A CLASS OF MULTIPLIERS FOR $\mathcal{W}^\perp$

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Dedicated to Hillel Furstenberg on the occasion of his retirement

Abstract. Let $\mathcal{W}^\perp$ denote the class of ergodic probability preserving transformations which are disjoint from every weakly mixing system. Let $\mathcal{M}(\mathcal{W}^\perp)$ be the class of multipliers for $\mathcal{W}^\perp$, i.e. ergodic transformations whose all ergodic joinings with any element of $\mathcal{W}^\perp$ are also in $\mathcal{W}^\perp$. Fix an ergodic rotation T, a mildly mixing action S of a locally compact second countable group G and an ergodic cocycle ϕ for T with values in G. The main result of the paper is a sufficient (and also necessary by [LeP] when G is countable Abelian and S is Bernoullian) condition for the skew product build from T, ϕ and S to be an element of $\mathcal{M}(\mathcal{W}^\perp)$. Moreover, the self-joinings of such extensions of T are described with an application to study semisimple extensions of rotations.

0. Introduction

In 1967 H. Furstenberg introduced a concept of disjointness for ergodic transformations as a sort of "extreme nonsimilarity" for them [Fu1]. In particular, disjoint transformations are nonisomorphic and even more, they have no nontrivial common factors. A nontrivial problem coming from [Fu1] is to describe the class $\mathcal{W}^\perp$ of those ergodic transformations that are disjoint with every weakly mixing transformation. It was actually shown there that $\mathcal{W}^\perp$ contains the class $\mathcal{D}$ of distal transformations. The fact that this inclusion is proper was established only in 1989 by E. Glasner and B. Weiss [GlW]. Later E. Glasner introduced a class $\mathcal{M}(\mathcal{W}^\perp)$ of multipliers for $\mathcal{W}^\perp$, i.e. the class of transformations whose all ergodic joinings with any member of $\mathcal{W}^\perp$ are also in $\mathcal{W}^\perp$. We then have $\mathcal{D} \subset \mathcal{M}(\mathcal{W}^\perp) \subset \mathcal{W}^\perp$. Elaborating the ideas from [GlW], E. Glasner demonstrated that $\mathcal{D} \neq \mathcal{M}(\mathcal{W}^\perp)$ [Gl1]. Finally, in a recent paper of F. Parreau and the second named author [LeP] it was shown that $\mathcal{M}(\mathcal{W}^\perp) \neq \mathcal{W}^\perp$. We now give some details on the latter result. Let T be an ergodic measure preserving transformation of a standard probability space $(X, \mathfrak{B}_X, \mu)$, $S = (S_g)_{g\in G}$ a measure

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preserving action of a locally compact second countable (l.c.s.c.) group G on a standard probability space $(Y, \mathfrak{B}_Y, \nu)$ and ϕ: $X \to G$ a Borel map. Throughout the paper we assume that G is not compact. Define two measure preserving transformations T_ϕ and $T_{\phi,S}$ of the product spaces $(X \times G, \mu \times \lambda_G)$ and $(X \times Y, \mu \times \nu)$ respectively by setting

$$T_\phi(x, g) = (Tx, \phi(x)g), \quad \text{and} \quad T_{\phi,S}(x, g) = (Tx, S_{\phi(x)}y),$$

where λ_G stands for a left Haar measure on G. Note that T_ϕ is infinite measure preserving. The following result was proved in [LeP]: if $T \in \mathcal{W}^\perp$, G is countable Abelian, S Bernoullian, ϕ is ergodic (i.e. T_ϕ is ergodic) and the group $e(T_\phi) \subset \mathbb{T}$ of $L^\infty(X \times G, \mu \times \lambda_G)$-eigenvalues of T_ϕ is uncountable then $T_{\phi,S} \in \mathcal{W}^\perp$ and for any weakly mixing transformation R whose (reduced) maximal spectral type does not vanish on $e(T_\phi)$ there exists an ergodic self-joining η of $T_{\phi,S}$ such that $(T_{\phi,S} \times T_{\phi,S}, \eta)$ is not disjoint from R. In this connection a question arises: what happens if $e(T_\phi)$ is countable? The answer is the main result of the paper (see Section 8):

Theorem 0.1. *Let T be an ergodic transformation with pure point spectrum and let G be an amenable l.c.s.c. group without nontrivial compact normal subgroups. Assume that S is a mildly mixing action of G. If ϕ: $X \to G$ is an ergodic cocycle of T for which $e(T_\phi)$ is countable then $T_{\phi,S}$ belongs to $\mathcal{M}(\mathcal{W}^\perp)$.*

This finally explains a relationship between Glasner-Weiss' generic techniques and our construction. Actually, we show that the set of ergodic cocycles with $e(T_\phi)$ countable is generic in the Polish space of all measurable maps from X to G. Moreover, the same is true for the subspace Φ_0 of continuous zero mean $\mathbb{R}$-valued cocycles of any irrational rotation (Φ_0 is furnished with the topology of uniform convergence). Taking any horocycle flow as S we then get as a corollary an extension of the main result from [Gl1]: $T_{\phi,S} \in \mathcal{M}(\mathcal{W}^\perp) \setminus \mathcal{D}$ for a generic ϕ from Φ_0 (there were some further restrictions on S and the rotation in [Gl1]).

Moreover, we obtain a full description of possible ergodic self-joinings of $T_{\phi,S}$ (under the assumptions of Theorem 0.1). This problem was already examined in [LMN] for Abelian G and some ergodic cocycles ϕ with the property that $\phi \times \phi \circ R$ is regular for each transformation R commuting with T. In this paper we make a step forward and analyze the general case of G and ergodic ϕ (but S is still mildly mixing). The case $\phi \times \phi \circ R$ is regular is treated similarly to the Abelian one. However, quite surprisingly it turns out that the case of nonregular $\phi \times \phi \circ R$ is easily handled due to a special property of its Mackey $G \times G$-action. In fact, the relatively independent extension of the graph joining μ_R is the only extension of μ_R to a self-joining of $T_{\phi,S}$ (see Theorem 7.3 for the precise statement).

Thus, as appears, the description of self-joinings of $T_{\phi,S}$ is very similar to what we have in the classical case of compact G (cf. [LeM], [Me]). As an application, we extend the main result of [LMN]:

Theorem 0.2. *Let T, G, S satisfy the assumptions of Theorem 0.1. If S is in addition 2-fold-extra-simple (i.e. for each continuous group automorphism θ of G, every ergodic joining of S and $S \circ \theta$ is either the product measure or a graph-joining) then $T_{\phi,S}$ is semisimple and the extension $T_{\phi,S} \to T$ is relatively weakly mixing for every ergodic cocycle ϕ: $X \to G$.*

Notice that in the present paper we bypass the use of the spectral theory which played a crucial role in [LeL], [LeP] and [LMN]. That enables us to get rid of the commutativity assumption on G which was standing in those papers.

Finally, we would like to note that even though $T_{\phi,S} \to T$ seems to be a very special case of a general extension (see a theorem of L. Abramov and V. Rokhlin [AbR]), however one of our first observations is that each Rokhlin cocycle is cohomologous to a "locally compact" one. In other words, each extension is isomorphic as extension to one of the form $T_{\phi,S} \to T$. In fact, G can be taken as countable and amenable (see Proposition 2.1).

The outline of the paper is as follows. Section 1 contains a background on nonsingular group actions, joinings and measurable orbit theory. In Section 2 we show that any extension can be given by an amenable countable group action. Sections 3–6 are of technical nature. Group self-joinings and their connection with type I actions are considered in Section 3. Some specific properties of the Mackey actions associated to $\phi \times \phi \circ R$ are discussed in Section 4. In Section 5 we introduce a concept of relatively finite measure preserving extensions and investigate their properties. A useful link between some simplices of invariant and quasi-invariant measures is discussed in Section 6. The main results of the paper are collected in Sections 7–9: the ergodic self-joinings of $T_{\phi,S}$ are described in Section 7, the theorem on multipliers for $\mathcal{W}^{\perp}$ is proved in Section 8 and semisimplicity of $T_{\phi,S}$ is studied in the final Section 9.

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1. Notation. Preliminaries

Nonsingular transformations and group actions. Let $(X, \mathfrak{B}_X, \mu)$ be a standard probability space. The group of μ-nonsingular transformations of X will be denoted by $\mathrm{Aut}(X, \mu)$. There exists a natural embedding $T \mapsto U_T$ of $\mathrm{Aut}(X, \mu)$ into the unitary group of $L^2(X, \mu)$ given by

$$U_T f(x) = f(T^{-1}x) \sqrt{\frac{d\mu \circ T^{-1}}{d\mu}(x)}, \ f \in L^2(X, \mu), \ x \in X.$$

Then $\mathrm{Aut}(X, \mu)$ endowed with the weak operator topology is a Polish group. The subgroup $\mathrm{Aut}_0(X, \mu)$ of μ-preserving transformations is closed in $\mathrm{Aut}(X, \mu)$.

Let G be a l.c.s.c. group. An ergodic nonsingular action $S = (S_g)_{g\in G}$ of G on $(X, \mathfrak{B}_X, \mu)$ is called **of type** I if μ is supported by a single orbit of S. Otherwise S is called **properly ergodic**. Given two nonsingular G-actions $S = (S_g)_{g\in G}$ and $Q = (Q_g)_{g\in G}$ on $(X, \mathfrak{B}_X, \mu)$ and $(Y, \mathfrak{B}_Y, \nu)$ respectively, we denote by $S \times Q$ (resp. $S \otimes Q$) the following G- (resp. $G \times G$-) action on the product space $(X \times Y, \mathfrak{B}_X \otimes \mathfrak{B}_Y, \mu \times \nu)$:

$$(S \times Q)(g) = S(g) \times Q(g), \quad (S \otimes Q)(g, h) = S(g) \times Q(h), \quad g, h \in G.$$

A properly ergodic action S is called **mildly mixing** (see [FuW], [SWa]) if for any properly ergodic G-action Q, the action $S \times Q$ is ergodic. As was shown in [SWa], such an S preserves an equivalent invariant probability measure. Moreover, a probability preserving S is mildly mixing if and only if for any sequence $g_n \to \infty$ in G and a measurable subset $A \in \mathfrak{B}_X$ with $\lim_{n\to\infty} \mu(S_{g_n} A \triangle A) = 0$, we have $\mu(A) = 0$ or $\mu(A) = 1$. Hence for any noncompact closed subgroup $H \subset G$, the action $S(H)$ is also mildly mixing.

For an action S of G, we denote by $C(S)$ the **centralizer** of S, i.e.

$$C(S) = \{T \in \mathrm{Aut}(X, \mu) |\ TS_g = S_g T \text{ for all } g \in G\}.$$

For a single transformation T, $C(T)$ denotes $C(\{T^n \mid n \in \mathbb{Z}\})$.

By a **cocycle** of a nonsingular transformation T on $(X, \mathfrak{B}_X, \mu)$ with values in G we mean a measurable map from X to G. The set of all such cocycles is denoted by $Z^1(T, G)$. Endowed with the topology of convergence in measure it is a Polish space. Two cocycles $\phi, \psi \in Z^1(T, G)$ are called **cohomologous** if

$$\phi(x) = a(x)\psi(x)a(Tx)^{-1}$$

for some measurable map $a \colon X \to G$ at a.a. $x \in X$.

