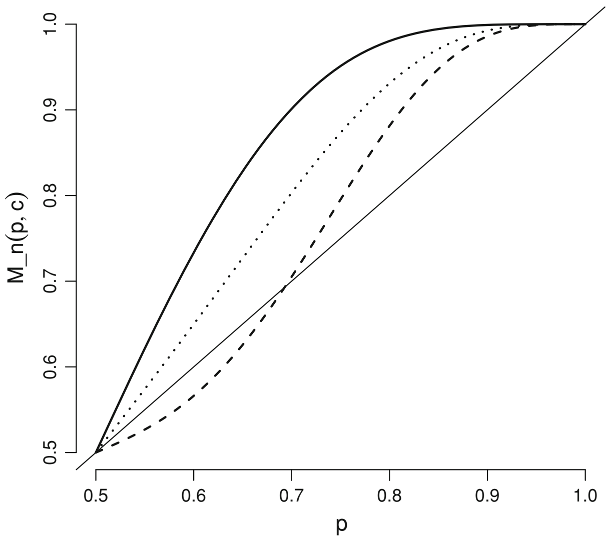

Fig. 3 The function $M_n(p, c)$. The figure compares the function $M_n(p, c)$ for $n = 9$, $c = 0$ (*solid line*), $c = 0.1$ (*dotted line*), and $c = 0.2$ (*dashed line*) to the 45° line. Positive correlation decreases the jury's competence under simple majority. The jury performs worse than a single juror unless the jurors are very competent indeed

Theorem. Twice differentiating $M_n(p, 0)$ with respect to p shows that $M_n(p, 0)$ is concave:

$$a_n M_n''(p, 0) = (n-1)(0.5-p)(p-p^2)^{\frac{n-3}{2}} < 0,$$
$$\text{where } a_n = B\left(\frac{n+1}{2}, \frac{n+1}{2}\right) > 0. \tag{20}$$

Together with $M_n(0.5, 0) = 0.5$ and $M_n(1, 0) = 1$, this implies $M_n(p, 0) > p$ for $p \in (0.5, 1)$. It is not the case under unanimity, as $U_n(p, 0) = p^n < p$ for all $p \in (0.5, 1)$.

4.2 Correlated votes: simple majority versus single juror

Consider the function $M_n(p, c)$. Since both $p = 0.5$ and $p = 1$ satisfy the equation $M_n(p, c) = p$, the function $M_n(p, c)$ meets the 45° line at the endpoints of the interval $(0.5, 1)$ (Fig. 3). We shall show that $M_n(p, c)$ has at most one interior inflection point and, therefore, if an interior solution to $M_n(p, c) = p$ exists, it is unique. The case $n = 3$ must be treated separately.

Theorem 2 *Let $n \geq 3$ and $c \geq 0$ be given and $(p, c) \in \mathcal{B}_n$. Then,*

$$n = 3,\ c < \tfrac{1}{3}, \qquad M_3(p, c) > p;$$
$$n = 3,\ c = \tfrac{1}{3}, \qquad M_3(p, c) = p;$$

$$\begin{aligned} &n = 3,\ c > \tfrac{1}{3}, && M_3(p,c) < p;\\ &n \ge 5,\ c \le \tfrac{2}{n-1}\left(1 - \tfrac{1}{f(0.5)}\right), && M_n(p,c) > p;\\ &n \ge 5,\ c > \tfrac{2}{n-1}\left(1 - \tfrac{1}{f(0.5)}\right), && M_n(p,c) < p \ \ \textit{for } p \in (0.5, \bar{p});\\ & && M_n(p,c) > p \ \ \textit{for } p \in (\bar{p}, 1), \end{aligned}$$

where $\bar{p} \in (0.5, 1)$ *is the solution to the equation* $M_n(p,c) = p$ *and* $f(p) = \frac{\partial I_p(\frac{n+1}{2}, \frac{n+1}{2})}{\partial p}$.

Proof Consider the case $n = 3$. We have $M_3(p,c) = p^2(3-2p)+3p(1-p)(1-2p)c$. In order to complete the proof, we observe that $M_3(p,c) - p = p(1-p)(2p-1)(1-3c)$.

