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On asymptotic properties of multidimensional α-stable densities

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A class of multidimensional α-stable distributions is considered. The Poisson spectral measure of each distribution is assumed to be absolutely continuous with respect to the surface Lebesgue measure. The author concentrates his attention on the asymptotic behavior of the α-stable densities $s(x)$ as $|x| \to \infty$ and $|x| \to 0$.

1 Introduction

Univariate stable distributions arose within the context of the central limit theorem as limit laws for sums of i.i.d. random variables. Except for the gaussian laws, all the stable distributions are heavy-tailed. This explains why stable distribution models are very attractive for statisticians. The fact that stable laws are limiting says a lot about possible genesis of the corresponding random variables. Analytical properties of univariate stable distributions are well studied (see e.g. [31] and [5, 10, 11, 29]).

The notion of stability can be carried over to higher dimensions in various ways. The simplest way is the following. Let $\eta, \eta^{(1)}, \eta^{(2)}, \ldots$ be i.i.d. random vectors. It is said that the distribution of η is *α-stable* in $\mathbb{R}^d$ if for any $n > 1$ there exists $a^{(n)} \in \mathbb{R}^d$ and a number $\alpha \in (0, 2]$ such that

$$\eta \overset{d}{=} n^{-1/\alpha}\big(\eta^{(1)} + \cdots + \eta^{(n)}\big) + a^{(n)} \tag{1.1}$$

(see e.g. [29, Corollary 2.1.3]). If in (1.1) $a^{(n)} \equiv \mathbf{0}$, then the α-stable distribution is called *strictly α-stable.*

Assume that $\alpha \in (0, 2)$, that is, η has a stable non-gaussian distribution, and denote by $s(x)$ the density of this distribution. Its characteristic function $\widehat{s}(y)$ admits the following representation (see e.g. [29, Remark 2.3.3]):

$$\ln \widehat{s}(y) = i\langle \gamma, y\rangle + \int\limits_{\mathbb{S}^{d-1}} \int\limits_0^\infty \left(e^{ir\langle e,y\rangle} - 1 - i\frac{r\langle e, y\rangle}{1+r^2}\right) \frac{\mathrm{d}r}{r^{1+\alpha}}\, \mu(\mathrm{d}e), \tag{1.2}$$

where $\gamma \in \mathbb{R}^d$ is a location parameter while μ is a finite Borel measure on the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$. We refer to μ as a *Poisson spectral* or simply *spectral measure*. Here, $\langle \cdot, \cdot \rangle$ stands for the standard inner product in $\mathbb{R}^d$.

One can transform (1.2) into the following, more convenient form (see e.g. [29, Theorem 2.3.1]):

$$-\ln \widehat{s}(y) = \begin{cases} i\langle \gamma, y\rangle + \displaystyle\int\limits_{\mathbb{S}^{d-1}} |\langle y, e\rangle|^\alpha (1 - i\tan(\pi\alpha/2)\operatorname{sign}\langle y, e\rangle)\, \mu(\mathrm{d}e) & \text{if} \quad \alpha \neq 1, \\ i\langle \gamma, y\rangle + \displaystyle\int\limits_{\mathbb{S}^{d-1}} |\langle y, e\rangle|\, (1 + i(2/\pi)\operatorname{sign}\langle y, e\rangle \ln|\langle y, e\rangle|)\, \mu(\mathrm{d}e) & \text{if} \quad \alpha = 1. \end{cases} \tag{1.3}$$

Therefore, each α-stable distribution is uniquely determined by the triple (α, μ, γ). So, the class of α-stable distributions is very rich. In contrast to its one-dimensional analog, it is not parametric. As such, this class

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forms an object inappropriate for a comprehensive analysis. Thus, it is promising to start with studying various subclasses of the class of α-stable distributions.

If the spectral measure μ is uniform or spherically invariant, then the α-stable distribution is called *sub-gaussian.* The sub-gaussian distributions constitute a parametric family parameterized by (α, b, γ), where $b = \mu\big(\mathbb{S}^{d-1}\big)$ is the *scale* parameter. The α-stable distributions, whose the spectral measures are absolutely continuous with respect to the surface Lebesgue measure σ, form a non-parametric subclass. A more narrow subclass, that is under the consideration here, is formed by distributions with the property

$$a(e) = \frac{\mathrm{d}\mu}{\mathrm{d}\sigma}(e) \in L_2\big(\mathbb{S}^{d-1}\big), \tag{1.4}$$

where $L_2\big(\mathbb{S}^{d-1}\big)$ stands for the class of all functions defined on $\mathbb{S}^{d-1}$ and squared integrable with respect to σ.

One can find the majority of known facts regarding multidimensional α-stable distributions in [29] (see Chapter 2 therein). For further development of the theory see e.g. [1, 2, 4, 11, 21, 23, 24, 25]. However, the referred works do not touch problems related to asymptotic properties of α-stable densities. It should be emphasized that such problems constitute a considerable part of the theory of univariate stable densities. Asymptotic formulae play significant role, say, in numerical analysis where they determine the region in which numerical computations should be implemented. On the other hand, the shape of the large deviation local theorems for sums of i.i.d. random vectors in the case, where limit distributions are stable, is closely related to asymptotic properties of the limiting distribution.

One of the basic goals of this paper is to fill up the gap in the asymptotic theory of multidimensional α-stable distributions. To the moment, this theory is rather weakly developed. The matter is that the method of analytical functions, which proved to be very efficient within the framework of one-dimensional stable distributions, is not so efficient in higher dimensions.

The first works on asymptotic analysis of multivariate α-stable densities were published by Kalinauskaite (see [12, 13, 14, 15]). In the first two papers sub-gaussian α-stable distributions are studied. In the third one, the author deals with the phenomenon of super-exponential decays of stable densities (typical for unilateral Poisson spectral measures). The last paper is devoted to investigation of the asymptotic behaviour of α-stable distributions with respect to α.

Approximately at the same time (1972), Fristedt [6] implemented an asymptotic analysis of the distribution function $\mathrm{P}(|\eta| < r)$ where a random vector η was α-stable. The case $\alpha = 1$, under the same setting, was studied in detail by Lisitsky [16] in 1989.

In the same year Arkhipov [3] suggested an approach based on the Fourier series method. This rather general approach helped to understand the real scale of complexity of the problem. The presentation in [3] was, however, far from being perfect. In our opinion, this explains in part why this undoubtedly pioneering paper has been ignored by experts. Even the latest of the referred works, except for [9], contain no reference to [3]. One of the goals of the present paper is to give a comprehensive and correct version of the Arkhipov's work, presenting the results in clear formulations and with readable proofs.

The paper is organized as follows. Two results are at the core of Section 2. The first one is a version of the Arkhipov theorem and deals with asymptotic properties of α-stable densities as $|x| \to \infty$. The second result is new and deals with asymptotic properties of α-stable densities as $|x| \to 0$. The proofs are given in Section 4. Section 3 contains a number of various comments concerning the referred papers of Kalinauskaite and Fristedt as well as some other known results.

2 Asymptotic formulas for α-stable densities

In what follows we set, for simplicity, $\gamma = \mathbf{0}$ in (1.3) (we can consider the random vector $\eta - \gamma$ instead of η). For $\alpha \neq 1$ this implies that we deal with strictly α-stable distributions. However, strictly 1-stable distributions are characterized by another condition:

$$\int\limits_{\mathbb{S}^{d-1}} \langle b, e\rangle \, \mu(\mathrm{d}e) = 0 \quad \text{for all} \quad b \in \mathbb{R}^d. \tag{2.1}$$

Further on we assume that (2.1) holds if $\alpha = 1$.

Assuming that (1.4) is fulfilled, we can rewrite (1.3) as follows

$$\widehat{s}(y) = \exp\{-|y|^{\alpha} g(e_y)\}, \quad y \neq \mathbf{0}, \quad e_y = |y|^{-1} y, \tag{2.2}$$

where

$$g(e) = \begin{cases} \int\limits_{\mathbb{S}^{d-1}} |\langle e, \varepsilon\rangle|^{\alpha}(1 - i\tan(\pi\alpha/2)\operatorname{sign}\langle e, \varepsilon\rangle) a(\varepsilon)\,\sigma(\mathrm{d}\varepsilon) & \text{if} \quad \alpha \neq 1, \\ \int\limits_{\mathbb{S}^{d-1}} (|\langle e, \varepsilon\rangle| + (2i/\pi)\langle e, \varepsilon\rangle \ln|\langle e, \varepsilon\rangle|) a(\varepsilon)\,\sigma(\mathrm{d}\varepsilon) & \text{if} \quad \alpha = 1. \end{cases} \tag{2.3}$$

Naturally, we assume that the distribution is essentially d-dimensional, that is $\inf_{e\in\mathbb{S}^{d-1}} \mathcal{R}e\, g(e) > 0$.

Observe that the kernels $|\langle e, \varepsilon\rangle|^{\alpha}(1 - i\tan(\pi\alpha/2)\operatorname{sign}\langle e, \varepsilon\rangle)$ and $|\langle e, \varepsilon\rangle| + (2i/\pi)\langle e, \varepsilon\rangle \ln|\langle e, \varepsilon\rangle|$ determine linear operators acting in the Hilbert space $L_2(\mathbb{S}^{d-1})$. Basic properties of such operators are given e.g. in [8, Chapter 3.4]. Those properties are established by means of the Fourier analysis method.