Joinings and disjointness. Given two transformations $T_i \in \mathrm{Aut}_0(X_i, \mu_i)$, we denote by $J(T_1, T_2)$ the set of **joinings** of T_1 and T_2, i.e. the set of $T_1 \times T_2$-invariant measures η on $\mathfrak{B}_{X_1} \otimes \mathfrak{B}_{X_2}$ whose marginal on $\mathfrak{B}_{X_i}$ is μ_i, $i = 1, 2$. The corresponding dynamical system $(X_1 \times X_2, \mathfrak{B}_{X_1} \otimes \mathfrak{B}_{X_2}, \eta, T_1 \times T_2)$ is also called a joining of T_1 and T_2. By $J^e(T_1, T_2) \subset J(T_1, T_2)$ we denote the subset of ergodic joinings (it is nonempty whenever T_1 and T_2 are ergodic). Considering three transformations T_1, T_2 and T_3 we define in a similar way $J(T_1, T_2, T_3)$ and $J^e(T_1, T_2, T_3)$. If $J(T_1, T_2) = \{\mu_1 \times \mu_2\}$ then T_1 and T_2 are called **disjoint** [Fu1]. This will be denoted by $T_1 \perp T_2$. If $T_1 = T_2 =: T$ we speak about self-joinings of T and use notation $J_2(T)$ for $J(T_1, T_2)$. Given an extension

$$(X, \mathfrak{B}_X, \mu, T) \to (Y, \mathfrak{B}_Y, \nu, S),$$

consider the desintegration of μ with respect to ν: $\mu = \int_Y \mu_y \, d\nu(y)$. If now $\eta \in J_2(S)$ then the measure $\widetilde{\eta} := \int_{Y\times Y} \mu_y \times \mu_{y'} \, d\eta(y, y')$ is a self-joining of T.

It is called the **relatively independent extension** of η. Let Δ_Y stand for the diagonal self-joining of S. Assuming that S is ergodic, the extension $T \to S$ is called **relatively weakly mixing** if the relatively independent extension of Δ_Y is ergodic. An ergodic transformation T of $(X, \mathfrak{B}, \mu)$ is called **semisimple** [JLM] if for each $\eta \in J_2^e(T)$, the extension $(T \times T, \eta) \to (T, \mu)$ is relatively weakly mixing. Recall also that T is **2-fold simple** [JRu] if every $\eta \in J_2^e(T)$ is either the product measure $\mu \times \mu$ or a graph joining, i.e. the joining supported by the graph of some $R \in C(T)$.

Given a class $\mathcal{A}$ of ergodic transformations, by $\mathcal{M}(\mathcal{A})$ we denote the class of **multipliers** of $\mathcal{A}$ [Gl1], i.e. the class of transformations whose all ergodic joinings with an arbitrary element of $\mathcal{A}$ give rise to a transformation that is still in $\mathcal{A}$. Let $\mathcal{W}$ and $\mathcal{D}$ stand for the classes of weakly mixing transformations and distal transformations respectively, see [Fu2]. Summarizing the results on disjointness from [Fu1], [GlW], [Gl1] and [LeP] we can write

$$\mathcal{D} \subsetneqq \mathcal{M}(\mathcal{W}^\perp) \subsetneqq \mathcal{W}^\perp.$$

For a detailed account on joinings and related things we refer to [JRu], [Th] and [Gl2].

Orbit theory and cocycles. We will now briefly recall basics of the orbit theory. The facts we present below can be found in [Sc], [FM], [GS2], [Da2]. The reader should be aware that these facts are not all obvious.

Assume that T is an ergodic nonsingular transformation of $(X, \mathfrak{B}_X, \mu)$. Let $\mathcal{R}$ stand for the T-orbital equivalence relation. We recall definitions of the full group $[\mathcal{R}]$ of $\mathcal{R}$ and its normalizer $N[\mathcal{R}]$:

$$[\mathcal{R}] = \{S \in \operatorname{Aut}(X, \mu) |\ (x, Sx) \in \mathcal{R} \text{ for } \mu\text{-a.a. } x\},$$
$$N[\mathcal{R}] = \{S \in \operatorname{Aut}(X, \mu) |\ S[\mathcal{R}]S^{-1} = [\mathcal{R}]\}.$$

We will also use the notation $[T]$ for $[\mathcal{R}]$. A measurable map $\alpha: \mathcal{R} \to G$ is called a **cocycle** of $\mathcal{R}$ if

$$\alpha(x, y)\alpha(y, z) = \alpha(x, z) \text{ for all } (x, y), (y, z) \in \mathcal{R}.$$

Two cocycles $\alpha, \beta: \mathcal{R} \to G$ are said to be **cohomologous** (we then write $\alpha \approx \beta$) if there exists a measurable map $a: X \to G$ such that $\alpha(x, y) = a(x)\beta(x, y)a(y)^{-1}$ for a.a. $(x, y) \in \mathcal{R}$. Two cocycles $\alpha, \beta: \mathcal{R} \to G$ are called **weakly equivalent** if $\alpha \approx \beta \circ \theta$ for some $\theta \in N[\mathcal{R}]$. (The cocycle $\beta \circ \theta$ is defined by $\beta \circ \theta(x, y) = \beta(\theta x, \theta y)$.) Given a cocycle α of $\mathcal{R}$, we set $\phi_\alpha(x) := \alpha(Tx, x)$, $x \in X$. It is easy to check that the map $\alpha \mapsto \phi_\alpha$ is a bijection between the $\mathcal{R}$-cocycles and the T-cocycles. Moreover, $\alpha \approx \beta$ if and only if ϕ_α is cohomologous to ϕ_β.

Recall that λ_G denotes a left Haar measure on G. Let us fix a probability measure λ on G equivalent to λ_G. We define the following nonsingular transformations on $(X \times G, \mathfrak{B}_X \otimes \mathfrak{B}_G, \mu \times \lambda)$:

$$T_\phi(x, g) = (Tx, \phi(x)g), \quad R_h(x, g) = (x, gh^{-1}), \ h \in G.$$

The cocycle ϕ is called **recurrent** (resp. **ergodic**) if T_ϕ is conservative (resp. ergodic). Notice that $(R_h)_{h\in G}$ is a G-action commuting with T_ϕ. Hence it induces a nonsingular G-action $W_\phi = (W_\phi(g))_{g\in G}$ on the space $(\Omega_\phi, \mathfrak{B}_{\Omega_\phi}, \nu_\phi)$ of T_ϕ-ergodic components. This space is just $(X \times G, \mathfrak{F}, (\mu \times \lambda) \restriction \mathfrak{F})$, where $\mathfrak{F} \subset \mathfrak{B}_X \otimes \mathfrak{B}_G$ denotes the σ-algebra of T_ϕ-invariant subsets. W_ϕ is called the **Mackey action** (or the **associated action**) of ϕ. Since T is ergodic, so is W_ϕ.

If there exists a closed subgroup $H \subset G$ such that ϕ is cohomologous to an ergodic cocycle with values in H then ϕ is called **regular**. The subgroup H turns out to be determined by ϕ up to conjugacy and it is always amenable. Moreover, H is equal to the stabilizer of a point from Ω_ϕ. It can be shown that ϕ is regular if and only if ν_ϕ is supported by a single orbit (i.e. W_ϕ is of type I). Clearly, ϕ is ergodic if and only if W_ϕ is the trivial action on a singleton.

Next, if ϕ corresponds to a cocycle α of $\mathcal{R}$ (i.e. $\phi = \phi_\alpha$) then we will also write W_α for W_ϕ and call α recurrent, regular or ergodic if so is ϕ. Notice that if α is weakly equivalent to β and α is recurrent, regular or ergodic then so is β. Moreover, if α and β are weakly equivalent then W_α and W_β are isomorphic. A theorem of Golodets and Sinelshchikov states that conversely, if T is measure preserving, α and β are both recurrent with W_α and W_β isomorphic then α and β are weakly equivalent [GS2].

2. Rokhlin extensions and locally compact group extensions

Let $\widetilde{T}$ be an ergodic measure preserving transformation on a standard probability space $(Z, \mathfrak{B}_Z, \kappa)$ and let $\mathfrak{F} \subset \mathfrak{B}_Z$ be a factor of $\widetilde{T}$. By a classical theorem of Abramov-Rokhlin [AbR], the dynamical system $(Z, \mathfrak{B}_Z, \kappa, \widetilde{T})$ can be represented in a skew product form as follows:

$$(Z, \mathfrak{B}_Z, \kappa) = (X, \mathfrak{B}_X, \mu) \otimes (Y, \mathfrak{B}_Y, \nu) \quad \text{and} \quad \widetilde{T}(x, y) = (Tx, \psi(x)y),$$

where T is an ergodic transformation of $(X, \mathfrak{B}_X, \mu)$ and $\psi\colon X \to \mathrm{Aut}_0(Y, \nu)$ is a measurable map (sometimes called **Rokhlin cocycle** of T). In such a representation $\mathfrak{F}$ corresponds to $\mathfrak{B}_X$ (or, more precisely to $\mathfrak{B}_X \otimes \mathfrak{N}_Y$, where $\mathfrak{N}_Y$ stands for the trivial sub-σ-algebra of $\mathfrak{B}_Y$).

In this paper we mainly study extensions $\widetilde{T} \to T$ of a special form. Namely, let $S = (S_g)_{g\in G}$ be an ergodic measure preserving action of a l.c.s.c. group G on a standard probability space $(Y, \mathfrak{B}_Y, \nu)$. Take $\phi \in Z^1(X, G)$. Then we set

$$\widetilde{T}(x, y) := (Tx, S_{\phi(x)}y)$$

and denote this extension by $T_{\phi,S}$. The case of compact G was deeply investigated by a number of authors (see e.g. the bibliography in [LeL]). It will not be considered in this paper. In case $G = \mathbb{Z}^n$ or $\mathbb{R}^n$, extensions $T_{\phi,S} \to T$ were studied in [An], [Ki], [Ru], [GlW], [Gl1], [Ro], etc. Later a more general case of Abelian G was under consideration in [LeL], [LeP], [LNM]. As far as we know, non-Abelian G were not studied in this context (except of some simple facts from [LeL]).

We start with an observation that $T_{\phi,S} \to T$ is not a 'special' extension. In fact, every extension is isomorphic (as an extension) to such a one.

Proposition 2.1. *Let $\widetilde{T} \to T$ be an ergodic extension and let ψ be the corresponding Rokhlin cocycle of T as above. Then there exist a countable amenable group Σ (it does not depend on ψ) which acts ergodically on $(Y, \mathfrak{B}_Y, \nu)$ and a measurable cocycle ϕ: $X \to \Sigma$ such that ψ is cohomologous to ϕ in $\mathrm{Aut}_0(Y,\nu)$ (the natural embedding $\Sigma \subset \mathrm{Aut}_0(Y,\nu)$ is implicit here). Thus $\widetilde{T} \to T$ is isomorphic to $T_{\phi,\Sigma} \to T$.*

Proof. It is easy to see that $\mathrm{Aut}_0(Y,\nu)$ contains a dense countable subgroup Σ which is amenable in the discrete topology. Actually, if ν has an atom then (Y,ν) is measurably isomorphic to a finite cyclic group endowed with Haar measure. Therefore $\mathrm{Aut}_0(Y,\nu)$ is finite and hence amenable. If ν is nonatomic then we can represent (Y,ν) as $\bigotimes_{n=1}^{\infty}(\{0,1\},\lambda)$ with $\lambda(0) = \lambda(1) = 0.5$. Let Σ_{2^n} denote the permutation group of $\{0,1\}^n$. It acts on (Y,ν) permutating the first n coordinates. Then we have $\Sigma_2 \subset \Sigma_4 \subset \cdots \subset \mathrm{Aut}_0(Y,\nu)$. Clearly, the locally finite countable (and hence amenable) group $\Sigma := \bigcup_{n=1}^{\infty} \Sigma_{2^n}$ is dense in $\mathrm{Aut}_0(Y,\nu)$. Hence Σ is an ergodic transformation group. By [Da2, Proposition 1.6], ψ is cohomologous to a cocycle ϕ taking values in Σ. □

3. Group self-joinings

A closed subgroup $H \subset G \times G$ is called a **group self-joining** of G if the two coordinate projections of H to G are onto. Put $H_1 := \{g \in G |\ (g, 1_G) \in H\}$ and $H_2 := \{g \in G |\ (1_G, g) \in H\}$. Then H_1 and H_2 are closed normal subgroups of G and, moreover, there exists a topological group isomorphism θ: $G/H_2 \to G/H_1$ such that

$$H = \{(g_1, g_2) \in G^2 |\ \theta(g_2 H_2) = g_1 H_1\}. \tag{3-1}$$

Conversely, given two closed normal subgroups H_1, H_2 of G and a topological group isomorphism θ: $G/H_2 \to G/H_1$, by (3-1), we obtain a group self-joining H of G.