Let $n \ge 5$. Twice differentiating $M_n(p,c)$ with respect to p, we get

$$\begin{aligned} a_n M_n'(p,c) &= (p-p^2)^{(n-3)/2}[(1-0.5c(n-1))(p-p^2)\\ &\quad + 0.5c(n-1)^2(0.5-p)^2] > 0;\\ a_n M_n''(p,c) &= (n-1)(0.5-p)(p-p^2)^{(n-5)/2} \times [(1-0.5cn(n-1))(p-p^2)\\ &\quad + 0.125c(n-1)(n-3)], \quad a_n = B\left(\frac{n+1}{2}, \frac{n+1}{2}\right), \end{aligned}$$

shows that $M_n(p,c)$ is increasing for $p \in (0.5, 1)$, and

$$M_n''(p,c) = 0 \iff p - p^2 = \frac{0.125c(n-1)(n-3)}{0.5cn(n-1)-1}. \tag{21}$$

Since $p - p^2$ for $p \in (0.5, 1)$ decreases monotonically from 0.25 to 0, it follows from (21) that an interior inflection point for $M_n(p,c)$ exists, iff $\frac{0.125c(n-1)(n-3)}{0.5cn(n-1)-1} \in (0, 0.25)$, and, if it exists, then it is unique. By a simple calculation, we obtain

$$\begin{aligned} &\frac{0.125c(n-1)(n-3)}{0.5cn(n-1)-1} \in (0, 0.25)\\ &\iff 0.5cn(n-1) - 1 > 0.5c(n-1)(n-3) \iff c > \frac{2}{3(n-1)}. \end{aligned}$$

In this case, the curvature of $M_n(p,c)$ changes from convex to concave at an interior point. Moreover, both $M_n(1,c) = 1$ and $M_n'(1,c) = 0$ imply $M_n(p,c) > p$ for $p \approx 1$.

In order to complete the investigation of the function $M_n(p,c)$, we consider its behavior for $p \approx 0.5$. From $M_n(0.5,c) = 0.5$, we infer that $M_n(p,c)$ intersects the 45° line at an interior point $\bar{p}$ if $M_n'(0.5,c) = f(0.5)[1 - 0.5c(n-1)] < 1$, that is if $c > \frac{2}{n-1}\left(1 - \frac{1}{f(0.5)}\right)$. Note that $1 - \frac{1}{f(0.5)} > \frac{1}{3}$ for $n \ge 5$.[3]

[3] Using the approximation, $B(a,b) \approx \sqrt{2\pi}\frac{a^{a-0.5}b^{b-0.5}}{(a+b)^{a+b-0.5}}$ it can be shown that $f(0.5) \approx \sqrt{\frac{2(n+1)}{\pi}}$.

For $c \leq \frac{2}{3(n-1)}$, $M_n(p, c)$ is concave for $p \in (0.5, 1)$. The inequality $M_n(p, c) > p$ for $p \in (0.5, 1)$ follows from $M_n(0.5, c) = 0.5$ and $M_n(1, c) = 1$.

Finally, for $\frac{2}{3(n-1)} < c \leq \frac{2}{n-1}\left(1 - \frac{1}{f(0.5)}\right)$, an interior inflection point exists, but $M_n(p, c)$ lies above the 45° line, i.e., $M_n(p, c) > p$ for $p \in (0.5, 1)$. □

The above theorem shows that a single juror will outperform a jury under simple majority rule when the individual competence is low, but the correlation is high. Since simple majority rule maximizes the collective competence, the single juror will outperform the jury under any conceivable voting rule.

Contrary to the case of independent votes, increasing the size of a jury when the votes are correlated will not necessarily improve its competence. For positive correlation, an enlargement of the jury can be detrimental up to a certain size, beyond which it becomes beneficial.