Let $\{H_{n,j}\}$, $n = 0, 1, 2, \ldots$, $j = 1, 2, \ldots, N(d, n)$, be an orthonormal system of spherical harmonics in $L_2(\mathbb{S}^{d-1})$. Here n is the so-called order of the harmonic $H_{n,j}$ and j is the ordinal number of the harmonic within the orthonormal system of spherical harmonics in the subspace of all harmonics of the order n, while

$$N(d, n) = \begin{cases} 1 & \text{if} \quad n = 0, \\ \dfrac{(2n + d - 2)(n + d - 3)!}{(d - 2)!n!} & \text{if} \quad n > 0 \end{cases}$$

(see e.g. [8, Chapters 3.1 and 3.2]). For instance, $N(d, 1) = d$ and for $n \geq 1$

$$N(2, n) = 2, \quad N(3, n) = 2n + 1.$$

Since for any fixed $l \in \mathbb{N}$

$$\frac{(n + l)!}{n!} = n^l\,(1 + o(1)) \quad \text{as} \quad n \longrightarrow \infty,$$

one can easily obtain that for any fixed $d \geq 3$

$$N(d, n) = \frac{2n^{d-2}}{(d - 2)!}\,(1 + o(1)) \quad \text{as} \quad n \longrightarrow \infty.$$

Given $M \in \mathbb{N}$, define

$$I_d = \{(n, j) : \ n \geq 0,\ 1 \leq j \leq N(d, n)\}, \quad I_{d,M} = \{(n, j) : \ (n, j) \in I_d,\ n \leq M\}.$$

Each $f \in L_2(\mathbb{S}^{d-1})$ can be expanded in the Fourier series with respect to $\{H_{n,j}\}$, that is the formula

$$\lim_{M\to\infty} \int\limits_{\mathbb{S}^{d-1}} \Bigg| f(e) - \sum_{(n,j)\in I_{d,M}} f_{n,j} H_{n,j}(e) \Bigg|^2 \sigma(\mathrm{d}e) = 0$$

holds where $\{f_{n,j}\}$ are the Fourier coefficients corresponding to f defined as

$$f_{n,j} = \int\limits_{\mathbb{S}^{d-1}} f(e) H_{n,j}(e)\,\sigma(\mathrm{d}e), \quad (n, j) \in I_d.$$

That series is usually called the *harmonic expansion* for the function f. We denote it by $\widetilde{f}$.

Further, the functions $g^k(e),\ e \in \mathbb{S}^{d-1}$, that is the k-th power of $g(e),\ k \in \mathbb{N}$, are of importance. Denote the Fourier coefficients of $g^k(e)$ by $\{g^{(k)}_{n,j}\}$, i.e.

$$g^{(k)}_{n,j} = \int\limits_{\mathbb{S}^{d-1}} g^k(e) H_{n,j}(e)\, \sigma(\mathrm{d}e), \quad (n,j) \in I_d.$$

Denote also

$$c(\alpha) = \int\limits_0^\infty \frac{1-\cos u}{u^{1+\alpha}}\, \mathrm{d}u = \frac{\pi}{2\Gamma(1+\alpha)\sin(\pi\alpha/2)}, \quad \alpha \in (0,2).$$

Let $L^q_2(\mathbb{S}^{d-1}),\ q \geq 0$, be the space of all the functions defined on $\mathbb{S}^{d-1}$, whose derivatives, up to order q, exist and belong to $L_2(\mathbb{S}^{d-1})$. As in [3], we admit q to be non-integer, that is we admit derivatives of non-integer orders.

Further on $[z]$ means the integer part of z.

The first theorem establishes an expansion of a multidimensional α-stable density, which is useful for $|x| > 1$.

Theorem 2.1 *Let the α-stable density $s(x)$ be determined by* (2.2) *and* (2.3)*. If the spectral density $a \in L^q_2(\mathbb{S}^{d-1})$ and $q \geq \alpha$, then for any $x \neq \mathbf{0}$*

$$s(x) = \sum_{k=1}^m \frac{\widetilde{a}_k(e_x)}{|x|^{k\alpha+d}} + R_m(x),$$

where $m = [q/\alpha]$ and

$$\widetilde{a}_k(e) = \sum_{(n,j)\in I_d} \gamma_{k,n} g^{(k)}_{n,j} H_{n,j}(e), \quad e \in \mathbb{S}^{d-1}, \tag{2.4}$$

$$\gamma_{k,n} = \begin{cases} \dfrac{(-1)^k(-i)^n 2^{k\alpha}\Gamma((n+d+k\alpha)/2)}{\pi^{d/2}k!\Gamma((n-k\alpha)/2)} & \textit{if} \quad \dfrac{k\alpha - n}{2} \notin \mathbb{N}\cup\{0\}, \\ 0 & \textit{otherwise.} \end{cases} \tag{2.5}$$

Moreover, $\widetilde{a}_1(e) = \widetilde{a}(e)/c(\alpha)$. The residual term R_m admits the upper bound

$$\left(\int\limits_{\mathbb{S}^{d-1}} R^2_m(te)\, \sigma(\mathrm{d}e) \right)^{1/2} \leq Ct^{-(m+1)\alpha-d}, \tag{2.6}$$

where C does not depend on $t > 0$.

If $q = \infty$, then $m = \infty$, and we have an asymptotic series for $s(x)$. Sometimes this series is convergent (see [6, 13] and Section 3).

Observe that in the case of a symmetric α-stable density, where $s(-x) = s(x)$, the coefficients $\{\gamma_{k,n}\}$ in (2.5) vanish for odd n.

In general, the functions $\{a_k(e)\}$ are not uniquely determined by (2.4). In order to guarantee the uniform convergence of $\widetilde{a}_k(e)$, we have to assume additional smoothness of $a(e)$. For example, if $a \in L^q_2(\mathbb{S}^{d-1})$ and $q > \alpha + d/2$, then the functions $a_k(e),\ 1 \leq k \leq [(q-d/2)/\alpha]$, are continuous (see [22, Chapter 6.1]). Then we also have instead of (2.6)

$$\max_{e\in\mathbb{S}^{d-1}} |R_m(te)| \leq Ct^{-(m+1)\alpha-d}, \quad t > 0.$$

Let

$$h^{(k)}_{n,j} = \int\limits_{\mathbb{S}^{d-1}} h^{(k)}(e) H_{n,j}(e)\, \sigma(\mathrm{d}e), \quad (n,j) \in I_d,$$

be the Fourier coefficients of the function $h^{(k)}(e) = g^{-(k+d)/\alpha}(e)$, $e \in \mathbb{S}^{d-1}$, $k = 0, 1, 2, \ldots$ In contrast to Theorem 2.1, the next statement provides a power expansion of a multidimensional α-stable density which is useful for $|x| < 1$.

Theorem 2.2 *Let the α-stable density $s(x)$ be determined by* (2.2) *and* (2.3). *If $\alpha \in (0, 1]$, then for any $m \in \mathbb{N}$*

$$s(x) = \sum_{k=0}^{m} b_k(e_x)\,|x|^k + R_m(x), \quad x \neq \mathbf{0},$$

where

$$b_k(e) = \sum_{(n,j)\in I_{d,k}} \beta_{k,n} h^{(k)}_{n,j} H_{n,j}(e), \quad e \in \mathbb{S}^{d-1}. \tag{2.7}$$

Here,

$$\beta_{k,n} = \frac{(-i)^k \Gamma((k+d)/\alpha)}{\alpha \pi^{d/2} 2^{k+d-1} \Gamma((k+2-n)/2) \Gamma((n+k+d)/2)}$$

if k and n are both even (*zero is also assumed to be even*) *or odd, and zero otherwise. The residual term R_m admits the upper bound*

$$\left(\int\limits_{\mathbb{S}^{d-1}} R_m^2(te)\,\sigma(\mathrm{d}e) \right)^{1/2} \leq C t^{m+1},$$

where C does not depend on $t > 0$.

If $\alpha \in (1, 2)$, then for any $x \neq \mathbf{0}$

$$s(x) = \sum_{k=0}^{\infty} b_k(e_x)\,|x|^k.$$

This result extends the analogous statements in [6, 13].

3 Remarks and comments

In this section we give several remarks concerning the stated theorems.

Remark 3.1 Let us compare Theorems 2.1 and 2.2 with the results established in [6, 12, 13]. We will see that our results are in agree with [6], but [12] and [13] contain errors.

Let the spectral measure μ be spherically invariant, i.e. $\mu = \lambda\sigma$, $\lambda > 0$. Then the α-stable distribution is necessarily symmetric and

$$g(e) = \lambda \int\limits_{\mathbb{S}^{d-1}} |\langle \varepsilon, e \rangle|^\alpha \,\sigma(\mathrm{d}\varepsilon), \quad e \in \mathbb{S}^{d-1}.$$

By straightforward calculations

$$g(e) \equiv \lambda \frac{2\pi^{(d-1)/2} \Gamma((1+\alpha)/2)}{\Gamma((d+\alpha)/2)}.$$

If we choose

$$\lambda = \frac{\Gamma((d+\alpha)/2)}{2\pi^{(d-1)/2} \Gamma((1+\alpha)/2)},$$

then $g(e) \equiv 1$ and we arrive at the case considered in [13].