We denote the set of all group self-joinings of G by $J_2(G)$. Given $H \in J_2(G)$, we have a natural topological G^2-action Q_H on G/H_1:

$$Q_H(g_1, g_2) g H_1 = g_1 g H_1 \theta(g_2 H_2)^{-1} \quad \text{for all } g, g_1, g_2 \in G.$$

Clearly, a left Haar measure λ_{G/H_1} is Q_H-quasi-invariant. Slightly abusing notation, we will denote the coordinate G-actions given by the subgroups $G \times \{1_G\}$ and $\{1_G\} \times G$ by $Q(G \times \{1_G\})$ and $Q(\{1_G\} \times G)$ respectively. Notice that these actions are transitive. Now we prove a converse to that.

Lemma 3.1. *Let Q be a nonsingular action of G^2 on a standard probability space $(Z, \mathfrak{B}_Z, \kappa)$ such that the G-actions $Q(G \times \{1_G\})$ and $Q(\{1_G\} \times G)$ are ergodic and of type I. Then there exists $H \in J_2(G)$ such that Q is isomorphic to Q_H.*

Proof. Denote the G-actions $Q(G \times \{1_G\})$ and $Q(\{1_G\} \times G)$ by Q_1 and Q_2 respectively. Since Q_1 is ergodic and of type I, there exists a closed subgroup $H_1 \subset G$ such that Z is measurably isomorphic to the homogeneous space G/H_1 and Q_1 is the action by left translations; moreover, κ is equivalent to a Haar measure on G/H_1. Denote by $N_G(H_1)$ the normalizer of H_1 in G, i.e.

$$N_G(H_1) = \{g \in G| \, g^{-1} H_1 g = H_1\}.$$

Then the quotient group $N_G(H_1)/H_1$ acts on $(G/H_1, \kappa)$ by inverted right translations:

$$(nH_1) \cdot (gH_1) = gH_1 n^{-1}, \quad \text{for all } g \in G \text{ and } n \in N_G(H_1).$$

Notice that $C(Q_1) = N_G(H_1)/H_1$ (see, for example, [Da1]). Since $Q_2(G) \subset C(Q_1)$ and Q_2 is ergodic and of type I, we conclude that $N_G(H_1)/H_1$ acts transitively on G/H_1. It is easy to verify that this happens if and only if H_1 is normal in G. Moreover, Q_2 determines an epimorphism θ' of G onto G/H_1 such that

$$Q_2(g) \cdot g' H_1 = g' H_1 \theta'(g)^{-1} \quad \text{for all } g, g' \in G.$$

It remains to set $H_2 := \operatorname{Ker} \theta'$ and $H := \{(g_1, g_2) \in G^2 | \, \theta'(g_2) = g_1 H_1\}$. □

4. Mackey actions for $\phi \times \phi \circ R$

Let T be an ergodic measure preserving transformation of $(X, \mathfrak{B}_X, \mu)$ and $\phi, \psi \in Z^1(T, G)$. The associated actions W_ϕ, W_ψ and $W_{\phi \times \psi}$ are connected by the following duality.

Lemma 4.1.

(i) *W_ϕ is isomorphic to the restriction of $W_{\phi \times \psi}(G \times \{1_G\})$ to the σ-algebra of $W_{\phi \times \psi}(\{1_G\} \times G)$-invariant subsets, and*

(ii) *W_ψ is isomorphic to the restriction of $W_{\phi \times \psi}(\{1_G\} \times G)$ to the σ-algebra of $W_{\phi \times \psi}(G \times \{1_G\})$-invariant subsets.*

Proof. We only need to demonstrate (i). Let $\mathfrak{F}_\phi \subset \mathfrak{B}_X \otimes \mathfrak{B}_G$ and $\mathfrak{F}_{\phi \otimes \psi} \subset \mathfrak{B}_X \otimes \mathfrak{B}_G \otimes \mathfrak{B}_G$ stand for the σ-algebras of T_ϕ- and $T_{\phi \times \psi}$-invariant subsets respectively. Consider the sub-σ-algebra $\mathfrak{S}$ of those subsets $A \in \mathfrak{F}_{\phi \times \psi}$ which are invariant under all translations along the 'third' coordinate. It is easy to see that $A = A' \times G$ for a subset $A' \in \mathfrak{B}_X \otimes \mathfrak{B}_G$. Since $A \in \mathfrak{F}_{\phi \times \psi}$, it follows that $A' \in \mathfrak{F}_\phi$. Thus we obtain a Boolean isomorphism $\mathfrak{F}_\phi \ni A' \mapsto A' \times G \in \mathfrak{S}$ intertwining $W_\phi(g)$ with $W_{\phi \times \psi}(g, 1_G)$ for all $g \in G$. □

By an immediate use of the lemma we get the following.

Proposition 4.2. *If ϕ is ergodic and $R \in C(T)$ then the coordinate G-actions $W_{\phi\times\phi\circ R}(G \times \{1_G\})$ and $W_{\phi\times\phi\circ R}(\{1_G\} \times G)$ are both ergodic.*

We intend to prove a converse to Proposition 4.2 under an additional assumption that $R^n \notin [T]$ for all $n \neq 0$. It is easy to check that this is equivalent to the following: $R^n \neq T^m$ for all $n, m \in \mathbb{Z}$ with $n^2 + m^2 \neq 0$. In turn, this means that the joint $\mathbb{Z}^2$-action generated by R and T is free.

Proposition 4.3. *Let G be amenable and let V be a nonsingular ergodic G^2-action. Suppose that the G-actions $V(G \times \{1_G\})$ and $V(\{1_G\} \times G)$ are both ergodic. Then under the above assumption on R there exists an ergodic T-cocycle ϕ: $X \to G$ such that V is conjugate to the G^2-action associated to the product T-cocycle $\phi \times \phi \circ R$.*

Proof. It is convenient to make use of the language of the orbit theory in the proof. Let $\mathcal{R}$ stand for the T-orbit equivalence relation. By a theorem of Golodets-Sinelshchikov [GS1], there exists a recurrent cocycle

$$\alpha = \alpha_1 \times \alpha_2 \colon \mathcal{R} \to G \times G$$

such that the associated action W_α is conjugate to V. By Lemma 4.1, the Mackey G-action W_{α_1} is just the restriction of $W_\alpha(G \times \{1_G\})$ to the σ-algebra of $W_\alpha(\{1_G\} \times G)$-invariant subsets. However this σ-algebra is trivial since $V(\{1_G\} \times G)$ is ergodic. Thus W_{α_1} is the trivial action on a singleton. Hence α_1 is ergodic. In a similar way, α_2 is ergodic as well. Then by the uniqueness theorem for ergodic cocycles [GS2], there exists a transformation $Q \in N[\mathcal{R}]$ such that the cocycles $\alpha_1 \circ Q$ and α_2 are cohomologous. By a standard trick in the orbit theory (see [GS2], [Da1]) replacing, if necessary, α by a weakly equivalent cocycle we can assume without loss of generality that $Q^n \notin [\mathcal{R}]$ for all nonzero $n \in \mathbb{Z}$, i.e. Q is outer aperiodic in the sense of [CK]. On the other hand, by the assumptions, R is also outer aperiodic. Then the Connes-Krieger outer conjugacy theorem [CK] implies that $Q = tLRL^{-1}$ for some transformations $t \in [\mathcal{R}]$ and $L \in N[\mathcal{R}]$. Now we have

$$\begin{aligned}\alpha &= \alpha_1 \times \alpha_2 \approx \alpha_1 \times \alpha_1 \circ Q = \alpha_1 \times \alpha_1 \circ t \circ (LRL^{-1}) \approx \alpha_1 \times \alpha_1 \circ (LRL^{-1}) \\ &= (\alpha_1 \circ L \times \alpha_1 \circ L \circ R) \circ L^{-1}.\end{aligned}$$

Denote the cocycle $\alpha_1 \circ L$ by β. Then α is weakly equivalent to $\beta \times \beta \circ R$. Since the isomorphism class of the associated Mackey action is invariant under the weak equivalence of the underlying cocycles, the action $W_{\beta\times\beta\circ R}$ of G^2 is conjugate to V. It remains to define ϕ: $X \to G$ by setting $\phi(x) := \beta(x, Tx)$ and notice that

$$\beta \circ R(x, Tx) = \phi(Rx) \text{ for a.a. } x \in X. \quad \square$$

Remark 4.4. Using the same argument one can extend Proposition 4.3 as follows. Let V be a nonsingular ergodic G^2-action. Then there exists a recurrent T-cocycle ϕ: $X \to G$ such that $W_{\phi\times\phi\circ R}$ is conjugate to V if and only if the restriction of $V(G \times \{1_G\})$ to the σ-algebra of $V(\{1_G\} \times G)$-invariant subsets is isomorphic to the restriction of $V(\{1_G\} \times G)$ to the σ-algebra of $V(G \times \{1_G\})$-invariant subsets.

5. Ergodic decomposition and r.f.m.p. factors

Let $S = (S_g)_{g \in G}$ be a Borel action of a l.c.s.c. group G on a standard Borel space $(Y, \mathfrak{B}_Y)$. Let α: $G \times Y \to \mathbb{R}^*_+$ be a Borel map satisfying the following cocycle identity

$$\alpha(g_2 g_1, y) = \alpha(g_2, S_{g_1} y)\alpha(g_1, y) \quad \text{for all } y \in Y \text{ and } g_1, g_2 \in G.$$

Denote by $\mathcal{P}$ the set of S-quasi-invariant probability measures on $(Y, \mathfrak{B}_Y)$. Given $\nu \in \mathcal{P}$, we set

$$\mathcal{P}_\alpha := \left\{ \lambda \in \mathcal{P} \;\middle|\; \frac{d\lambda \circ S_g}{d\lambda}(y) = \alpha(g, y) \text{ at } \lambda\text{-a.e. } y \text{ for every } g \in G \right\} \text{ and}$$
$$\mathcal{E}_\alpha := \{\lambda \in \mathcal{P}_\alpha |\; S \text{ is ergodic with respect to } \lambda\}.$$

Notice that $\mathcal{P}_\alpha$ can be empty. Suppose this is not the case. Then clearly, $\mathcal{P}_\alpha$ is convex and $\mathcal{E}_\alpha$ is the set of extremal points of $\mathcal{P}_\alpha$. Notice that $\mathcal{P}_\alpha$ furnished with the natural Borel σ-algebra $\mathfrak{B}_{\mathcal{P}_\alpha}$ (making the map $\mathcal{P}_\alpha \ni \lambda \mapsto \lambda(B) \in \mathbb{R}$ Borel for any $B \in \mathfrak{B}_Y$) is a standard Borel space and $\mathcal{E}_\alpha$ is a Borel subset of it [GrS]. In view of the following lemma, $\mathcal{P}_\alpha$ can be interpreted as a Borel 'simplex' of nonsingular measures.

Lemma 5.1 [GrS]**.** *Given $\nu \in \mathcal{P}$ fix a Borel variant α_ν: $G \times Y \to \mathbb{R}^*_+$ of the Radon-Nikodym derivative of (S, ν). Then there exists a unique probability measure κ on $\mathcal{E}_{\alpha_\nu}$ such that*

$$\nu = \int_{\mathcal{E}_{\alpha_\nu}} \epsilon \, d\kappa(\epsilon). \tag{5-1}$$

Moreover, if $\mathfrak{F}$ stands for the σ-algebra of S-invariant subsets then $(\mathcal{E}_{\alpha_\nu}, \mathfrak{B}_{\mathcal{E}_{\alpha_\nu}} \kappa)$ is identified naturally with $(Y, \mathfrak{F}, \nu \restriction \mathfrak{F})$.

For a measure $\nu \in \mathcal{P}$, let $\mathfrak{F}$ be a factor of $(Y, \mathfrak{B}_Y, \nu, S)$. If S preserves ν and S is ergodic on $\mathfrak{F}$ then $\epsilon \restriction \mathfrak{F} = \nu \restriction \mathfrak{F}$ for κ-a.e. ϵ in (5-1). This 'good projection' property no longer holds for an arbitrary S-quasi-invariant measure ν. However, we will show that it survives in an important special 'nonsingular' case.