Corollary 4 *Let $p \in (0.5, 1)$ and $c \geq 0$ be given such that $(p, c) \in \mathcal{B}_n$. Under simple majority, enlarging the jury by two jurors has a positive effect on the collective competence regardless of c if n satisfies the condition*

$$\frac{n^2 - 1}{n^2 + 2n + 3} > 4p(1 - p). \tag{22}$$

If (22) does not hold, the effect of enlarging the jury by two jurors on the collective competence can be positive or negative depending on c.

Proof According to Boland (1989), $M_{n+2}(p, 0) - M_n(p, 0) = (2p - 1)C_n^{\frac{n+1}{2}}(p - p^2)^{\frac{n+1}{2}}$. Substituting this difference into $M_{n+2}(p, c) - M_n(p, c)$ and using the identity $(n + 1)C_n^{\frac{n+1}{2}} = 4nC_{n-2}^{\frac{n-1}{2}}$, we obtain

$$\begin{aligned} M_{n+2}(p, c) - M_n(p, c) &= (2p - 1)C_n^{\frac{n+1}{2}}(p - p^2)^{\frac{n+1}{2}} \\ &\times \left[1 - 0.5c(n + 1)\left(n + 2 - \frac{n - 1}{4p(1 - p)}\right)\right]. \end{aligned} \tag{23}$$

By simple calculations,

$$\begin{aligned} &\frac{n^2 - 1}{n^2 + 2n + 3} > 4p(1 - p) \Longleftrightarrow \frac{(n + 1)^2 + 2}{n^2 - 1} < \frac{1}{4p(1 - p)} \\ &\Longleftrightarrow n + 2 - \frac{n - 1}{4p(1 - p)} < 1 - \frac{2}{n + 1} \\ &\Longleftrightarrow 0.5c(n + 1)\left(n + 2 - \frac{n - 1}{4p(1 - p)}\right) < 0.5c(n - 1) \leq 1. \end{aligned}$$

If inequality (22) does not hold, then for $c > 0$

$$M_{n+2}(p,c) - M_n(p,c) > 0 \quad \text{if} \quad (n+1)\left(n+2-\frac{n-1}{4p(1-p)}\right) < \frac{2}{c};$$
$$M_{n+2}(p,c) - M_n(p,c) < 0 \quad \text{if} \quad (n+1)\left(n+2-\frac{n-1}{4p(1-p)}\right) > \frac{2}{c}.$$

□

When p is low but c is high, enlarging the jury may decrease its collective competence. For $n \geq 3$, the term $\frac{n^2-1}{n^2+2n+3}$ increases monotonically from $\frac{4}{9}$ to 1, so that even a small enlargement of a very competent jury will improve collective competence regardless of c.

For given p and $c > 0$, the value of n is bounded from above at least by $\frac{2}{c}+1$ (Appendix). Note that if for a given n_0 inequality (22) holds, then it will also hold for any $n > n_0$. On the other hand, if for some n (22) does not hold, then n_0 exists such that it holds for all $n \geq n_0$.

4.3 Correlated votes: unanimity versus single juror

A single juror is always more likely to be correct than a jury to be unanimously correct. Indeed,

$$U_n(p,c) = P\left\{\cap_{i=1}^n V_i = 1\right\} \leq P\{V_i = 1\} \leq p \Longrightarrow U_n(p,c) < p \quad \text{for } p \in (0.5, 1).$$

The effect of correlation on the probability $U_n(p,c)$ is illustrated in Fig. 4. Although positive correlation improves the jury's competence under unanimity, it cannot raise it above that of a single juror. This is true for independent and correlated votes, as well as for homogeneous and heterogeneous juries.

4.4 Numerical examples

The following numerical examples illustrate our results. Let $n = 9$. This jury size corresponds, for example, to the full bench in the U.S. Supreme Court and the Supreme Court of Canada.

Depending on the individual competence p, the upper bound on $c \geq 0$ ranges from $1/(n-1) = 0.125$ to $2/(n-1) = 0.25$ (Fig. 1).