Then, obviously, in Theorem 2.1 $m = \infty$ and, therefore, we have the asymptotic series for the density $s(x)$. Moreover, in the sum (2.4) determining $\widetilde{a}_k(e)$ all the summands, except for that corresponding to $n = 0$ and $j = 1$, vanish and we obtain for $k \geq 1$

$$a_k(e) \equiv \gamma_{k,0} = \frac{(-1)^{k+1} 2^{k\alpha} \Gamma((d+k\alpha)/2) \Gamma(1+k\alpha/2) \sin(\pi k\alpha/2)}{\pi^{d/2+1} k!}. \tag{3.1}$$

The asymptotic series for $s(x)$ is convergent provided that $\alpha \in (0, 1)$.

It turns out that the coefficients $\{a_k\}$ suggested in [13, Theorems 2 and 3] should be multiplied by 2. In order to justify this, we compare the statement of Theorem 2.1 with one of the main results of [6]. Assume that the random vector η has the spherically invariant α-stable distribution with $g(e) \equiv 1$. Then the density $q(r)$ of $|\eta|$ takes the form

$$q(r) = r^{d-1} \int\limits_{\mathbb{S}^{d-1}} s(re)\, \sigma(\mathrm{d}e), \quad r > 0.$$

From Theorem 2.1 it follows that

$$\widetilde{q}(r) = \frac{\sigma_d \lambda}{c(\alpha) r^{\alpha+1}} + \sigma_d \sum_{k=2}^{\infty} \frac{a_k}{r^{k\alpha+1}}, \quad r > 0,$$

where

$$\sigma_d = \sigma\big(\mathbb{S}^{d-1}\big) = \frac{2\pi^{d/2}}{\Gamma(d/2)},$$

and the series is convergent provided that $\alpha \in (0, 1)$. Hence, we arrive at the particular case of the statement given in [6, Theorem 1]. This serves as an additional confirmation that [13] contains the error. It is of interest to note that the coefficients $\{a_k\}$ given by (3.1) are in accord with those in the expansion of univariate stable densities (see e.g. [10, Chapter 2.4]).

In a more general case where $g(e)$ is not a constant, it can be verified that under the conditions of Theorem 2.1 for $r > 0$ and for any integer $m \geq 2$ the density of $|\eta|$ takes the form

$$q(r) = \frac{\mu\big(\mathbb{S}^{d-1}\big)}{c(\alpha) r^{\alpha+1}} + \sum_{k=2}^{m} \frac{a_k}{r^{k\alpha+1}} \int\limits_{\mathbb{S}^{d-1}} g^k(e)\, \sigma(\mathrm{d}e) + O\big(r^{-(m+1)\alpha-1}\big),$$

that is we obtain the first m terms of the asymptotic expansion given in [6, Theorem 1]. If $\alpha \in (0, 1)$, then letting $m = \infty$ we obtain the convergent infinite series for $q(r)$.

By the way, if $\alpha \in (0, 1)$, then for any $t > 0$

$$P(|\eta| > t) = \int\limits_t^{\infty} q(r)\, \mathrm{d}r = \frac{\mu\big(\mathbb{S}^{d-1}\big)}{\alpha c(\alpha) t^{\alpha}} + \sum_{k=2}^{\infty} \frac{a_k}{k\alpha t^{k\alpha}} \int\limits_{\mathbb{S}^{d-1}} g^k(e)\, \sigma(\mathrm{d}e).$$

Therefore, the limit of $P\big(|\eta| > t^{-1/\alpha}\big)$ as $\alpha \to 0$ exists for any $t > 0$, since

$$P\big(|\eta| > t^{-1/\alpha}\big) \longrightarrow \frac{1}{\sigma_d} \int\limits_{\mathbb{S}^{d-1}} \big(1 - e^{-tg(e)}\big)\, \sigma(\mathrm{d}e) = 1 - \frac{1}{\sigma_d} \int\limits_{\mathbb{S}^{d-1}} e^{-tg(e)}\, \sigma(\mathrm{d}e).$$

This result extends those obtained in [30] and in [15, Theorem 1].

Now, we compare Theorem 2.2 with the results obtained in [13]. It turns out that in [13, Theorem 4] the statement analogous to Theorem 2.2 again contains an error. Putting $g(e) \equiv 1$ in (2.7) yields

$$b_{2k}(e) \equiv \beta_{2k,0} = \frac{(-1)^k \Gamma((2k+d)/\alpha)}{\alpha \pi^{d/2} 2^{2k+d-1} k! \Gamma(k+d/2)},$$

while $b_{2k+1}(e) \equiv 0,\ k \geq 0$. This implies that the corresponding coefficients obtained in [13] should be multiplied by $2^{1-d/2}$, while those suggested in [12] for the case $d = 2$ should be divided by 2.

In order to justify these corrections, we again utilize [6]. For the density $q(r)$ of the spherically invariant α-stable distribution with $g(e) \equiv 1$, from Theorem 2.2 it follows that for $r > 0$ and for any integer $m \geq 0$

$$q(r) = \sigma_d \sum_{k=0}^{m} b_{2k} r^{2k+d-1} + O\big(r^{2m+d+1}\big)$$

if $\alpha \in (0, 1]$, and

$$q(r) = \sigma_d \sum_{k=0}^{\infty} b_{2k} r^{2k+d-1}$$

if $\alpha \in (1, 2)$. These expansions are also in accord with the classical one-dimensional expansions (see e.g. [10, Chapter 2.4]).

In a more general case where $g(e)$ is not a constant, under the conditions of Theorem 2.2 for $r > 0$ and for any integer $m \geq 0$ we have

$$q(r) = \sum_{k=0}^{m} b_{2k} r^{2k+d-1} \int_{\mathbb{S}^{d-1}} h^{(2k)}(e)\, \sigma(\mathrm{d}e) + O\big(r^{2m+d+1}\big)$$

if $\alpha \in (0, 1]$, and

$$q(r) = \sum_{k=0}^{\infty} b_{2k} r^{2k+d-1} \int_{\mathbb{S}^{d-1}} h^{(2k)}(e)\, \sigma(\mathrm{d}e)$$

if $\alpha \in (1, 2)$. This fits completely the statement of [6, Theorem 3].

Finally, observe that the statement of Theorem 2.2 agrees with [2, Corollary 4a] where it is stated that

$$s(\mathbf{0}) = \frac{\Gamma(d/\alpha)}{\alpha(2\pi)^d} \int_{\mathbb{S}^{d-1}} g^{-d/\alpha}(e)\, \sigma(\mathrm{d}e).$$

Remark 3.2 Let

$$E_0 = \big\{e \in \mathbb{S}^{d-1} :\ a(e) = 0\big\}.$$

The question concerning the asymptotic behaviour of $s(te)$, $e \in E_0$, as $t \to \infty$, seems to be rather delicate. From Theorem 2.1 it follows that $s(te) = O\big(t^{-d-2\alpha}\big)$, provided that the function $a(e)$ is sufficiently smooth. Obviously, such a result is not very informative. So, we arrive at an interesting problem: *what is the precise asymptotics of* $s(te)$, $e \in E_0$, *as* $t \to \infty$?

The answer is given in [20]. If $e \in E_0$, then the term $a_2(e)$ in the asymptotic expansion for $s(x)$ from Theorem 2.1 is strictly positive and

$$s(te) = \frac{a_2(e)}{t^{d+2\alpha}}\,(1 + o(1))\,, \quad t \longrightarrow \infty.$$

The following example confirms this result.

Let $d = 2$. For the sake of simplicity we write $a(\phi)$ instead of $a(\cos\phi, \sin\phi)$, $\phi \in [0, 2\pi]$, for the function $a(e_1, e_2)$ defined on $\mathbb{S}^1$.

In this case $N(2, 0) = 1$, $N(2, n) = 2$, $n \geq 1$, and the orthonormal sequence of spherical harmonics looks like

$$H_{0,1}(\phi) = \frac{1}{\sqrt{2\pi}}, \quad H_{n,1}(\phi) = \frac{1}{\sqrt{\pi}} \cos n\phi, \quad H_{n,2}(\phi) = \frac{1}{\sqrt{\pi}} \sin n\phi, \quad \phi \in [0, 2\pi], \quad n \geq 1.$$

Take

$$a(\phi) = 1 + \cos 2\phi, \quad \phi \in [0, \pi],$$

that is $a(e_1, e_2) = 2e_1^2$. Since $a(\phi) = a(\phi + \pi)$, $\phi \in [0, \pi]$, the spectral measure μ is symmetric. Clearly, the conditions of Theorem 2.1 hold with $m = \infty$. Note that $E_0 = \{e',\ -e'\}$, where $e' = (0, 1)$.