Definition 5.2. Given a measure $\nu \in \mathcal{P}$, a factor $\mathfrak{F}$ (and the extension $S \to S \restriction \mathfrak{F}$) is called **relatively finite measure preserving** (r.f.m.p.) if the Radon-Nikodym derivative $\frac{d\nu \circ S_g}{d\nu}$ is $\mathfrak{F}$-measurable for all $g \in G$.

In particular, $S \to S \restriction \mathfrak{N}_Y$ is r.f.m.p. if and only if S preserves ν. (Recall that $\mathfrak{N}_Y$ stands for the trivial sub-σ-algebra of $\mathfrak{B}_Y$.) Moreover, it is easy to verify that if $S \to S \restriction \mathfrak{F}$ is r.f.m.p. and $S \restriction \mathfrak{F}$ admits an equivalent invariant (finite or σ-finite) measure then so does S (it also follows from (5-2) below).

We can restate Definition 5.2 in an equivalent way. Denote the dynamical system $(Y, \mathfrak{F}, \nu \restriction \mathfrak{F}, S \restriction \mathfrak{F})$ by $(Z, \mathfrak{B}_Z, \kappa, V)$. Let $\pi\colon Y \to Z$ stand for the corresponding projection and $\nu = \int_Z \nu_z \, d\kappa(z)$ be the desintegration of ν with respect to κ. Then

$$\frac{d\nu \circ S_g}{d\nu}(y) = \frac{d\kappa \circ V(g)}{d\kappa}(\pi y) \frac{d\nu_{V(g)\pi(y)} \circ S_g}{d\nu_{\pi(y)}}(y)$$

at ν-a.e. y for all $g \in G$. Hence $\mathfrak{F}$ is r.f.m.p. if and only if

$$\frac{d\nu_{V(g)\pi(y)} \circ S_g}{d\nu_{\pi(y)}}(y) = 1 \ \text{ at } \nu\text{-a.e. } y \text{ for all } g \in G, \text{ i.e.}$$

$$\nu_{V(g)\pi(y)} \circ S_g = \nu_{\pi(y)} \text{ for all } g \in G. \tag{5-2}$$

Now we see that if $\mathfrak{F}$ is r.f.m.p. (with respect to ν) then by Lemma 5.1 the Radon-Nikodym derivative $\frac{d\epsilon \circ S_g}{d\epsilon}$ is $\mathfrak{F}$-measurable for κ-a.e. ϵ in (5-1) and all $g \in G$. Suppose in addition that $S \restriction \mathfrak{F}$ is ergodic. Since $\nu \restriction \mathfrak{F} = \int_{\mathcal{E}_{\alpha_\nu}} \epsilon \restriction \mathfrak{F} \, d\kappa(\epsilon)$, it follows from the uniqueness part of Lemma 5.1 that $\epsilon \restriction \mathfrak{F} = \nu \restriction \mathfrak{F}$ for κ-a.e. ϵ. Thus we have proved the following.

Proposition 5.3. *Let $\nu \in \mathcal{P}$. If $\mathfrak{F}$ is an ergodic r.f.m.p. factor of $(Y, \mathfrak{B}_Y, \nu, S)$ then for κ-a.e. ϵ from* (5-1)*, the restriction of ϵ to $\mathfrak{F}$ is equal to $\nu \restriction \mathfrak{F}$.*

We will also need the following simple lemma about r.f.m.p. extensions.

Lemma 5.4. *Let S be an ergodic nonsingular G-action on a standard probability space $(Y, \mathfrak{B}_Y, \nu)$ and let ρ be an $(S \times \mathrm{Id})$-quasi-invariant measure on $(Y \times Z, \mathfrak{B}_Y \otimes \mathfrak{B}_Z)$. Assume that $(Y \times Z, \mathfrak{B}_Y \otimes \mathfrak{B}_Z, \rho, S \times \mathrm{Id}) \to (Y, \mathfrak{B}_Y, \nu, S)$ is an r.f.m.p. extension. Then $\rho = \nu \times \kappa$ for a probability measure κ on $\mathfrak{B}_Z$.*

Proof. Passing, if necessary, to a dense countable subgroup we may assume without loss of generality that G is countable. Let $\rho = \int (\delta_y \times \rho_y) \, d\nu(y)$ be the desintegration of ρ with respect to ν. It follows from (5-2) that $\rho_{S_g y} = \rho_y$ a.e. in ν for all $g \in G$. Since S is ergodic and the map $Y \ni y \mapsto \rho_y$ is measurable, the result follows. □

Now we give a natural example of r.f.m.p. factors.

We will need the following nonsingular version of the Abramov-Rokhlin theorem on factors (see [Ra]):

Let V be an ergodic nonsingular action of G on a standard probability space $(Z, \mathfrak{B}_Z, \kappa)$ and $\mathfrak{F}$ a factor of V (i.e. a V-invariant sub-σ-algebra). Then there exist a measure space isomorphism Λ of $(Z, \mathfrak{B}_Z, \kappa)$ onto a product measure space $(X, \mathfrak{B}_X, \mu) \times (Y, \mathfrak{B}_Y, \nu)$, a nonsingular action W of G on $(X, \mathfrak{B}_X, \mu)$ and a Borel cocycle

$$F\colon G \times X \ni (g, x) \mapsto F(g, x) \in \mathrm{Aut}(Y, \nu)$$

such that $\{\Lambda(F) |\, F \in \mathfrak{F}\} = \{B \times Y |\, B \in \mathfrak{B}_X\}$ (mod 0) and

$$\Lambda V(g) \Lambda^{-1}(x, y) = (W(g)x, F(g, x)y)$$

at a.a. (x, y) for all $g \in G$.

Proposition 5.5. *Let V be an ergodic nonsingular action of G on a standard probability space $(Z, \mathfrak{B}_Z, \kappa)$ and let R be a κ-preserving transformation from the centralizer $C(V)$. Then the σ-algebra $\mathfrak{F}$ of R-invariant sets is an r.f.m.p. factor of V.*

Proof. By the nonsingular version of the Abramov-Rokhlin theorem, we may assume

$$(Z, \mathfrak{B}_Z, \kappa) = (X, \mathfrak{B}_X, \mu) \otimes (Y, \mathfrak{B}_Y, \nu)$$
$$V(g)(x, y) = (W(g)x, F(g, x)y), \ g \in G,$$

where $\mathfrak{F} = \mathfrak{B}_X \otimes \{\emptyset, Y\}$, W is the restriction of V to $\mathfrak{F}$ and

$$F \colon G \times X \ni (g, x) \mapsto F(g, x) \in \mathrm{Aut}(Y, \nu)$$

a Borel cocycle of W. Since $\mathfrak{F}$ is a factor of R as well and R acts as the identity on $\mathfrak{F}$, it follows that $R(x, y) = (x, R_x y)$ at a.a. (x, y) for a measurable field of nonsingular transformations $X \ni x \mapsto R_x \in \mathrm{Aut}(Y, \nu)$. Moreover, these transformations R_x are ergodic for a.a. x as the extension $\mathfrak{B}_Z \to \mathfrak{F}$ yields the R-ergodic decomposition. Since R preserves $\mu \times \nu$, we conclude immediately that R_x preserves ν for μ-a.e. x. Moreover, since

$$R^{-1}V(g)R(x, y) = (W(g)x, R^{-1}_{W(g)x}F(g, x)R_x y) = V(g)(x, y),$$

it follows that $R^{-1}_{W(g)x}F(g, x)R_x = F(g, x)$ at a.a. x for all $g \in G$. Hence

$$\frac{d\nu \circ F(g, x)}{d\nu}(y) = \frac{d\nu \circ F(g, x)}{d\nu}(R_x y)$$

at ν-a.e. y for μ-a.a. x and all $g \in G$. Therefore $\frac{d\nu \circ F(g,x)}{d\nu}$ is a constant ν-a.e. and this constant is obviously equal to 1, i.e. $F(g, x)$ preserves ν for μ-a.e. x and all $g \in G$. The latter is equivalent to the r.f.m.p. property of $\mathfrak{F}$ by (5-2). □

Notice that the above proposition is a natural generalization of the well-known fact that a nonsingular transformation commuting with an ergodic probability preserving transformation is itself measure preserving.

The proposition below will be used in the proof of the main result of the paper. Let T be an ergodic nonsingular transformation of $(X, \mathfrak{B}_X, \mu)$ and R a measure preserving transformation of $(Z, \mathfrak{B}_Z, \kappa)$ such that $T \times R$ is ergodic. Let $\phi \in Z^1(T, G)$. By $\phi \otimes 1$ we denote the following cocycle of $T \times R$:

$$\phi \otimes 1(x, z) = \phi(x), \quad (x, z) \in X \times Z.$$

Recall that a probability measure λ equivalent to a left Haar measure on G is fixed and that $(\Omega_\phi, \mathfrak{B}_{\Omega_\phi}, \nu_\phi)$ stands for the space of the Mackey G-action W_ϕ. Notice that since $T \times R$ is ergodic, the Mackey action $W_{\phi \otimes 1}$ is well defined on its measure space $(\Omega_{\phi\otimes 1}, \mathfrak{B}_{\Omega_{\phi\otimes 1}}, \nu_{\phi\otimes 1})$.

Proposition 5.6. *Assume that T, R and ϕ are as above. Denote by R' the restriction of the transformation* $\mathrm{Id} \times R \times \mathrm{Id} \in \mathrm{Aut}_0(X \times Z \times G, \mu \times \kappa \times \lambda)$ *to the σ-algebra of $(T \times R)_{\phi\otimes 1}$-invariant subsets. Then*

(i) *$R' \in C(W_{\phi\otimes 1})$ and it is a conservative transformation of $(\Omega_{\phi\otimes 1}, \nu_{\phi\otimes 1})$,*

(ii) *the natural projection π: $(\Omega_{\phi\otimes 1}, \nu_{\phi\otimes 1}) \to (\Omega_\phi, \nu_\phi)$ intertwining $W_{\phi\otimes 1}$ with W_ϕ yields the R'-ergodic decomposition.*

Proof. (i) The transformation $\mathrm{Id}\times R\times \mathrm{Id}$ is conservative since it preserves a finite measure. Hence R' is conservative (as a factor of a conservative map). Clearly, it commutes with $W_{\phi\otimes 1}$.

(ii) It suffices to notice that any R'-invariant subset A' is of the form

$$\{(x, z, g) |\, (x, g) \in A, z \in Z\}$$

for some subset $A \subset X \times G$. Clearly, A' is $(T \times R)_{\phi\otimes 1}$-invariant if and only if A is T_ϕ-invariant. $\square$

We deduce from Propositions 5.6 and 5.5 the following.

Corollary 5.7. *Under the assumptions of Proposition 5.6, the natural projection π is r.f.m.p.*

Using Corollary 5.7 and the remark just after Definition 5.2 we obtain the following.

Corollary 5.8. *Under the assumptions of Proposition 5.6:*

(i) *If W_ϕ admits an equivalent invariant finite (or σ-finite) measure then so does $W_{\phi\otimes 1}$.*

(ii) *If ϕ is ergodic (and hence W_ϕ is trivial) then $W_{\phi\otimes 1}$ preserves $\nu_{\phi\otimes 1}$.*

We note that the assertion (ii) of Corollary 5.8 was established in [LeP] for finite measure preserving T and Abelian G.

6. R.f.m.p. extensions $T_{\phi,S} \to T$ and associated Mackey actions

Let S be a Borel action of G on a standard Borel space $(Y, \mathfrak{B}_Y)$. For an invariant sub-σ-algebra $\mathfrak{F} \subset \mathfrak{B}_Y$ and a quasi-invariant measure κ on $\mathfrak{F}$ we let

$$\mathcal{P}(S, \mathfrak{F}, \kappa) := \{\nu \in \mathcal{P} |\, \nu \restriction \mathfrak{F} = \kappa \text{ and } \mathfrak{F} \text{ is an r.f.m.p. factor of } (Y, \mathfrak{B}_Y, \nu, S)\}.$$

Given an ergodic nonsingular transformation T of $(X, \mathfrak{B}_X, \mu)$ and a cocycle ϕ: $X \to G$ of T we are interested in the simplex $\mathcal{P}(T_{\phi,S}, \mathfrak{B}_X, \mu)$. Let $R = (R_g)_{g\in G}$ denote the nonsingular G-action on $(G, \mathfrak{B}_G, \lambda)$ by inverted right translations.