For $p = 0.55$, the correlation-robust voting quota $\alpha = p(n-1)+1 = 5.4$. Since $\alpha \notin \mathbb{N}$, we consider the two correlation-robust voting rules $\lfloor\alpha\rfloor = 5$ and $\lceil\alpha\rceil = 6$. The $\lfloor\alpha\rfloor$-rule coincides with simple majority rule and maximizes the jury's competence. A check of inequality (12) shows that the jury's competence is less sensitive to correlation under the $\lfloor\alpha\rfloor$-rule than under the $\lceil\alpha\rceil$-rule. For $p = 0.75$, the correlation-robust quota $\alpha = 7$ defines a valid voting rule. Under this voting rule, the jury is immune to correlation, as $P_9^7(p,c) = P_9^7(p,0)$ for all $(p,c) \in \mathcal{B}_9$. Setting $p = 0.95$ yields

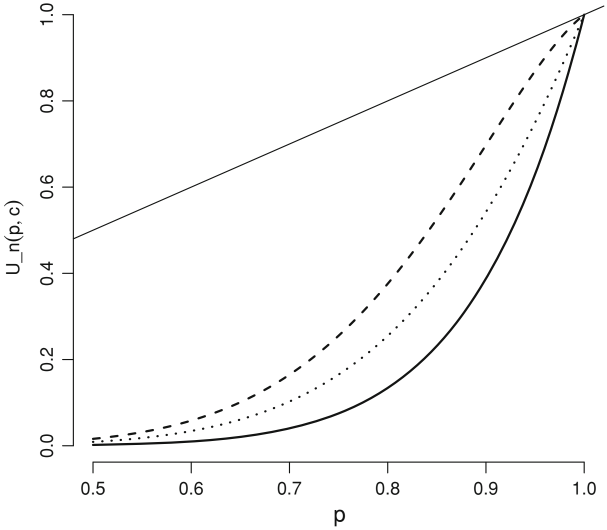

Fig. 4 The function $U_n(p, c)$. The figure compares the function $M_n(p, c)$ for $n = 9, c = 0$ (*solid line*), $c = 0.1$ (*dotted line*), and $c = 0.2$ (*dashed line*) to the 45° line. Positive correlation increases the jury's competence under unanimity but the jury remains less competent than a single juror

$\alpha = 8.6$. A check of inequality (12) again shows that under the $\lfloor \alpha \rfloor$-rule the jury is less sensitive to correlation than under the $\lceil \alpha \rceil$-rule.

Enlarging a jury under simple majority may decrease the jury's competence if n is such that inequality (22) does not hold, i.e., if $\frac{n^2-1}{n^2+2n+3} \leq 4p(1-p)$. For $n = 9$, we have $\frac{n^2-1}{n^2+2n+3} \approx 0.8$. Therefore, for very competent juries (with $p = 0.75$ or $p = 0.95$), inequality (22) holds and enlarging a jury increases its competence. However, enlarging a barely competent jury ($p = 0.55$) may lower its competence if c is not too small, e.g., if $c = 0.1$. For a lower correlation, e.g., for $c = 0.05$, the effect of enlarging the jury is positive again.

Theorem 2 tells us that if $c \leq \frac{2}{n-1}\left(1 - \frac{1}{f(0.5)}\right) \approx 0.148$, the jury under simple majority rule would outperform a single juror for all $(p, c) \in \mathcal{B}_9$. For $c > 0.148$, the jury will be more competent than a single juror if $p > \bar{p}$, where $M_9(\bar{p}, c) = \bar{p}$, e.g., if $c = 0.2$, then $\bar{p} \approx 0.693$.

5 Concluding remarks

In this article, we discuss the optimal size and voting rule of a homogeneous jury with a given individual competence. The model directly extends the setting of the classic Condorcet's Jury Theorem to correlated votes.

The effect of correlation on the jury's competence is negative for voting rules close to simple majority and positive for voting rules close to unanimity. The collective competence under simple majority may fall below that of a single juror. This will be the case when the individual competence is low but the correlation is high. The optimal jury then consists of a single juror. The collective competence under simple majority will not, however, fall below the "competence" offered by a fair coin flip, while under unanimity, it will not exceed the individual competence of a single juror.