From Theorem 2.1 it follows that for $e \neq e'$

$$s(te) = \frac{2e_1^2}{c(\alpha)t^{2+\alpha}}\,(1 + o(1))\,, \quad t \longrightarrow \infty. \tag{3.2}$$

Let $e = e'$. In order to establish the precise asymptotics of $s(te')$, we should calculate $a_2(e')$. From Lemma 4.3 below it follows that

$$g(\phi) = g_{1,0} + g_{1,2}\cos 2\phi, \quad \phi \in [0, \pi],$$

where

$$g_{1,0} = \frac{4\pi^{1/2}\Gamma((1+\alpha)/2)}{\alpha\Gamma(\alpha/2)}, \quad g_{1,2} = \frac{4\pi^{1/2}\Gamma((1+\alpha)/2)}{(2+\alpha)\Gamma(\alpha/2)}.$$

Therefore,

$$g^2(\phi) = g_{2,0} + g_{2,2}\cos 2\phi + g_{2,4}\cos 4\phi, \quad \phi \in [0, \pi],$$

where

$$g_{2,0} = \frac{8\pi(3\alpha^2 + 8\alpha + 8)\Gamma^2((1+\alpha)/2)}{\alpha^2(2+\alpha)^2\Gamma^2(\alpha/2)},$$
$$g_{2,2} = \frac{32\pi\Gamma^2((1+\alpha)/2)}{\alpha(2+\alpha)\Gamma^2(\alpha/2)},$$
$$g_{2,4} = \frac{8\pi\Gamma^2((1+\alpha)/2)}{(2+\alpha)^2\Gamma^2(\alpha/2)}.$$

From Theorem 2.1 we obtain

$$a_2(\phi) = a_{2,0} + a_{2,2}\cos 2\phi + a_{2,4}\cos 4\phi, \quad \phi \in [0, \pi],$$

where

$$a_{2,0} = -\frac{2^{2\alpha+2}(1-\alpha)(3\alpha^2 + 8\alpha + 8)\Gamma^2((1+\alpha)/2)\Gamma(1+\alpha)}{\alpha(2+\alpha)^2\Gamma^2(\alpha/2)\Gamma(2-\alpha)},$$
$$a_{2,2} = -\frac{2^{2\alpha+4}(1-\alpha)\Gamma^2((1+\alpha)/2)\Gamma(2+\alpha)}{\alpha(2+\alpha)\Gamma^2(\alpha/2)\Gamma(2-\alpha)},$$
$$a_{2,4} = \frac{2^{2\alpha+2}\Gamma^2((1+\alpha)/2)\Gamma(2+\alpha)}{(2+\alpha)\Gamma^2(\alpha/2)\Gamma(2-\alpha)}.$$

Therefore,

$$a_2(e') = \frac{2^{2\alpha+3}\cdot 3\cdot\Gamma^2((1+\alpha)/2)\Gamma(1+\alpha)}{(2+\alpha)^2\Gamma^2(\alpha/2)\Gamma(2-\alpha)} > 0$$

and

$$s(te') = \frac{a_2(e')}{t^{2+2\alpha}}\,(1 + o(1))\,, \quad t \longrightarrow \infty.$$

This example shows that the asymptotic behaviour of $s(x)$ as $|x| \to \infty$, when $e_x \in E_0$, is atypical, that is differs from (3.2).

It remains to emphasize that Theorem 2.1 is not sensitive to the phenomenon of a super-exponential decay of $s(x)$ (see [14, 19]).

Remark 3.3 If in Theorem 2.1 $m \geq 1$, then for any Borel subset $E \subset \mathbb{S}^{d-1}$ such that $\mu(E) > 0$ we have

$$P\left(e_\eta \in E \mid |\eta| > t\right) = \frac{\mu(E)}{\mu(\mathbb{S}^{d-1})} + O\left(t^{-\alpha}\right). \tag{3.3}$$

Among various statistical problems related to the stable distributions, a problem of estimating $\mu(E)/\mu(\mathbb{S}^{d-1})$ in the case of known α is of interest. The asymptotic properties of a non-parameter estimator of $\mu(E)/\mu(\mathbb{S}^{d-1})$, based on the number of observations occuring in the part of the cone $\left\{x \in \mathbb{R}^d : \ e_x \in E\right\}$ that moves away from the origin, was studied in [18]. It turns out that if (3.3) holds, then this estimator is $n^{1/3}$-consistent, that is it has the best possible rate of consistency.

4 Proofs

In order to obtain an expansion for the α-stable density $s(x)$, we begin with the inversion formula

$$\begin{aligned} s(x) &= \frac{1}{(2\pi)^d} \int\limits_{\mathbb{R}^d} e^{-i\langle x,y\rangle} \widehat{s}(y)\,\mathrm{d}y \\ &= \frac{1}{(2\pi)^d} \int\limits_0^\infty r^{d-1}\,\mathrm{d}r \int\limits_{\mathbb{S}^{d-1}} e^{-ir\langle x,e\rangle - r^\alpha g(e)} \sigma(\mathrm{d}e) \\ &= \frac{1}{(2\pi)^d\,|x|^d} \int\limits_0^\infty r^{d-1}\,\mathrm{d}r \int\limits_{\mathbb{S}^{d-1}} e^{-ir\langle e_x,e\rangle - r^\alpha g(e)/|x|^\alpha} \sigma(\mathrm{d}e), \quad x \neq \mathbf{0}. \end{aligned} \tag{4.1}$$

P r o o f o f T h e o r e m 2.1. Consider the well-known Taylor expansion

$$\begin{aligned} e^{-r^\alpha g(e)/|x|^\alpha} = 1 &- \frac{r^\alpha g(e)}{|x|^\alpha} + \frac{r^{2\alpha} g^2(e)}{2!\,|x|^{2\alpha}} - \ldots + (-1)^m \frac{r^{m\alpha} g^m(e)}{m!\,|x|^{m\alpha}} \\ &+ (-1)^{m+1} \frac{r^{(m+1)\alpha} g^{m+1}(e)}{m!\,|x|^{(m+1)\alpha}} \int\limits_0^1 (1-u)^m e^{-r^\alpha g(e) u/|x|^\alpha}\,\mathrm{d}u. \end{aligned} \tag{4.2}$$

Substituting (4.2) in (4.1), we observe that the integrals

$$\int\limits_0^\infty r^{k\alpha+d-1}\,\mathrm{d}r \int\limits_{\mathbb{S}^{d-1}} e^{-ir\langle e_x,e\rangle} g^k(e)\,\sigma(\mathrm{d}e), \quad k = 0, 1, 2, \ldots,$$

do not converge absolutely. That is why we use the Abelian summation method for their calculation. In other words,

$$\begin{aligned} s(x) &= \frac{1}{(2\pi)^d\,|x|^d} \sum_{k=0}^m \frac{(-1)^k}{k!\,|x|^{k\alpha}} \lim_{\delta\to 0} \int\limits_0^\infty r^{k\alpha+d-1}\,\mathrm{d}r \int\limits_{\mathbb{S}^{d-1}} e^{-ir\langle e_x,e\rangle - \delta r} g^k(e)\,\sigma(\mathrm{d}e) \\ &\quad + \frac{(-1)^{m+1}}{(2\pi)^d m!\,|x|^{(m+1)\alpha+d}} \lim_{\delta\to 0} \int\limits_0^\infty r^{(m+1)\alpha+d-1}\,\mathrm{d}r \int\limits_{\mathbb{S}^{d-1}} e^{-ir\langle e_x,e\rangle - \delta r} g^{m+1}(e)\,\sigma(\mathrm{d}e) \\ &\quad \times \int\limits_0^1 (1-u)^m e^{-r^\alpha g(e) u/|x|^\alpha}\,\mathrm{d}u \\ &= \sum_{k=0}^m \frac{I_k(e_x)}{|x|^{k\alpha+d}} + R_m(x), \quad x \neq \mathbf{0}. \end{aligned} \tag{4.3}$$

At the moment, we do not specify the number of terms in the expansion. Later on we show that under the conditions of the theorem we should take $m \le q/\alpha$.

Now we establish several auxiliary facts.

Lemma 4.1 *Let $I_k(e)$ be defined by* (4.3)*. Then $I_0(e) \equiv 0,\ e \in \mathbb{S}^{d-1}$.*

P r o o f. By [8, Theorem 3.4.1] and [7, formula (8.411.10)], we have

$$\int\limits_{\mathbb{S}^{d-1}} e^{-ir\langle e_x, e\rangle}\sigma(\mathrm{d}e) = \sigma_{d-1}\int\limits_{-1}^{1} e^{-iru}\left(1-u^2\right)^{(d-3)/2}\mathrm{d}u = \frac{(2\pi)^{d/2}J_{d/2-1}(r)}{r^{d/2-1}},$$

where $J_{d/2-1}(r)$ denotes the Bessel function of the first kind. Therefore,

$$I_0(e) = \frac{1}{(2\pi)^{d/2}}\lim_{\delta\to 0}\int\limits_0^\infty e^{-\delta r} r^{d/2} J_{d/2-1}(r)\,\mathrm{d}r.$$

By [7, formula (6.621.1)], we obtain

$$I_0(e) = \frac{\Gamma(d)}{(2\pi)^{d/2}2^{d/2-1}\Gamma(d/2)}\lim_{\delta\to 0}\ \delta^{-d}\ {}_2F_1\left(\frac{d}{2},\frac{d+1}{2},\frac{d}{2},-\frac{1}{\delta^2}\right),$$

where ${}_2F_1(a,b,c,z)$ denotes the hypergeometric Gauss function. In view of [7, formula (9.121.1)], we have

$$I_0(e) = \frac{\Gamma(d)}{\pi^{d/2}2^{d-1}\Gamma(d/2)}\lim_{\delta\to 0}\frac{1}{\delta^d}\left(1+\frac{1}{\delta^2}\right)^{-(d+1)/2} = 0.$$

$\square$

Lemma 4.2 *Let $I_k(e)$ be defined by* (4.3)*. If $I_k \in L_2^{q-(k-1)\alpha}\big(\mathbb{S}^{d-1}\big)$, $1 \le k \le m$, then the harmonic expansion for $I_k(e)$ is given by* (2.4)*.*

P r o o f. Let

$$\widetilde{g^k}(e) = \sum_{(n,j)\in I_d} g_{n,j}^{(k)} H_{n,j}(e), \quad e \in \mathbb{S}^{d-1},$$

be the harmonic expansion for the function $g^k(e)$.