Our next statement is a slight modification and extension of a part of Proposition 2.1 from [LeP], where G was assumed Abelian and T measure preserving.

Consider the G-action $\mathrm{Id} \times R \times S$ on the product space $(X \times G \times Y, \mathfrak{B}_X \otimes \mathfrak{B}_G \otimes \mathfrak{B}_Y)$. It obviously commutes with the transformation $T_\phi \times \mathrm{Id}$. Hence their 'joint' $(\mathbb{Z} \times G)$-action, say V, is well defined on $X \times G \times Y$.

Proposition 6.1. *The simplices* $\mathcal{P}(V, \mathfrak{B}_X \otimes \mathfrak{B}_G, \mu \times \lambda)$*,* $\mathcal{P}(T_{\phi,S}, \mathfrak{B}_X, \mu)$ *and* $\mathcal{P}(W_\phi \times S, \mathfrak{B}_{\Omega_\phi}, \nu_\phi)$ *are pairwise affine isomorphic. Moreover, if* Λ *stands for the corresponding affine isomorphism of* $\mathcal{P}(T_{\phi,S}, \mathfrak{B}_X, \mu)$ *onto* $\mathcal{P}(W_\phi \times S, \mathfrak{B}_{\Omega_\phi}, \nu_\phi)$ *then* $\Lambda(\mu \times \nu) = \nu_\phi \times \nu$ *for any* S*-invariant measure* ν *on* Y*.*

Proof. Take any probability measure η on $X \times G \times Y$ projecting onto $\mu \times \lambda$ and let $\eta = \int_{X\times G} \delta_{(x,g)} \times \eta_{(x,g)}\, d\mu(x)d\lambda(g)$ be its desintegration. By definition, $\eta \in \mathcal{P}(V, \mathfrak{B}_X \otimes \mathfrak{B}_G, \mu \times \lambda)$ if and only if η is V-quasi-invariant and the extensions

$$(X \times G \times Y, \eta, (\mathrm{Id} \times R_g \times S_g)_{g\in G}) \to (X \times G, \mu \times \lambda, (\mathrm{Id} \times R_g)_{g\in G})$$
$$(X \times G \times Y, \eta, T_\phi \times \mathrm{Id}) \to (X \times G, \mu \times \lambda, T_\phi)$$

are r.f.m.p. By (5-2) this is equivalent to the following two equations on $\eta_{(x,g)}$:

$$\eta_{(x,gh^{-1})} = \eta_{(x,g)} \circ S_h^{-1} \tag{6-1}$$
$$\eta_{T_\phi(x,g)} = \eta_{(x,g)} \tag{6-2}$$

at a.e. (x, g) for every $h \in G$. It is a standard fact that the first equation admits a unique solution of the form $\eta_{(x,g)} = \eta^*_x \circ S_g$ at a.a. (x, g) for a measurable field $X \ni x \mapsto \eta^*_x$ of probability measures on Y. The second equation now means that $\eta^*_{Tx} = \eta^*_x \circ S^{-1}_{\phi(x)}$. We define a measure η^* on $X \times Y$ by setting $\eta^* := \int_X \delta_x \times \eta^*_x\, d\mu(x)$. By (5-2), $\eta^* \in \mathcal{P}(T_{\phi,S}, \mathfrak{B}_X, \mu)$. Clearly, the map $\eta \mapsto \eta^*$ is an affine isomorphism of $\mathcal{P}(V, \mathfrak{B}_X \otimes \mathfrak{B}_G, \mu \times \lambda)$ onto $\mathcal{P}(T_{\phi,S}, \mathfrak{B}_X, \mu)$.

Consider the T_ϕ-ergodic decomposition of $\mu \times \lambda$ (see Lemma 5.1): $\mu \times \lambda = \int_{\Omega_\phi} \omega\, d\nu_\phi(\omega)$. Then for any

$$\eta = \int_{X\times G} \delta_{(x,g)} \times \eta_{(x,g)} d\mu(x)d\lambda(g) \in \mathcal{P}(V, \mathfrak{B}_X \otimes \mathfrak{B}_G, \mu \times \lambda),$$

we have

$$\eta = \int_{\Omega_\phi} \int_{X\times G} \delta_{(x,g)} \times \eta_{(x,g)}\, d\omega(x, g)d\nu_\phi(\omega)$$

with $\eta_{(x,g)}$ satisfying (6-1) and (6-2). It follows from (6-2) that $\eta_{(x,g)} = \eta^\#_\omega$ at ω-a.a. (x, g) for a probability measure $\eta^\#_\omega$ on Y and ν_ϕ-a.a. ω. Now (6-1) implies that $\eta^\#_{W_\phi(g)\omega} = \eta^\#_\omega \circ S_g^{-1}$ at a.e. ω for all $g \in G$. Let $\eta^\#$ be a probability measure on $\Omega_\phi \times Y$ given by $\eta^\# = \int_{\Omega_\phi} \delta_\omega \times \eta^\#_\omega\, d\nu_\phi(\omega)$. It follows from the construction and (5-2) that the map $\eta \mapsto \eta^\#$ is an affine isomorphism of $\mathcal{P}(V, \mathfrak{B}_X \otimes \mathfrak{B}_G, \mu \times \lambda)$ onto $\mathcal{P}(W_\phi \times S, \mathfrak{B}_{\Omega_\phi}, \nu_\phi)$.

The second claim of the proposition can be verified now by a straightforward calculation. □

Remark 6.2. Let $\mathfrak{L} \subset \mathfrak{B}_Y$ be an S-invariant sub-σ-algebra. Suppose that for some $\rho \in \mathcal{P}(T_{\phi,S}, \mathfrak{B}_X, \mu)$, we have $\rho \restriction (\mathfrak{B}_X \otimes \mathfrak{L}) = \mu \times \nu_1$, where ν_1 is an S-invariant probability on $(Y, \mathfrak{L})$. Then by the proof of the second claim of Proposition 6.1, $\Lambda(\rho) \restriction (\mathfrak{B}_{\Omega_\phi} \otimes \mathfrak{L}) = \nu_\phi \times \nu_1$.

Remark 6.3 (on functorial properties of $*$ and $\#$). Let A be a measure preserving transformation of a standard probability space $(Z, \mathfrak{B}_Z, \kappa)$ such that the product $T \times A$ is ergodic. Then the map

$$\phi \otimes 1 \colon X \times Z \ni (x, z) \mapsto \phi(x) \in G$$

is a cocycle of $T \times A$. Next, we can define a $\mathbb{Z} \times G$-action V' on $(X \times Z \times G \times Y, \mu \times \kappa \times \lambda \times \nu)$ in perfect analogy with V. Since A preserves κ, the natural restrictions of measures induce the following affine onto maps:

$$\begin{aligned}
&\pi_1 \colon \mathcal{P}(V', \mathfrak{B}_X \otimes \mathfrak{B}_Z \otimes \mathfrak{B}_G, \mu \times \kappa \times \lambda) \to \mathcal{P}(V, \mathfrak{B}_X \otimes \mathfrak{B}_G, \mu \times \lambda),\\
&\pi_2 \colon \mathcal{P}((T \times A)_{\phi \otimes 1, S}, \mathfrak{B}_X \otimes \mathfrak{B}_Z, \mu \times \kappa) \to \mathcal{P}(T_{\phi, S}, \mathfrak{B}_X, \mu) \ \text{ and}\\
&\pi_3 \colon \mathcal{P}(W_{\phi \otimes 1} \times S, \mathfrak{B}_{\Omega_{\phi \otimes 1}}, \nu_{\phi \otimes 1}) \to \mathcal{P}(W_\phi \times S, \mathfrak{B}_{\Omega_\phi}, \nu_\phi).
\end{aligned}$$

We claim that they respect the maps $*$ and $\#$ constructed in the proof of Proposition 6.1, i.e. $\pi_1(\eta)^* = \pi_2(\eta^*)$ and $\pi_1(\eta)^\# = \pi_3(\eta^\#)$ for all

$$\eta \in \mathcal{P}(V', \mathfrak{B}_X \otimes \mathfrak{B}_Z \otimes \mathfrak{B}_G, \mu \times \kappa \times \lambda).$$

We only briefly prove the second formula (the first one is easier and we leave its verification to the reader). Take any $\eta \in \mathcal{P}(V', \mathfrak{B}_{X \times Z \times G}, \mu \times \kappa \times \lambda)$. Then

$$\eta = \int_{\Omega_{\phi \otimes 1}} \omega' \times \eta^\#_{\omega'} \, d\nu_{\phi \otimes 1}(\omega'). \tag{6-3}$$

Next, desintegrate $\nu_{\phi \otimes 1}$ with respect to ν_ϕ as follows

$$\nu_{\phi \otimes 1} = \int_{\tau^{-1}(\omega)} \xi_\omega \, d\nu_\phi(\omega), \tag{6-4}$$

where $\tau \colon (\Omega_{\phi \otimes 1}, \nu_{\phi \otimes 1}) \to (\Omega_\phi, \nu_\phi)$ is the natural projection intertwining $W_{\phi \otimes 1}$ with W_ϕ and substitute this into (6-3). By the uniqueness of desintegration, we obtain

$$\int_{\tau^{-1}(\omega)} \eta^\#_{\omega'} \, d\xi_\omega(\omega') = \pi_1(\eta)^\#_\omega \ \text{ for a.a. } \omega \in \Omega_\phi.$$

In a similar way, substituting (6-4) into

$$\eta^\# = \int_{\Omega_{\phi \otimes 1}} \delta_{\omega'} \times \eta^\#_{\omega'} \, d\nu_{\phi \otimes 1}(\omega')$$

we deduce that

$$\int_{\tau^{-1}(\omega)} \eta^\#_{\omega'} \, d\xi_\omega(\omega') = \pi_3(\eta^\#)_\omega \ \text{ for a.a. } \omega \in \Omega_\phi.$$

Hence $\pi_1(\eta)^\#_\omega = \pi_3(\eta^\#)_\omega$ for a.a. ω and we are done.

7. Lifting of joinings

We recall that the definitions of $J_2(G)$ and Q_H for an element $H \in J_2(G)$ were given in Section 3. We also notice that an $(S \otimes S)(H)$-invariant measure is both $S(H_1) \otimes \mathrm{Id}$- and $\mathrm{Id} \otimes S(H_2)$-invariant. In order to prove the main result of this section—Theorem 7.3—we need two auxiliary lemmas.

Lemma 7.1. *Let S_i be an ergodic measure preserving G-action on $(Y_i, \mathfrak{B}_{Y_i}, \nu_i)$, $i = 1, 2$. Assume that Q is a nonsingular G^2-action on a standard probability space $(Z, \mathfrak{B}_Z, \kappa)$ such that the coordinate G-actions $Q(\{1_G\} \times G)$ and $Q(G \times \{1_G\})$ are both ergodic.*

(i) *If S_2 is mildly mixing and $Q(\{1_G\} \times G)$ is properly ergodic then*

$$\{\rho \in \mathcal{P}((S_1 \otimes S_2) \times Q, \mathfrak{B}_Z, \kappa) |\ \rho \restriction (\mathfrak{B}_{Y_2} \otimes \mathfrak{B}_Z) = \nu_2 \times \kappa \text{ and } \rho \restriction \mathfrak{B}_{Y_1} = \nu_1\} = \{\nu_1 \times \nu_2 \times \kappa\},$$

(ii) *If $Q(\{1_G\} \times G)$ and $Q(G \times \{1_G\})$ are both of type I then*

$$\rho \in \mathcal{P}((S_1 \otimes S_2) \times Q, \mathfrak{B}_Z, \kappa)$$

if and only if there exists $H \in J_2(G)$ and an $(S \otimes S)(H)$-invariant measure ρ^* on $Y_1 \times Y_2$ such that (up to isomorphism) $Q = Q_H$, $Z = G/H_1$, κ is equivalent to a left Haar measure λ_{G/H_1} and

$$\rho = \int_Z \rho^* \circ (S_1(g) \times \mathrm{Id}) \times \delta_{gH_1}\, d\kappa(gH_1)$$

is the desintegration of ρ relative to κ.