If the individual competence is low, it may be beneficial and presumably more economical to hire one expert rather than several. In all other cases, simple majority rule is the optimal decision rule for the group. A jury operating under simple majority rule will not necessarily benefit from an enlargement, unless the enlargement is substantial. The higher the individual competence, the sooner an enlargement will begin to be beneficial.

We derive a voting rule which minimizes the effect of correlation on collective competence by making it as close as possible to that of a jury of independent jurors. The optimal correlation-robust voting rule should be preferred to simple majority rule if mitigating the effect of correlation is more important than maximizing the accuracy of the collective decision, e.g., as a means of fostering the image of impartiality.

Another reason to choose the correlation-robust voting rule is predictability. Suppose one knows the individual competences of each juror but not the correlations between their votes, e.g., because the jurors had never before jointly provided expertise, or the voting record is not comprehensive enough to reliably estimate the correlation coefficient. Under the correlation-robust rule the jury's performance will be close to its performance in the case of independent votes, which is easily computable using the binomial distribution. The correlation robust voting rule would make the jury's performance predictable, but it would not necessarily maximize it.

Knowing the joint probability distribution allows us to compute the probability of any event of interest. This makes our approach potentially useful in other situations in which there is a need to model correlated dichotomous choice. These situations include modeling the probability of casting a decisive vote as a measure of voting power, the Value at Risk of a portfolio of assets with dependent default risks, the robustness of computer networks against both random failure and intentional attack, or consumer choice among complements and substitutes.

Appendix

Lemma 1 *If* $(p, c) \in \mathcal{B}_n$*, then the upper bound for* $0.5c(n-1)$ *satisfies the conditions:*

$$0.5 < \frac{(n-1)p(1-p)}{(n-1)p(1-p) + 0.25 - \gamma} \leq 1, \tag{24}$$

$$\frac{(n-1)p(1-p)}{(n-1)p(1-p) + 0.25 - \gamma} \to 0.5, \quad p \to 1, \tag{25}$$

where $\gamma = \min_{0 \leq t \leq n}\{[t - (n-1)p - 0.5]^2\} \leq 0.25$.

Proof The upper bound in (24) follows as $0.25 - \gamma \geq 0$. In order to derive the lower bound in (24) and (25) rewrite

$$0.25 - \min_{0\le t\le n}\{[t-(n-1)p-0.5]^2\} = \max_{0\le t\le n}\{(t-(n-1)p)(1-t+(n-1)p)\},$$
$$= (\lfloor (n-1)p\rfloor + 1 - (n-1)p)((n-1)p - \lfloor (n-1)p\rfloor).$$

Consider two cases: $0.5 < p < \frac{n-2}{n-1}$ and $\frac{n-2}{n-1} \le p < 1$. Let

$$\frac{(n-1)p(1-p)}{(n-1)p(1-p)+0.25-\gamma} = \frac{1}{1+A(z)},$$

where $z = (n-1)p$ and $A(z) = \frac{(\lfloor z\rfloor+1-z)(z-\lfloor z\rfloor)}{z(1-\frac{z}{n-1})}$. For $\frac{n-2}{n-1} \le p < 1$, we have $n-2 \le z < n-1$ and $\lfloor z\rfloor = n-2$. Therefore, $A(z) = \frac{(n-1)(z-n+2)}{z}$. This function is increasing for $z \in [n-2, n-1)$, and $A(z) \to 1$ as $z \to n-1$ or, equivalently, as $p \to 1$. This proves (25).

In order to derive the lower bound in (24), it suffices to show that $A(z) < 1$ for $z \in (\frac{n-1}{2}, n-2)$. Clearly,

$$(\lfloor z\rfloor + 1 - z)(z - \lfloor z\rfloor) \le 0.25. \tag{26}$$

On the other hand, the function $z(1-\frac{z}{n-1})$ is decreasing for $z \in (\frac{n-1}{2}, n-2)$ and, therefore,

$$\frac{1}{z(1-\frac{z}{n-1})} < \frac{n-1}{n-2}. \tag{27}$$

Together, inequalities (26) and (27) imply $A(z) < 0.25\frac{n-1}{n-2} < 1$. □

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