The crucial point of the proof is making use of the Funk–Hecke formula (see e.g. [8, Theorem 3.4.1]):

$$\int\limits_{\mathbb{S}^{d-1}} \Phi(\langle e,\varepsilon\rangle)H_{n,j}(\varepsilon)\,\sigma(\mathrm{d}\varepsilon) = \alpha_{d,n}(\Phi)H_{n,j}(e), \quad e \in \mathbb{S}^{d-1}, \tag{4.4}$$

that holds for any $(n,j) \in I_d$ and any bounded integrable function Φ on $[-1,1]$. Here

$$\alpha_{d,n}(\Phi) = \sigma_{d-1}\int\limits_{-1}^{1}\Phi(u)P_n^d(u)\big(1-u^2\big)^{(d-3)/2}\,\mathrm{d}u, \tag{4.5}$$

where $\{P_n^d\}$ are the Legendre polynomials of dimension d and degree n (see e.g. [8, p. 98]). In particular, $\{P_n^2\}$ are the Chebyshev polynomials, $\{P_n^3\}$ are the classical Legendre polynomials. Observe that the polynomials $\{P_n^d\}$ are even functions if n is even and odd functions if n is odd (see e.g. [8, p. 80]).

It is also known that for $d \ge 3$ the Legendre polynomials can be expressed through the Gegenbauer polynomials (see e.g. [8, p. 97]):

$$P_n^d(u) = c_{d,n}C_n^{(d-2)/2}(u), \quad u \in [-1,1],$$

where $\{C_n^\lambda\}$ are the Gegenbauer polynomials of degree n and

$$c_{d,n} = \frac{\Gamma(n+1)\Gamma(d-2)}{\Gamma(n+d-2)}.$$

Therefore,

$$\alpha_{d,n}(\Phi) = \sigma_{d-1} c_{d,n} \int_{-1}^{1} \Phi(u) C_n^{(d-2)/2}(u) \left(1-u^2\right)^{(d-3)/2} \mathrm{d}u. \tag{4.6}$$

Later on we will use (4.5) or (4.6) depending on whether $d = 2$ or $d \geq 3$.

Given $r > 0$ consider the function

$$f_r^{(k)}(e) = \int_{\mathbb{S}^{d-1}} e^{-ir\langle e,\varepsilon\rangle} g^k(\varepsilon)\,\sigma(\mathrm{d}\varepsilon), \quad e \in \mathbb{S}^{d-1}.$$

By (4.4)–(4.6) and [8, Proposition 3.4.3], we have

$$\widetilde{f}_r^{(k)}(e) = \sum_{(n,j)\in I_d} \alpha_{d,n}(r) g_{n,j}^{(k)} H_{n,j}(e), \quad e \in \mathbb{S}^{d-1}, \tag{4.7}$$

where, in view of [7, formulae (7.355.1–2)],

$$\alpha_{2,n}(r) = 2\int_{-1}^{1} e^{-iru} P_n^2(u)\left(1-u^2\right)^{-1/2} \mathrm{d}u = (-i)^n 2\pi J_n(r), \tag{4.8}$$

while for $d \geq 3$, by virtue of [7, formula (7.321)],

$$\begin{aligned} \alpha_{d,n}(r) &= \sigma_{d-1} c_{d,n} \int_{-1}^{1} e^{-iru} C_n^{(d-2)/2}(u)\left(1-u^2\right)^{(d-3)/2} \mathrm{d}u \\ &= (-i)^n (2\pi)^{d/2} r^{1-d/2} J_{n+d/2-1}(r). \end{aligned} \tag{4.9}$$

After substitution (4.9) or (4.8) into (4.7) we obtain

$$\widetilde{f}_r^{(k)}(e) = (2\pi)^{d/2} r^{1-d/2} \sum_{(n,j)\in I_d} (-i)^n J_{n+d/2-1}(r) g_{n,j}^{(k)} H_{n,j}(e), \quad e \in \mathbb{S}^{d-1}.$$

Since, by definition,

$$I_k(e) = \frac{(-1)^k}{(2\pi)^d k!} \lim_{\delta\to 0} \int_0^\infty f_r^{(k)}(e) e^{-\delta r} r^{k\alpha+d-1} \,\mathrm{d}r,$$

we conclude that

$$\widetilde{I}_k(e) = \frac{(-1)^k}{(2\pi)^{d/2} k!} \sum_{(n,j)\in I_d} (-i)^n \delta_{k,n} g_{n,j}^{(k)} H_{n,j}(e), \quad e \in \mathbb{S}^{d-1}, \tag{4.10}$$

where

$$\delta_{k,n} = \lim_{\delta\to 0} \int_0^\infty e^{-\delta r} r^{k\alpha+d/2} J_{n+d/2-1}(r)\,\mathrm{d}r.$$

[7, formula (6.621.1)] implies

$$\delta_{k,n} = \frac{\Gamma(n+d+k\alpha)}{2^{n+d/2-1}\Gamma(n+d/2)} \lim_{\delta\to 0} \frac{1}{(1+\delta^2)^{(n+d+k\alpha)/2}} {}_2F_1\left(\frac{n+d+k\alpha}{2}, \frac{n-k\alpha-1}{2}, n+\frac{d}{2}, \frac{1}{1+\delta^2}\right)$$

or

$$\delta_{k,n} = \frac{\Gamma(n+d+k\alpha)}{2^{n+d/2-1}\Gamma(n+d/2)}\, {}_2F_1\left(\frac{n+d+k\alpha}{2}, \frac{n-k\alpha-1}{2}, n+\frac{d}{2}, 1\right). \tag{4.11}$$

If $(k\alpha - n)/2 \notin \mathbb{N}\cup\{0\}$, then, by [7, formula (9.122.1)], we have

$${}_2F_1\left(\frac{n+d+k\alpha}{2}, \frac{n-k\alpha-1}{2}, n+\frac{d}{2}, 1\right) = \frac{\Gamma(n+d/2)\Gamma(1/2)}{\Gamma((n-k\alpha)/2)\Gamma((n+k\alpha+d+1)/2)}. \tag{4.12}$$

Otherwise, $(k\alpha - n)/2 = m \geq 0$ and, by virtue of [28, formula (7.3.1.198)], we get

$${}_2F_1\left(\frac{d}{2}+n+m, -m-\frac{1}{2}, n+\frac{d}{2}, 1\right) = 0. \tag{4.13}$$

We obtain (2.4) by substitution (4.12) and (4.13) into (4.11) and then into (4.10). □

Lemma 4.3 *If the conditions of Theorem* 2.1 *hold, then* $g \in L_2^{q+\alpha+d/2}(\mathbb{S}^{d-1})$ *and*

$$\widetilde{g}(e) = \sum_{(n,j)\in I_d} \frac{a_{n,j}}{c(\alpha)\gamma_{1,n}} H_{n,j}(e), \quad e \in \mathbb{S}^{d-1}, \tag{4.14}$$

where $a_{n,j} = \int\limits_{\mathbb{S}^{d-1}} a(e)H_{n,j}(e)\,\sigma(\mathrm{d}e)$. *For* $\alpha = 1$ *the terms in* (4.14) *corresponding to* $n = 1$ *vanish.*

Proof. Let

$$\widetilde{a}(e) = \sum_{(n,j)\in I_d} a_{n,j} H_{n,j}(e), \quad e \in \mathbb{S}^{d-1},$$

be the harmonic expansion for the function $a(e)$.