Proof. (i) Take $\rho \in \mathcal{P}((S_1 \otimes S_2) \times Q, \mathfrak{B}_Z, \kappa)$. Then

$$\frac{d\rho \circ (S_1(g_1) \times S_2(g_2) \times Q(g_1, g_2))}{d\rho}(y_1, y_2, z) = \frac{d\kappa \circ Q(g_1, g_2)}{d\kappa}(z) \tag{7-1}$$

for ρ-a.e. (y_1, y_2, z), and all $(g_1, g_2) \in G^2$. Assume additionally that

$$\rho \restriction (\mathfrak{B}_{Y_2} \otimes \mathfrak{B}_Z) = \nu_2 \times \kappa.$$

It follows that the G-action $((S_2(g) \times Q(1_G, g))_{g \in G}, \rho \restriction (\mathfrak{B}_{Y_2} \otimes \mathfrak{B}_Z))$ is ergodic since S_2 is mildly mixing while $Q(\{1_G\} \times G)$ properly ergodic. Now put $g_1 = 1_G$ in (7-1) and apply Lemma 5.4 to deduce that $\rho = \nu' \times (\nu_2 \times \kappa)$ for a measure ν' on $\mathfrak{B}_{Y_1}$. If we assume in addition that $\rho \restriction \mathfrak{B}_{Y_1} = \nu_1$ then $\nu' = \nu_1$ and (i) follows.

(ii) By Lemma 3.1, there exists $H \in J_2(G)$ such that (up to isomorphism) $Z = G/H_1$, $Q = Q_H$ and κ is equivalent to λ_{G/H_1}. Let

$$\rho = \int_{G/H_1} \rho_{gH_1} \times \delta_{gH_1}\, d\kappa(\rho H_1)$$

be the desintegration of ρ. By (5-2),

$$\rho_{Q_H(g_1,g_2)gH_1} = \rho_{gH_1} \circ (S_1(g_1) \times S_2(g_2))$$

for κ-a.a. $gH_1 \in G/H_1$ and all $g_1, g_2 \in G$. Without loss of generality we may assume that this holds for all $g, g_1, g_2 \in G$. Let $\rho^* := \rho_{H_1}$. Since $Q_H(G \times \{1_G\})$ is transitive, we obtain that

$$\rho_{g_1H_1} = \rho^* \circ (S_1(g_1) \times \mathrm{Id}) \text{ for all } g_1 \in G. \tag{7-2}$$

Moreover, $\rho^* \circ (S_1(g_1) \times S_2(g_2)) = \rho^*$ for all $(g_1, g_2) \in H$ since H is the Q_H-stabilizer of the point $H_1 \in G/H_1$. The converse is also true: every $S_1 \otimes S_2(H)$-invariant measure ρ^* gives rise to a measure $\rho \in \mathcal{P}((S_1 \otimes S_2) \times Q, \mathfrak{B}_Z, \kappa)$ by (7-2). □

The lemma below was formulated in [LeL] only in the Abelian case but the proof in the non-Abelian case remains unchanged. It also follows immediately from Proposition 6.1.

Lemma 7.2. *Let G be amenable and let ϕ: $X \to G$ be an ergodic cocycle of an ergodic measure preserving transformation T of $(X, \mathfrak{B}_X, \mu)$. Assume that S is a Borel G-action on $(Y, \mathfrak{B}_Y)$. Suppose that ρ is an ergodic $T_{\phi,S}$-invariant measure on $X \times Y$ whose marginal onto X equals μ. Then $\rho = \mu \times \nu$ for an ergodic S-invariant measure ν.*

The following theorem provides a full description for the ergodic self-joinings of $T_{\phi,S}$ when T has pure point spectrum and S is mildly mixing.

Theorem 7.3. *Let T be an ergodic measure preserving transformation of the space $(X, \mathfrak{B}_X, \mu)$ with pure point spectrum and let $\eta \in J_2^e(T)$. Assume that S is a mildly mixing measure preserving action of G on $(Y, \mathfrak{B}_Y, \nu)$. Assume, moreover, that a cocycle ϕ: $X \to G$ is ergodic. If the cocycle*

$$\phi \otimes \phi \colon X \times X \ni (x_1, x_2) \mapsto (\phi(x_1), \phi(x_2)) \in G^2$$

of $(X \times X, \mathfrak{B}_X \otimes \mathfrak{B}_X, \eta, T \times T)$ is regular and cohomologous to an ergodic cocycle ψ with values in some $H \in J_2(G)$ then there exists an affine isomorphism Λ of the simplex

$$J_2(T_{\phi,S}, \eta) := \{\eta' \in J_2(T_{\phi,S}) |\, \eta' \upharpoonright (\mathfrak{B}_X \otimes \mathfrak{B}_X) = \eta\}$$

onto the simplex of $S \otimes S(H)$-invariant measures on $Y \times Y$. More precisely, if

$$\phi \otimes \phi(x_1, x_2) = f(x_1, x_2)\psi(x_1, x_2)f(Tx_1, Tx_2)^{-1} \quad \eta\text{-a.e.}$$

for a measurable function f: $X^2 \to G^2$, we define a map $A : (X \times Y)^2 \to X^2 \times Y^2$ by setting

$$A(x_1, y_1, x_2, y_2) = (x_1, x_2, S \otimes S(f(x_1, x_2))(y_1, y_2)).$$

Then $\eta' \circ A^{-1} = \eta \times \Lambda(\eta')$ for all $\eta' \in J_2(T_{\phi,S}, \eta)$.

Otherwise $J_2(T_{\phi,S}, \eta)$ consists of only one measure—the relatively independent extension of η.

Proof. Consider the first case. It has been studied in [LMN]. Though it was assumed that G is Abelian, this commutativity was not really used there. Therefore we only briefly sketch the idea of proof. Without loss of generality we may assume that $\phi \otimes \phi$ itself takes values in H. Indeed, changing a Rokhlin cocycle by a cohomologous one we always obtain an isomorphic extension. Then it remains to apply Lemma 7.2 and the first case easily follows.

Now we pass to the second case. Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ denote the $S \otimes S$-invariant sub-σ-algebras $\mathfrak{B}_Y \otimes \mathfrak{N}_Y$ and $\mathfrak{N}_Y \otimes \mathfrak{B}_Y$ of $\mathfrak{B}_Y \otimes \mathfrak{B}_Y$ respectively, where $\mathfrak{N}_Y$ stands for the trivial σ-algebra on Y. Since T has pure point spectrum, η is supported on the graph of a transformation $R \in C(T)$, i.e. $\eta(A \times B) = \mu(A \cap R^{-1}B)$ for all $A, B \in \mathfrak{B}_X$. Hence we may consider any measure $\eta' \in J_2(T_{\phi,S}, \eta)$ as a measure on $X \times Y \times Y$ invariant under $T_{\phi \times \phi \circ R, S \otimes S}$ and whose restriction to $\mathfrak{B}_X \otimes \mathfrak{L}_i$ is equal to $\mu \times \nu$, $i = 1, 2$. We have assumed that $\phi \otimes \phi$ is either nonregular or $\phi \otimes \phi$ is regular but the corresponding group $H \notin J_2(G)$. Therefore this assumption, Proposition 4.2 and Lemma 3.1 imply that at least one of the coordinate actions $W_{\phi \times \phi \circ R}(G \times \{1_G\})$ or $W_{\phi \times \phi \circ R}(G \times \{1_G\})$ is not of type I. It follows from Remark 6.2 that the affine isomorphism

$$\Lambda \colon \mathcal{P}(T_{\phi, \phi \circ R, S \otimes S}, \mathfrak{B}_X, \mu) \to \mathcal{P}(W_{\phi \times \phi \circ R}, \mathfrak{B}_{\Omega_{\phi \times \phi \circ R}}, \nu_{\phi \times \phi \circ R})$$

has the property that $\Lambda(\eta') \restriction (\mathfrak{B}_{\Omega_{\phi \times \phi \circ R}} \otimes \mathfrak{L}_i) = \nu_{\phi \times \phi \circ R} \times \nu$ for $i = 1, 2$. We can now apply Lemma 7.1(i) to conclude that the set

$$\{\rho \in \mathcal{P}(W_{\phi \times \phi \circ R} \times (S \otimes S), \mathfrak{B}_{\Omega_{\phi \times \phi \circ R}}, \nu_{\phi \times \phi \circ R}) |\ \rho \restriction (\mathfrak{B}_{\Omega_{\phi \times \phi \circ R}} \otimes \mathfrak{L}_i) = \nu_{\phi \times \phi \circ R} \times \nu,\ i = 1, 2\}$$

is a singleton. Hence the set

$$\mathcal{Q} := \{\eta' \in J_2(T_{\phi,S}) |\ \eta' \restriction (\mathfrak{B}_X \otimes \mathfrak{L}_i) = \mu \times \nu,\ i = 1, 2\}$$

is a singleton as well. It remains to notice that the relatively independent extension of η belongs to $\mathcal{Q}$. $\square$

Remark 7.4. It is worthwhile to note that the second case in Theorem 7.3 with nonregular $\phi \times \phi$ (which was not considered in [LMN]) is not vacuous. Actually, let T and S be as above and V any nonsingular G^2-action such that the G-actions $V(\{1_G\} \times G)$ and $V(G \times \{1_G\})$ are both ergodic. Suppose that at least one of the latter two actions is properly ergodic. Next, fix a transformation $R \in C(T)$ such that the joint $\mathbb{Z}^2$-action with generators T and R is free (notice that such a transformation always exists since T has pure point spectrum). Denote by η the self-joining of T supported by the graph of R. Then by Proposition 4.3 and Theorem 7.3 there exists an ergodic cocycle $\phi \in Z^1(T, G)$ such that $J_2(T_{\phi,S}, \eta)$ is a singleton and the Mackey action associated to the cocycle $\phi \otimes \phi$ of $(X \times X, \mathfrak{B}_X \otimes \mathfrak{B}_X, \eta, T \times T)$ is isomorphic to V.

8. Multipliers of $\mathcal{W}^{\perp}$

In this section the actions T, R, V and S considered below are assumed to be measure preserving. We need an auxiliary lemma from [LeP].

Lemma 8.1 [LeP, Proposition 5.1]**.** *Let T and R be ergodic transformations. If R is weakly mixing and $R\times R$ is disjoint from any ergodic self-joining of T then $T\in\mathcal{M}(\{R\}^{\perp})$.*

It follows immediately that in order to prove that $T\in\mathcal{M}(\mathcal{W}^{\perp})$ it is enough to show that every ergodic self-joining of T is disjoint from $\mathcal{W}$.

Let T be an ergodic transformation on $(X,\mathfrak{B}_X,\mu)$ such that $T\in\mathcal{W}^{\perp}$. Let $\phi\colon X\to G$ be an ergodic cocycle of T and S be an ergodic action of G on $(Y,\mathfrak{B}_Y,\nu)$. Assume that V is a weakly mixing transformation on $(Z,\mathfrak{B}_Z,\kappa)$. We claim that if $e(T_\phi)$ is countable then $T_{\phi,S}\perp V$. To prove this claim we notice first of all that $T\perp V$. Then observe that the cocycle $\phi\otimes 1\in Z^1(T\times V,G)$ is ergodic. Indeed, the skew product extension

$$(T\times V)_{\phi\otimes 1}=T_\phi\times V\in \mathrm{Aut}(X\times G\times Z,\mu\times\lambda_G\times\kappa)$$

is ergodic if and only if $\sigma_V(e(T_\phi))=0$ (see [Aa, p. 81]), where σ_V denotes the measure of maximal spectral type of V on $L^2(Z,\kappa)\ominus\mathbb{C}1$. It suffices now to notice that σ_V is continuous and $e(T_\phi)$ countable. In view of Lemma 7.2, our claim follows. Thus we have proved the following.

Proposition 8.2. *If T, ϕ, S are as above and $e(T_\phi)$ is countable then $T_{\phi,S}\in\mathcal{W}^{\perp}$.*

Now we are ready to prove the main result of the paper, i.e. Theorem 0.1 stated in Introduction.