First, consider the case $\alpha \neq 1$. In view of (4.4)–(4.6) and [8, Proposition 3.4.3], formula (2.3) yields

$$\widetilde{g}(e) = \sum_{(n,j)\in I_d} \alpha_{d,n} a_{n,j} H_{n,j}(e), \quad e \in \mathbb{S}^{d-1}, \tag{4.15}$$

where for $d \geq 3$

$$\alpha_{d,n} = \sigma_{d-1} c_{d,n} \int\limits_{-1}^{1} |u|^\alpha (1 - i\tan(\pi\alpha/2)\,\mathrm{sign}\, u) C_n^{(d-2)/2}(u)\big(1-u^2\big)^{(d-3)/2}\,\mathrm{d}u.$$

It is clear that

$$\alpha_{d,n} = \begin{cases} A_{d,2l} & \text{if} \quad n = 2l, \\ -i\tan(\pi\alpha/2) A_{d,2l+1} & \text{if} \quad n = 2l+1, \end{cases} \tag{4.16}$$

where

$$A_{d,n} = 2\sigma_{d-1} c_{d,n} \int\limits_0^1 u^\alpha C_n^{(d-2)/2}(u)\,\big(1-u^2\big)^{(d-3)/2}\,\mathrm{d}u.$$

By [27, formula (2.21.2.5)], we obtain for $d \geq 3$

$$A_{d,2l} = \frac{(-1)^l 2\pi^{(d-1)/2}\Gamma(l-\alpha/2)\Gamma((1+\alpha)/2)}{\Gamma(-\alpha/2)\Gamma(l+(d+\alpha)/2)}, \tag{4.17}$$

$$A_{d,2l+1} = \frac{(-1)^l 2\pi^{(d-1)/2}\Gamma(l-(\alpha-1)/2)\Gamma((2+\alpha)/2)}{\Gamma(-(\alpha-1)/2)\Gamma(l+(d+\alpha+1)/2)}. \tag{4.18}$$

Since

$$\frac{(-1)^l\Gamma(l-\alpha/2)}{\Gamma(-\alpha/2)} = \frac{\Gamma(\alpha/2+1)}{\Gamma(\alpha/2+1-l)},$$

formulas (4.17) and (4.18) are rewritten as

$$A_{d,2l} = \frac{2\pi^{(d-1)/2}\Gamma((1+\alpha)/2)\Gamma(\alpha/2+1)}{\Gamma(l+(d+\alpha)/2)\Gamma(\alpha/2+1-l)} = \frac{\pi^{d/2}\Gamma(1+\alpha)}{2^{\alpha-1}\Gamma(l+(d+\alpha)/2)\Gamma(\alpha/2+1-l)}, \tag{4.19}$$

$$\begin{aligned} A_{d,2l+1} &= \frac{2\pi^{(d-1)/2}\Gamma(\alpha/2+1)\Gamma((1+\alpha)/2)}{\Gamma(l+(d+\alpha+1)/2)\Gamma((\alpha+1)/2-l)} \\ &= \frac{\pi^{d/2}\Gamma(1+\alpha)}{2^{\alpha-1}\Gamma(l+(d+\alpha+1)/2)\Gamma((\alpha+1)/2-l)}. \end{aligned} \tag{4.20}$$

We obtain the same result for $d = 2$ thanks to [7, formula (7.346)].

Now one can unite (4.19) and (4.20) in a single formula:

$$A_{d,n} = \frac{\pi^{d/2}\Gamma(1+\alpha)}{2^{\alpha-1}\Gamma((n+d+\alpha)/2)\Gamma((\alpha+2-n)/2)}. \tag{4.21}$$

Substituting (4.21) into (4.16) and then into (4.15) we obtain (4.14). Indeed, we have (4.14) for $n = 2l$ due to

$$\Gamma(l-\alpha/2)\Gamma(1-(l-\alpha/2)) = \frac{(-1)^{l+1}\pi}{\sin(\pi\alpha/2)},$$

while for $n = 2l+1$ due to

$$\Gamma(l+(1-\alpha)/2)\Gamma(1-(l+(1-\alpha)/2)) = \frac{(-1)^l\pi}{\cos(\pi\alpha/2)}.$$

From (4.14), (4.17), (4.18), [17, Theorem 1.31] and the asymptotic relation

$$\frac{\Gamma(n+(d+\alpha)/2)}{\Gamma(n-\alpha/2)} = n^{\alpha+d/2}(1+o(1)), \quad n \longrightarrow \infty, \tag{4.22}$$

it follows that $a \in L_2^q(\mathbb{S}^{d-1})$ yields $g \in L_2^{q+\alpha+d/2}(\mathbb{S}^{d-1})$.

Now consider the case $\alpha = 1$. Let us see what follows from condition (2.1). By (4.4)–(4.6) and [8, Proposition 3.4.3], we have

$$\int\limits_{\mathbb{S}^{d-1}} \langle\varepsilon, e\rangle a(e)\,\sigma(\mathrm{d}e) = \sum_{(n,j)\in I_d} \alpha_{d,n} a_{n,j} H_{n,j}(\varepsilon), \quad \varepsilon \in \mathbb{S}^{d-1},$$

where for $d \geq 3$

$$\alpha_{d,n} = \sigma_{d-1} c_{d,n} \int\limits_{-1}^{1} u C_n^{(d-2)/2}(u)\left(1-u^2\right)^{(d-3)/2}\,\mathrm{d}u.$$

Observe that $\alpha_{d,2l} = 0$ since the function under the integral is odd. For odd n by virtue of [27, formulas (2.21.2.5–6)], we get $\alpha_{d,2l+1} = 0$ for all $l \geq 1$ but

$$\alpha_{d,1} = \frac{\pi^{d/2}}{\Gamma(d/2+1)}.$$

Therefore,

$$\int\limits_{\mathbb{S}^{d-1}} \langle \varepsilon, e\rangle a(e)\, \sigma(\mathrm{d}e) = \alpha_{d,1} \sum_{j=1}^{d} a_{1,j} H_{1,j}(\varepsilon), \quad \varepsilon \in \mathbb{S}^{d-1},$$

and condition (2.1) implies $a_{1,j} = 0,\ j = 1, \ldots, d.$

We obtain the same result for $d = 2$ due to [7, formula (7.346)].

Therefore, we again have (4.15) but now

$$\alpha_{d,n} = \sigma_{d-1} c_{d,n} \int\limits_{-1}^{1} (|u| + (2i/\pi) u \ln|u|) C_n^{(d-2)/2}(u) \left(1-u^2\right)^{(d-3)/2} \mathrm{d}u.$$

It is clear that

$$\alpha_{d,n} = \begin{cases} A_{d,2l} & \text{if} \quad n = 2l, \\ (2i/\pi) B_{d,2l+1} & \text{if} \quad n = 2l+1, \end{cases} \tag{4.23}$$

where

$$A_{d,2l} = 2\sigma_{d-1} c_{d,2l} \int\limits_{0}^{1} u C_{2l}^{(d-2)/2}(u) \left(1-u^2\right)^{(d-3)/2} \mathrm{d}u,$$

$$B_{d,2l+1} = 2\sigma_{d-1} c_{d,2l+1} \int\limits_{0}^{1} u \ln u\, C_{2l+1}^{(d-2)/2}(u) \left(1-u^2\right)^{(d-3)/2} \mathrm{d}u.$$

We have already calculated $A_{d,2l}$ (see (4.17) with $\alpha = 1$):

$$A_{d,2l} = \frac{(-1)^l 2\pi^{(d-1)/2} \Gamma(l-1/2)}{\Gamma(-1/2)\Gamma(l+(d+1)/2)} = \frac{(-1)^{l+1} \pi^{d/2-1} \Gamma(l-1/2)}{\Gamma(l+(d+1)/2)}. \tag{4.24}$$

Hence, it remains to calculate $B_{d,2l+1}$ for $l \geq 1$ (recall that $a_{1,j} = 0,\ j = 1, \ldots, d$). First, take $d = 2$. Then

$$\begin{aligned} B_{2,2l+1} &= 4 \int\limits_{0}^{1} u \ln u \cos((2l+1) \arccos u) \left(1-u^2\right)^{-1/2} \mathrm{d}u \\ &= 4 \int\limits_{0}^{\pi/2} \cos t \ln(\cos t) \cos((2l+1)t)\, \mathrm{d}t \\ &= 2 \int\limits_{0}^{\pi/2} \ln(\cos t) \cos((2l+2)t)\, \mathrm{d}t + 2 \int\limits_{0}^{\pi/2} \ln(\cos t) \cos(2lt)\, \mathrm{d}t. \end{aligned}$$

Owing to [26, formula (2.6.34.18)], we have

$$B_{2,2l+1} = \frac{\pi}{2} \left(\frac{(-1)^{l+2}}{l+1} + \frac{(-1)^{l+1}}{l} \right) = \frac{(-1)^{l+1}\pi}{2} \left(\frac{1}{l} - \frac{1}{l+1} \right) = \frac{(-1)^{l+1}\pi}{2l(l+1)}.$$

By straightforward calculations one can show (see Appendix) that for $d \geq 3$ and $l \geq 1$

$$B_{d,2l+1} = \frac{(-1)^{l+1}\pi^{d/2}\Gamma(l)}{2\Gamma(l+1+d/2)}. \tag{4.25}$$

Substituting (4.24) and (4.25) into (4.23), we get

$$\alpha_{d,n} = -\frac{i^n\pi^{d/2-1}\Gamma((n-1)/2)}{\Gamma((n+d+1)/2)}.$$

Thus, substituting $\alpha_{d,n}$ in (4.15) we obtain (4.14). The arguments utilized in the case $\alpha \neq 1$ imply $g \in L_2^{q+1+d/2}(\mathbb{S}^{d-1})$. □

Now we continue the proof of Theorem 2.1. Since $g \in L_2^{q+\alpha+d/2}(\mathbb{S}^{d-1})$, then also $g^k \in L_2^{q+\alpha+d/2}(\mathbb{S}^{d-1})$. By (2.5), (4.22) and [17, Theorem 1.31], this implies $I_k \in L_2^{q-(k-1)\alpha}(\mathbb{S}^{d-1})$ and, therefore, the conditions of Lemma 4.2 are fulfilled. Observe that Lemma 4.3 yields $\widetilde{a}_1(e) = \widetilde{a}(e)/c(\alpha),\ e \in \mathbb{S}^{d-1}$.