Proof of Theorem 0.1. Let η be an ergodic self-joining of $T_{\phi,S}$. Take a weakly mixing transformation V of a standard probability space $(Z,\mathfrak{B}_Z,\kappa)$. Consider a joining $\eta'\in J^e(T_{\phi,S},T_{\phi,S},V)$ projecting onto η. In view of Lemma 8.1, to prove the theorem it is enough to show that $\eta'=\eta\times\kappa$.

Since T has pure point spectrum, the projection of η onto $X\times X$ is supported by the graph of a transformation $R\in C(T)$. Hence we can consider η and η' as measures on $X\times Y\times Y$ and $X\times Y\times Y\times Z$ invariant under the transformations $T_{\phi\times\phi\circ R,S\otimes S}$ and $T_{\phi\times\phi\circ R,S\otimes S}\times V$ respectively. Since T and V are disjoint, the projection of η' onto $X\times Z$ is $\mu\times\kappa$. Moreover, $R':=R\times\mathrm{Id}\in C(T\times V)$ and we can rewrite $T_{\phi\times\phi\circ R,S\otimes S}\times V$ as $(T\times V)_{\phi\otimes1\times(\phi\otimes1)\circ R',S\otimes S}$. Thus η' belongs to the simplex

$$\mathcal{P}((T\times V)_{\phi\otimes1\times(\phi\otimes1)\circ R',S\otimes S},\mathfrak{B}_X\otimes\mathfrak{B}_Z,\mu\times\kappa). \tag{8-1}$$

Moreover, by Proposition 8.2,

$$\begin{aligned}&\eta'\restriction(\mathfrak{B}_X\otimes\mathfrak{B}_Z\otimes\mathfrak{B}_Y\otimes\mathfrak{N}_Y)=\mu\times\kappa\times\nu\quad\text{and}\\&\eta'\restriction(\mathfrak{B}_X\otimes\mathfrak{B}_Z\otimes\mathfrak{N}_Y\otimes\mathfrak{B}_Y)=\mu\times\kappa\times\nu.\end{aligned}\tag{8-2}$$

Let $W_{\phi\times\phi\circ R}$ and $W_{(\phi\otimes 1)\times(\phi\otimes 1)\circ R'}$ act on their measure spaces $(\Omega, \mathfrak{B}_\Omega, \rho)$ and $(\Omega', \mathfrak{B}_{\Omega'}, \rho')$ respectively. By Proposition 6.1, the simplex (8-1) is affine isomorphic (via Λ) to the nonsingular simplex

$$\mathcal{P}(W_{(\phi\otimes 1)\times(\phi\otimes 1)\circ R'} \times (S \otimes S), \mathfrak{B}_{\Omega'}, \rho'). \tag{8-3}$$

Furthermore, in view of (8-2) and Remark 6.2,

$$\begin{aligned} &\Lambda(\eta') \restriction (\mathfrak{B}_{\Omega'} \otimes \mathfrak{B}_Y \otimes \mathfrak{N}_Y) = \rho' \times \nu, \text{ and} \\ &\Lambda(\eta') \restriction (\mathfrak{B}_{\Omega'} \otimes \mathfrak{N}_Y \otimes \mathfrak{B}_Y) = \rho' \times \nu. \end{aligned} \tag{8-4}$$

It follows from Proposition 4.2 and the fact that $\phi \otimes 1$ is ergodic (see the proof of Proposition 8.2) that the G-actions

$$W_{\phi\otimes 1\times(\phi\otimes 1)\circ R'}(G \times \{1_G\}) \text{ and } W_{\phi\otimes 1\times(\phi\otimes 1)\circ R'}(\{1_G\} \times G)$$

are ergodic. If at least one of them is properly ergodic then by Lemma 7.1(i), there is only one measure satisfying (8-4) and belonging to the simplex (8-3). Hence there is only one measure satisfying (8-2) and belonging to the simplex (8-1). Since the measure $\eta \times \kappa$ satisfies these properties, we conclude that $\eta' = \eta \times \kappa$.

Consider now the case where the transformation groups

$$W_{\phi\otimes 1\times(\phi\otimes 1)\circ R'}(G \times \{1_G\}) \text{ and } W_{\phi\otimes 1\times(\phi\otimes 1)\circ R'}(\{1_G\} \times G)$$

are both of type I. By Lemma 7.1(ii), there exist $H \in J_2(G)$ and a measure ρ^* on $(Y \times Y, \mathfrak{B}_Y \otimes \mathfrak{B}_Y)$ invariant under $S \otimes S(H)$ such that (up to isomorphism) $\Omega' = G/H_1$, $\rho' \sim \lambda_{G/H_1}$, $W_{\phi\otimes 1\times(\phi\otimes 1)\circ R'} = Q_H$ and

$$\Lambda(\eta') = \int_{\Omega'} \rho^* \circ (S(g) \times \mathrm{Id}) \times \delta_{gH_1}\, d\rho'(gH_1).$$

It follows from (8-4) that the marginals of ρ^* are equal to ν. Clearly,

$$H \supset (H_1 \times \{1_G\}) \cup (\{1_G\} \times H_2).$$

If H_1 is nontrivial then it is noncompact by the assumption on G. Since S is mildly mixing, the transformation group $S(H_1)$ is also mildly mixing and, in particular, ergodic. Therefore by Lemma 5.4, ρ^* splits into a direct product $\nu \times \nu_1$. Clearly, $\nu_1 = \nu$ by our observation on the marginals of ρ^*. In a similar way, if H_2 is nontrivial then $\rho^* = \nu \times \nu$. Thus in both cases there exists only one measure satisfying (8-4) and belonging to the simplex (8-3). Thus we get again $\eta' = \eta \times \kappa$.

It remains to consider the case where $H_1 = H_2 = \{1_G\}$. Then the subset of measures satisfying (8-4) and belonging to (8-3) does not need to be a singleton. (Consider, for instance, the case where H is the diagonal subgroup of $G \times G$.

Then the measure $\rho' \times \xi$ satisfies the two properties for any self-joining ξ of S.) To settle this case consider the natural projection $(\Omega', \mathfrak{B}_{\Omega'}, \rho') \to (\Omega, \mathfrak{B}_\Omega, \rho)$ intertwining $W_{(\phi\otimes 1)\times(\phi\otimes 1)\circ R'}$ with $W_{\phi\times\phi\circ R}$. By Proposition 5.6(ii) (the cocycle $\phi \times \phi \circ R$ plays now the role of ϕ from that corollary), it yields the ergodic decomposition of a transformation

$$D \in C(W_{(\phi\otimes 1)\times(\phi\otimes 1)\circ R'}) = C(Q_H).$$

Since $C(Q_H)$ is just the center $Z(G)$ of G acting on G by translations, we can identify D with an element $d \in Z(G)$. Let $K := \overline{\{d^n | \, n \in \mathbb{Z}\}}$. It is well known that the the quotient map $G \to G/K$ yields the ergodic decomposition of D. Any monothetic locally compact group is either compact or infinite discrete (and hence isomorphic to $\mathbb{Z}$) [HR]. Since D is conservative by Proposition 5.6(i), the latter is impossible for K. Hence K is compact and therefore trivial by our assumption on G. Thus the natural projection $\Omega' \to \Omega$ is the identity. Hence the natural projection of (8-3) onto the simplex $\mathcal{P}(W_{\phi\times\phi\circ R} \times (S \otimes S), \mathfrak{B}_\Omega, \rho)$ is one-to-one. Therefore so is the natural projection of (8-1) onto $\mathcal{P}(T_{\phi\times\phi\circ R}, \mathfrak{B}_X \otimes \mathfrak{B}_Z, \mu \times \kappa)$ (see Remark 6.3). Thus we get again $\eta' = \eta \times \kappa$. □

Proposition 8.3. *Let G be amenable and let T be an ergodic transformation. Assume that there exists $R \in C(T) \setminus \{T^n | \, n \in \mathbb{Z}\}$. Then the subset*

$$\mathcal{L} := \{\phi \in Z^1(T, G) | \, \phi \text{ is ergodic and } e(T_\phi) = e(T)\}$$

is generic in $Z^1(T, G)$.

Proof. It follows from the proof of Theorem 4.2(i) from [Da1] that the subset

$$\mathcal{M} := \{\phi \in Z^1(T, G) | \, \phi \times \phi \circ R \text{ is ergodic }\}$$

is a dense G_δ in $Z^1(T, G)$. Next, if $\lambda \in e(T_\phi) \setminus e(T)$ then by [ALV] there exists a nontrivial continuous homomorphism (character) $\chi\colon G \to \mathbb{T}$ such that $\chi \circ \phi \approx \lambda$ in $Z^1(T, \mathbb{T})$. Since R commutes with T, we obtain $\chi \circ \phi \circ R \approx \lambda$ as well. Therefore the cocycle $(\chi \times \chi) \circ (\phi \times \phi \circ R)$ is cohomologous to a constant (λ, λ) in $Z^1(T, \mathbb{T} \times \mathbb{T})$. Since the group generated by this constant is not dense in $\mathbb{T} \times \mathbb{T}$, we obtain that $(\chi \times \chi) \circ (\phi \times \phi \circ R)$ is not ergodic. Hence $\phi \times \phi \circ R$ is not ergodic as well. Thus $\mathcal{L} \supset \mathcal{M}$ and we are done. □

Corollary 8.4. *Let G, T and S be as in Theorem 0.1. Then for a generic cocycle $\phi \in Z^1(T, G)$ we have $T_{\phi,S} \in \mathcal{M}(\mathcal{W}^\perp) \setminus \mathcal{D}$.*

Proof. Since T has pure point spectrum, the centralizer $C(T)$ is nontrivial. Moreover, $e(T)$ is countable since for the probability preserving transformations the L^∞-spectrum equals to the L^2-spectrum. It now follows from Theorem 0.1 and Proposition 8.2 that $T_{\phi,S} \in \mathcal{M}(\mathcal{W}^\perp)$. It follows from Lemma 9.1 below that the extension $T_{\phi,S} \to T$ is relatively weakly mixing. Then by [Fu2], $T_{\phi,S}$ is not distal. □

Now we show how to deduce from that the main results of [Gl1]. Let $G = \mathbb{R}$ and S a horocycle flow corresponding to a lattice Γ in $\mathrm{PSL}_2(\mathbb{R})$. Recall that S is mixing of all degrees [Ma]. Let $(X, \mathfrak{B}, \mu) = (\mathbb{T}, \mathfrak{B}_{\mathbb{T}}, \lambda_{\mathbb{T}})$ and $Tx = xe^{2\pi i\alpha}$, $x \in \mathbb{T}$ ($\mathbb{T}$ denotes the circle group), for an irrational number $\alpha \in (0, 1)$. Denote by $[0 : a_1, a_2, \dots]$ the continued fraction expansion of α. Let $(q_n)_{n\geq 0}$ stand for the sequence of denominators of α, i.e.

$$q_0 = 1,\ q_1 = a_1,\ q_{k+1} = a_{k+1}q_k + q_{k-1},\ k \geq 1.$$

We define a cocycle $\phi_0 \in Z^1(T, \mathbb{R})$ by setting $\phi_0(e^{2\pi it}) = t-0.5$, where $0 \leq t < 1$. Ergodicity of ϕ_0 was established e.g. in [Pa]. We need a stronger result.

Proposition 8.5. *There exists a transformation $R \in C(T)$ such that the cocycle $\phi_0 \times \phi_0 \circ R$ of T is ergodic.*

To prove this proposition we need an auxiliary fact from [LMN] (see the proof of Lemma 3 in [LMN]).