It remains to estimate $R_m(x)$. Obviously,

$$\begin{aligned}&\left(|x|^{(m+1)\alpha+d}R_m(x)\right)^2\\ &\leq \frac{1}{(m!)^2(2\pi)^{2d}}\left(\lim_{\delta\to 0}\int_0^\infty r^{(m+1)\alpha+d-1}\,\mathrm{d}r\int_{\mathbb{S}^{d-1}} e^{-ir\langle e_x,e\rangle-\delta r}g^{m+1}(e)\,\sigma(\mathrm{d}e)\right)^2 = I_{m+1}^2(e_x),\end{aligned}$$

that implies for any $t > 0$

$$t^{(m+1)\alpha+d}\left(\int_{\mathbb{S}^{d-1}} R_m^2(te)\,\sigma(\mathrm{d}e)\right)^{1/2} \leq \left(\int_{\mathbb{S}^{d-1}} I_{m+1}^2(e)\,\sigma(\mathrm{d}e)\right)^{1/2}.$$

Repeating the arguments, utilized in the proof of Lemma 4.2, for $I_{m+1}(e),\ m \leq q/\alpha$, we obtain $I_{m+1} \in L_2^{q-m\alpha}(\mathbb{S}^{d-1})$.

Thus, all the limits in (4.3) exist and, therefore, the theorem is proved. □

P r o o f o f T h e o r e m 2.2. Given $\varepsilon \in \mathbb{S}^{d-1}$ consider the function

$$\rho_\varepsilon(t) = (2\pi)^{-d}\int_{\mathbb{S}^{d-1}}\int_0^\infty e^{-irt\langle\varepsilon,e\rangle-r^\alpha g(e)}r^{d-1}\,\mathrm{d}r\,\sigma(\mathrm{d}e), \quad t \geq 0,$$

and decompose it, for any $m \in \mathbb{N}$, as

$$\rho_\varepsilon(t) = \sum_{k=0}^m \frac{\rho_\varepsilon^{(k)}(0)t^k}{k!} + \frac{\rho_\varepsilon^{(m+1)}(\delta t)t^{m+1}}{(m+1)!}, \quad 0 < \delta < 1, \tag{4.26}$$

where $\rho_\varepsilon^{(k)}(t)$ denotes the derivative of order k for $\rho_\varepsilon(t)$. Obviously, for $k \geq 0$

$$\rho_\varepsilon^{(k)}(0) = \frac{(-i)^k}{(2\pi)^d}\int_{\mathbb{S}^{d-1}}\int_0^\infty r^{k+d-1}e^{-r^\alpha g(e)}\langle\varepsilon,e\rangle^k\,\mathrm{d}r\,\sigma(\mathrm{d}e).$$

Since for any $k \geq 0$

$$\int_0^\infty r^{k+d-1}e^{-r^\alpha g(e)}\,\mathrm{d}r = \frac{\Gamma((k+d)/\alpha)}{\alpha(g(e))^{(k+d)/\alpha}},$$

we obtain

$$\rho_\varepsilon^{(k)}(0) = \frac{(-i)^k\Gamma((k+d)/\alpha)}{\alpha(2\pi)^d} \int\limits_{\mathbb{S}^{d-1}} \langle\varepsilon, e\rangle^k h^{(k)}(e)\,\sigma(\mathrm{d}e), \quad k \geq 0. \tag{4.27}$$

Let

$$\widetilde{h}^{(k)}(e) = \sum_{(n,j)\in I_d} h_{n,j}^{(k)} H_{n,j}(e), \quad e \in \mathbb{S}^{d-1}, \tag{4.28}$$

be the harmonic expansion for the function $h^{(k)}(e)$. By the Funk–Hecke formula and [27, formulas (2.21.2.5–6)] $(d \geq 3)$ and [7, formula (7.346)] $(d = 2)$, we have

$$\int\limits_{\mathbb{S}^{d-1}} \langle\varepsilon, e\rangle^k H_{n,j}(e)\,\sigma(\mathrm{d}e) = \frac{\pi^{d/2} k! H_{n,j}(\varepsilon)}{2^{k-1}\Gamma((n+k+d)/2)\Gamma((k+2-n)/2)}, \quad \varepsilon \in \mathbb{S}^{d-1}, \tag{4.29}$$

if k and n are both even or odd and $n \leq k$. Otherwise, this integral vanishes. Substituting (4.28) into (4.27) and taking into account (4.29) and [8, Proposition 3.4.3], we obtain

$$\frac{\rho_\varepsilon^{(k)}(0)}{k!} = \sum_{(n,j)\in I_{d,k}} \beta_{k,n} h_{n,j}^{(k)} H_{n,j}(\varepsilon), \quad k \geq 0. \tag{4.30}$$

Since $s(x) = \rho_{e_x}(|x|)$, the statement of the theorem follows from (4.26) and (4.30). It remains only to note that for $\alpha \in (1, 2)$

$$\sum_{k=0}^{\infty}\sum_{n=0}^{k} \beta_{k,n}^2 < \infty.$$

□

A Appendix

Here, we prove (4.25). Hence, our goal is to show that

$$2\sigma_{d-1} c_{d,2l+1} \int\limits_0^1 u \ln u C_{2l+1}^{(d-2)/2}(u) \left(1-u^2\right)^{(d-3)/2} \mathrm{d}u = \frac{(-1)^{l+1}\pi^{d/2}\Gamma(l)}{2\Gamma(l+1+d/2)}$$

or

$$\int\limits_0^1 u \ln u C_{2l+1}^{(d-2)/2}(u) \left(1-u^2\right)^{(d-3)/2} \mathrm{d}u = \frac{(-1)^{l+1}\pi\Gamma(l+(d-1)/2)}{8\Gamma((d-2)/2)\Gamma(l+3/2) l(l+d/2)}. \tag{A.1}$$

For the sake of brevity, we denote

$$I = \int\limits_0^1 u \ln u C_{2l+1}^{(d-2)/2}(u) \left(1-u^2\right)^{(d-3)/2} \mathrm{d}u = \int\limits_0^{\pi/2} \cos\phi \ln(\cos\phi) C_{2l+1}^{(d-2)/2}(\cos\phi)(\sin\phi)^{d-2}\mathrm{d}\phi.$$

By [7, formula (8.934.2)], we have

$$C_{2l+1}^{(d-2)/2}(\cos\phi) = \frac{2}{(\Gamma((d-2)/2))^2} \sum_{j=0}^{l} \frac{\Gamma(j+d/2-1)\Gamma(2l+d/2-j)}{j!(2l+1-j)!} \cos(2l+1-2j)\phi. \tag{A.2}$$

Making use of the trigonometric formulas, we obtain

$$\begin{aligned}\cos(2l+1-2j)\phi\cos\phi &= \frac{1}{2}(\cos(2(l-j)+2)\phi+\cos(2(l-j))\phi)\\ &= 2^{2(l-j)}\cos^{2(l-j+1)}\phi\\ &\quad+(2(l-j)+1)\sum_{k=0}^{l-j-1}\frac{(-1)^{k+1}\Gamma(2(l-j)-k)}{(k+1)!\Gamma(2(l-j)-2k)}\,2^{2(l-j-k-1)}\cos^{2(l-j-k)}\phi.\end{aligned}$$

Substituting (A.2) into I, we get

$$\begin{aligned}I=\frac{2}{(\Gamma((d-2)/2))^2}\Bigg[&\sum_{j=0}^{l-1}\frac{\Gamma(j+d/2-1)\Gamma(2l+d/2-j)}{j!(2l+1-j)!}\Bigg(2^{2(l-j)}I_{2(l-j+1)}\\ &+(2(l-j)+1)\sum_{k=0}^{l-j-1}\frac{(-1)^{k+1}\Gamma(2(l-j)-k)}{(k+1)!\Gamma(2(l-j)-2k)}2^{2(l-j-k-1)}I_{2(l-j-k)}\Bigg)\\ &+\frac{\Gamma(l+d/2-1)\Gamma(l+d/2)}{l!(l+1)!}I_2\Bigg],\end{aligned}$$

where, by virtue of [26, formula (2.6.34.16)],

$$\begin{aligned}I_{2m} &= \int_0^{\pi/2}\cos^{2m}\phi\ln(\cos\phi)(\sin\phi)^{d-2}\,\mathrm{d}\phi\\ &= \frac{1}{4}B\left(\frac{d-1}{2},\frac{2m+1}{2}\right)\left[\Psi\left(\frac{2m+1}{2}\right)-\Psi\left(\frac{2m+d}{2}\right)\right],\quad m\in\mathbb{N}.\end{aligned}\tag{A.3}$$

Here, $\Psi(x)$ denotes the logarithmic derivative of the gamma function.

Routine calculations of I, consisting in gathering coefficients under $I_2,\ldots,I_{2(l+1)}$, lead to the formula

$$I=\frac{1}{\Gamma((d-2)/2)}\sum_{j=0}^{l}\frac{(-1)^j2^{2l-2j+1}\Gamma(2l+d/2-j)}{j!\Gamma(2(l-j+1))}\,I_{2(l-j+1)}.$$

With substitution (A.3) in I, we obtain

$$\begin{aligned}I &= \frac{\Gamma((d-1)/2)}{\Gamma((d-2)/2)}\sum_{j=0}^{l}\frac{(-1)^j2^{2l-2j-1}\Gamma(2l+d/2-j)\Gamma(l-j+3/2)}{j!\Gamma(2(l-j+1))\Gamma(l-j+1+d/2)}\left[\Psi\left(l-j+3/2\right)-\Psi\left(l-j+1+d/2\right)\right]\\ &= \frac{\pi^{1/2}\Gamma((d-1)/2)}{4\Gamma((d-2)/2)}\sum_{j=0}^{l}\frac{(-1)^j\Gamma(2l+d/2-j)}{j!\Gamma(l-j+1)\Gamma(l-j+1+d/2)}\left[\Psi\left(l-j+3/2\right)-\Psi\left(l-j+1+d/2\right)\right]\\ &= \frac{\pi^{1/2}(-1)^l\Gamma((d-1)/2)}{4\Gamma((d-2)/2)}\sum_{m=0}^{l}\frac{(-1)^m\Gamma(l+m+d/2)}{m!(l-m)!\Gamma(m+1+d/2)}\left[\Psi\left(m+3/2\right)-\Psi\left(m+1+d/2\right)\right].\end{aligned}$$

Since

$$\begin{aligned}\Psi\left(m+3/2\right) &= \Psi\left(l+3/2\right)-\sum_{j=m}^{l-1}\frac{1}{j+3/2},\\ \Psi\left(m+1+d/2\right) &= \Psi\left(l+1+d/2\right)-\sum_{j=m}^{l-1}\frac{1}{j+1+d/2},\end{aligned}$$

we have

$$I = \frac{\pi^{1/2}(-1)^l \Gamma((d-1)/2)}{4\Gamma((d-2)/2)} \sum_{m=0}^{l} \frac{(-1)^m \Gamma(l+m+d/2)}{m!(l-m)!\Gamma(m+1+d/2)} \times \left[\Psi(l+3/2) - \Psi(l+1+d/2) - \sum_{j=m}^{l-1} \left(\frac{1}{j+3/2} - \frac{1}{j+1+d/2} \right) \right].$$

By [26, formula (4.2.5.47)],

$$\sum_{m=0}^{l} \frac{(-1)^m \Gamma(l+m+d/2)}{m!(l-m)!\Gamma(m+1+d/2)} = 0. \tag{A.4}$$

Therefore,

$$\begin{aligned} I &= \frac{\pi^{1/2}(-1)^l \Gamma((d-1)/2)}{4\Gamma((d-2)/2)} \sum_{m=0}^{l-1} \frac{(-1)^m \Gamma(l+m+d/2)}{m!(l-m)!\Gamma(m+1+d/2)} \left[-\sum_{j=m}^{l-1} \left(\frac{1}{j+3/2} - \frac{1}{j+1+d/2} \right) \right] \\ &= \frac{\pi^{1/2}(-1)^{l+1} \Gamma((d+1)/2)}{\Gamma((d-2)/2)} \sum_{m=0}^{l-1} \frac{(-1)^m \Gamma(l+m+d/2)}{m!(l-m)!\Gamma(m+1+d/2)} \sum_{j=m}^{l-1} \frac{1}{(2j+3)(2j+2+d)} \end{aligned}$$

or, changing the order of summation,

$$I = \frac{\pi^{1/2}(-1)^{l+1} \Gamma((d+1)/2)}{\Gamma((d-2)/2)} \sum_{j=0}^{l-1} \frac{1}{(2j+3)(2j+2+d)} \sum_{m=0}^{j} \frac{(-1)^m \Gamma(l+m+d/2)}{m!(l-m)!\Gamma(m+1+d/2)}.$$

By direct calculations, we have

$$\sum_{m=0}^{j} \frac{(-1)^m \Gamma(l+m+d/2)}{m!(l-m)!\Gamma(m+1+d/2)} = \frac{(-1)^j \Gamma(d/2+l+j+1)}{j!(l-1-j)!\Gamma(d/2+j+1) l(l+d/2)}$$

provided that $0 \le j \le l-1$. As we know, this sum equals zero for $j = l$ (see (A.4)). Therefore,

$$I = \frac{\pi^{1/2}(-1)^{l+1} \Gamma((d+1)/2)}{2\Gamma((d-2)/2) l(l+d/2)} \sum_{j=0}^{l-1} \frac{(-1)^j \Gamma(d/2+l+j+1)}{(2j+3) j!(l-1-j)!\Gamma(d/2+j+2)}. \tag{A.5}$$

It remains to calculate the sum in (A.5). Introduce the notation:

$$\begin{aligned} A_{n,m} &= \sum_{j=0}^{n} \frac{(-1)^j}{2j+2m+1} \cdot \frac{\Gamma(d/2+n+j+2m)}{j!(n-j)!\Gamma(d/2+j+2m)}, \\ B_{n,m} &= \sum_{j=1}^{n} \frac{(-1)^j j}{2j+2m+1} \cdot \frac{\Gamma(d/2+n+j+2m)}{j!(n-j)!\Gamma(d/2+j+2m)}. \end{aligned}$$

So, we need to calculate $A_{l-1,1}$.

By [26, formula (4.2.5.47)], we have

$$\sum_{j=0}^{n} \frac{(-1)^j \Gamma(d/2+n+j+2m)}{j!(n-j)!\Gamma(d/2+j+2m)} = (-1)^n.$$

Therefore,

$$\begin{aligned} A_{n,m} &= \frac{(-1)^n}{2m+1} + \sum_{j=1}^{n} (-1)^j \left(\frac{1}{2j+2m+1} - \frac{1}{2m+1} \right) \frac{\Gamma(d/2+n+j+2m)}{j!(n-j)!\Gamma(d/2+j+2m)} \\ &= \frac{(-1)^n}{2m+1} - \frac{2}{2m+1} \sum_{j=1}^{n} \frac{(-1)^j j}{2j+2m+1} \cdot \frac{\Gamma(d/2+n+j+2m)}{j!(n-j)!\Gamma(d/2+j+2m)} \\ &= \frac{1}{2m+1} ((-1)^n - 2B_{n,m}). \end{aligned} \tag{A.6}$$

Further, by (A.6),

$$\begin{aligned} B_{n,m} &= \sum_{j=1}^{n} \frac{(-1)^j}{2j+2m+1} \cdot \frac{\Gamma(d/2+n+j+2m)}{(j-1)!(n-j)!\Gamma(d/2+j+2m)} \\ &= \sum_{j=0}^{n-1} \frac{(-1)^{j+1}}{2j+2m+3} \cdot \frac{(j+2m+1+d/2)\Gamma(d/2+n+j+2m+1)}{j!(n-1-j)!\Gamma(d/2+j+2m+2)} \\ &= -(2m+1+d/2) A_{n-1,m+1} - B_{n-1,m+1} \\ &= (-1)^n \frac{4m+2+d}{4m+6} + \frac{2m+d-1}{2m+3} \, B_{n-1,m+1}. \end{aligned} \tag{A.7}$$

Making use of (A.6) and (A.7), we obtain

$$\begin{aligned} A_{l-1,1} &= \frac{1}{3} \big((-1)^{l-1} - 2B_{l-1,1} \big) \\ &= \frac{d+1}{3 \cdot 5} \big((-1)^{l-2} - 2B_{l-2,2} \big) = \frac{(d+1)(d+3)}{3 \cdot 5 \cdot 7} \big((-1)^{l-3} - 2B_{l-3,3} \big) \end{aligned}$$

or

$$A_{l-1,1} = \frac{\Gamma(5/2)\Gamma(2+(d+1)/2)}{3\Gamma(2+5/2)\Gamma((d+1)/2)} \big((-1)^{l-3} - 2B_{l-3,3} \big).$$

Therefore, after m steps we get

$$A_{l-1,1} = \frac{\Gamma(5/2)\Gamma(m+(d+1)/2)}{3\Gamma(m+5/2)\Gamma((d+1)/2)} \big((-1)^{l-m-1} - 2B_{l-m-1,m+1} \big).$$

For $m = l-1$ we have

$$A_{l-1,1} = \frac{\Gamma(5/2)\Gamma(l-1+(d+1)/2)}{3\Gamma(l+3/2)\Gamma((d+1)/2)} \big(1 - 2B_{0,l} \big) = \frac{\Gamma(5/2)\Gamma(l-1+(d+1)/2)}{3\Gamma(l+3/2)\Gamma((d+1)/2)} \tag{A.8}$$

since $B_{0,l} = 0$. Now we obtain (A.1) by substitution (A.8) into (A.5).

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