Lemma 8.6. *Given $\beta \in (0, 1)$, let $Rx := xe^{2\pi i\beta}$, $x \in \mathbb{T}$. If the sequence $(\{q_n\beta\})_n$ has infinitely many accumulation points then the cocycle $\phi_0 \times \phi_0 \circ R$ of T is ergodic.*

Proof of Proposition 8.5. Fix a sequence of positive reals $\epsilon_n \to 0$. Let $c_k \in (0, 1)$ be a sequence of reals which contains every rational from $(0, 1)$ infinitely many times. Then it is easy to select a sequence of positive integers l_k and a subsequence (q_{n_k}) of (q_n) such that the segments

$$I_k := \left[\frac{l_k + c_k - \epsilon_k}{q_{n_k}}, \frac{l_k + c_k + \epsilon_k}{q_{n_k}}\right]$$

form a nested sequence, i.e. $I_1 \supset I_2 \supset \cdots$. (Indeed it is suffices to notice that the distance between $[\frac{l+c_k-\epsilon_k}{q_n}, \frac{l+c_k+\epsilon_k}{q_n}]$ and $[\frac{l+1+c_k-\epsilon_k}{q_n}, \frac{l+1+c_k+\epsilon_k}{q_n}]$ tends to zero uniformly in l as $n \to \infty$ for each k.) Take $\beta \in \bigcap_{k=1}^{\infty} I_k$. Then $|\{q_{n_k}\beta\} - c_k| < \epsilon_k$ for all $k > 0$. Hence the sequence $\{q_n\beta\}$ has infinitely many accumulation points. Now we apply Lemma 8.6 and the result follows. $\square$

We also note that there exist ergodic cocycles ϕ of an irrational rotation T such that $\phi \times \phi \circ R$ is not ergodic for any $R \in C(T)$ (an example of such a cocycle is given in [LMN] for $G = \mathbb{Z}$).

Remark 8.7. Let us also notice that using a.a.c.c.p. method from [KwLR] one can construct smooth (even analytic) real valued cocycles ϕ (over irrational rotations under some Diophantine restrictions) satisfying the assertion of Proposition 8.6. Now, by putting a horocycle flow on the fiber we will obtain examples of nondistal smooth multipliers of $\mathcal{W}^{\perp}$.

Let Φ_0 stand for the family of continuous cocycles of T with zero mean. Endowed with the topology of uniform convergence Φ_0 is a Polish space. Since T is uniquely ergodic, we have

$$\Phi_0 = \overline{\{f - f \circ T |\ f \colon \mathbb{T} \to \mathbb{R} \text{ is continuous}\}}.$$

By [Ko] (see also [Ru]), ϕ_0 is cohomologous to a cocycle $\psi \in \Phi_0$. Then, of course, the set

$$\{f + \psi - f \circ T | \text{ for all continuous } f\colon \mathbb{T} \to \mathbb{R}\}$$

is dense in Φ_0. It is also a subset of $\mathcal{M}$. Since the uniform topology is stronger than the topology of convergence in measure and $\mathcal{M}$ is a G_δ in $Z^1(X, \mathbb{R})$, we conclude that $\Phi_0 \cap \mathcal{M}$ is a dense G_δ in Φ_0. Thus we have proved an extension of the most technically involved statement in [Gl1]—Theorem 5.1 (proved there under some Diophantine restrictions on α):

Proposition 8.8. *For any irrational number α, the subset*

$$\{\phi \in \Phi_0 | \ T_\phi \text{ is ergodic and } e(T_\phi) = e(T)\}$$

is generic in Φ_0.

The corollary below follows from this and Theorem 0.1.

Corollary 8.9. *For every ϕ from a dense G_δ-subset of Φ_0, the strictly ergodic homeomorphism $T_{\phi,S}$ of the compact manifold $X \times Y$ is in $\mathcal{M}(\mathcal{W}^\perp)$ but not in $\mathcal{D}$.*

This extends [Gl1, Theorem 4.1] where it was assumed additionally that Γ is maximal and nonarithmetic and α is rather special.

9. Semisimple extensions of transformations with pure point spectrum

We first extend an assertion on relative weak mixing from [LeL], where it was assumed that G is Abelian and spectral theory was used in the proof.

Lemma 9.1. *Let T be a measure preserving transformation and let $\phi\colon X \to G$ be a cocycle of T. Assume that S is a mildly mixing G-action. If $T_{\phi,S}$ is ergodic then the extension $T_{\phi,S} \to T$ is relatively weakly mixing.*

Proof. What we need in fact to prove is that the transformation $T_{\phi,S\times S}$ of the space $(X \times Y \times Y, \mathfrak{B}_X \otimes \mathfrak{B}_Y \otimes \mathfrak{B}_Y, \mu \times \nu \times \nu)$ is ergodic. By Proposition 6.1, the measure $\mu \times \nu \times \nu$ corresponds under an affine map to the measure

$$\nu_\phi \times \nu \times \nu \in \mathcal{P}(W_\phi \times S \times S, \mathfrak{B}_{\Omega_\phi}, \nu_\phi).$$

Suppose first that W_ϕ is properly ergodic. Since $S \times S$ is mildly mixing, we conclude that $\nu_\phi \times \nu \times \nu$ is ergodic for $W_\phi \times S \times S$. Hence $\mu \times \nu \times \nu$ is ergodic for $T_{\phi,S\times S}$.

Now let W_ϕ be of type I. This means that the cocycle ϕ is cohomologous to an ergodic cocycle with values in a closed subgroup H of G. Without loss of generality we may assume that ϕ itself enjoys this property (changing ϕ with a cohomologous cocycle we obtain an isomorphic extension). Since $T_{\phi,S} = T_{\phi,S(H)}$ is ergodic, so is $S(H)$. If H were compact then $S(H)$ and hence $S(G)$ would be of type I. That contradicts the mild mixing assumption on S. Hence H is not compact and therefore $S(H)$ is mildly mixing. Now ϕ is ergodic (as a cocycle with values in H), so W_ϕ is trivial and since $(S \times S)(H)$ is ergodic, we are done. □

Definition 9.2. A probability preserving action S of a l.c.s.c. group G on $(Y, \mathfrak{B}_Y, \nu)$ is called **2-fold-extra-simple** if for any continuous group automorphism θ: $G \to G$, every ergodic joining of S and $S \circ \theta$ is either the product $\nu \times \nu$ or a joining supported by the graph of a transformation $R \in \mathrm{Aut}_0(Y, \nu)$ such that $RS(g)R^{-1} = S(\theta(g))$ for all $g \in G$.

Notice that a 2-fold simple action S is 2-fold-extra-simple if and only if for any continuous group automorphism θ: $G \to G$, the G-action $S \circ \theta$ is either isomorphic to S or disjoint from it. Suppose that the center of G has no compact subgroups. If S is simple and prime (in particular, if it has the MSJ property, see [JRu, Theorem 3.1]) then S is 2-fold-extra-simple by [JRu, Corollary 4.3].

For example if $G = \mathbb{R}$, the horocycle flow corresponding to a maximal nonarithmetic lattice $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ and the Chacon flow are 2-fold-extra-simple since they have the MSJ property by [Rat] and [JPa] respectively.

Example 9.3 (simple but not 2-fold-extra-simple transformation)**.** Let K be a compact metric group. Suppose that T has the MSJ property and T and T^{-1} are conjugate via a transformation $R \in \mathrm{Aut}_0(X, \mu)$ (see [JRS] for examples of such maps). Denote by $\mathcal{R}$ the T-orbit equivalence relation. It is easy to see that $R \in N[\mathcal{R}] \setminus [\mathcal{R}]$. From the proof of [Da1, Theorem 4.2(i)] we deduce that the cocycles $\phi \in Z^1(T, K)$ such that $\alpha_\phi \times \alpha_\phi \circ R$ is ergodic form a dense G_δ subset of $Z^1(T, K)$. Recall that $\alpha_\phi(Tx, x) = \phi(x)$ for a.a. x (see Section 1). Fix such a ϕ. Next, as in the proof of Proposition 8.3 one can check that $e(T_\phi) = e(T)$. Since $e(T_\phi)$ and $e(T)$ are equal to the L^2-spectrum of T_ϕ and T respectively and T is weakly mixing, T_ϕ is also weakly mixing. Then by [JRu, Theorem 5.4], T_ϕ is simple. We claim that it is not 2-fold-extra-simple. Indeed, assume that the contrary holds. Since $T_\phi \not\perp (T_\phi)^{-1}$ (these transformations have a common factor—T), there exists a transformation $S' \in \mathrm{Aut}_0(X \times K, \mu \times \lambda_K)$ which conjugates T_ϕ and $(T_\phi)^{-1}$. Then by [GJLR, Theorem 5], there exists a transformation S of $(X, \mathfrak{B}_X, \mu)$ such that $S'(x, k) = (Sx, S_2(x, k))$ for a.a. $(x, k) \in X \times K$. (Though it was assumed in [GJLR] that K is commutative, the proof of the cited fact holds for noncommutative groups as well.) Clearly, S conjugates T and T^{-1}. Hence

$$SR^{-1} \in C(T) = \{T^n |\, n \in \mathbb{Z}\}$$

and therefore $\alpha_\phi \circ S \approx \alpha_\phi \circ R$. Moreover, by [GJLR, Proposition 7] (Abelian case) and [Da1, Theorem 5.3] (general case), there is a group automorphism l of K such that $\alpha_\phi \circ S \approx l \circ \alpha_\phi$. Thus the cocycle $l \circ \alpha_\phi \times \alpha_\phi \circ R$ is cohomologous to $l \circ \alpha_\phi \times \alpha_\phi \circ S$ which is in turn cohomologous to the cocycle $l \circ \alpha_\phi \times l \circ \alpha_\phi$ taking values in the diagonal subgroup of G^2. Hence it is never ergodic. Since the ergodicity of a cocycle is invariant under composition with a group automorphism, it follows that $\alpha_\phi \times \alpha_\phi \circ R$ is neither ergodic, a contradiction.

Now we are ready to give a proof of Theorem 0.2 stated in Introduction.

Proof of Theorem 0.2. Let η be any ergodic self-joining of $T_{\phi,S}$. As in the proof of Theorem 7.3 we may consider η as an ergodic $T_{\phi\times\phi\circ R, S\otimes S}$-invariant measure on $X \times Y \times Y$ such that

$$\text{(9-1)} \qquad \eta \restriction (\mathfrak{B}_X \otimes \mathfrak{B}_Y \otimes \mathfrak{N}_Y) = \mu \times \nu \text{ and } \eta \restriction (\mathfrak{B}_X \otimes \mathfrak{N}_Y \otimes \mathfrak{B}_Y) = \mu \times \nu,$$

where R is a transformation from $C(T)$. Suppose first that the cocycle $\phi \times \phi \circ R$ is not regular or is regular but cohomologous to an ergodic cocycle with values in a closed subgroup $H \notin J_2(G)$. Then $\eta = \mu \times \nu \times \nu$ by Theorem 7.3. It follows from Lemma 9.1 then the extension

$$\text{(9-2)} \qquad (T_{\phi\times\phi\circ R, S\otimes S}, \eta) = ((T_{\phi,S})_{(\phi\circ R)\otimes 1}, \mu \times \nu \times \nu) \to (T_{\phi,S}, \mu \times \nu)$$

is relatively weakly mixing and we are done.

In the remaining case we may assume that $\phi \times \phi \circ R$ is ergodic itself as a cocycle with values in $H \in J_2(G)$. By Theorem 7.3, $\eta = \mu \times \rho^*$, where ρ^* is an $S \otimes S(H)$-invariant measure. It follows from (9-1) that the marginals of ρ^* are both equal to ν. Arguing as in the proof of Theorem 0.1 we obtain that $\rho^* = \nu \times \nu$ whenever $H \cap (\{1_G\} \times G)$ or $H \cap (G \times \{1_G\})$ is nontrivial. Thus we come to the case considered above.

Finally, let H be the graph of a group automorphism $\theta\colon G \to G$. Since S is 2-fold-extra-simple, either $\rho^* = \nu \times \nu$ or ρ^* is supported by the graph of some ν-preserving transformation Q such that $QS(g)Q^{-1} = S(\theta(g))$ for all $g \in G$. In both cases (9-2) is relatively weakly mixing. Summarizing all the cases we see that $T_{\phi,S}$ is semisimple.

The relative weak mixing of $T_{\phi,S} \to T$ has been established in Lemma 9.1. □

Notice that if $\phi \circ R \not\approx \theta \circ \phi$ for all $R \in C(T)$ and nontrivial group automorphisms θ then we can replace (relax) the condition of 2-fold-extra-simplicity in Theorem 0.2 with the 2-fold-simplicity.

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