Special flows over irrational rotations with the simple convolutions property

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Abstract

Koopman representations associated to some smooth, or singular with finitely many singularities, measure-preserving flows on $\mathbb{T}^2$ are studied. It is shown that they enjoy so called simple convolution property, *i.e.* all Gaussian systems induced by the measures of the (reduced) maximal spectral types of the flows, have simple spectra.

We show that for a (continuous) unitary representation $\underline{U} = (U_t)_{t\in\mathbb{R}}$ on a separable Hilbert space the function which to $t \in \mathbb{R}$ associates the maximal spectral multiplicity of the unitary operator U_t is of the second Baire class, answering a question raised by J.-P. Thouvenot.

Introduction

Let H be a separable Hilbert space. Suppose that $\underline{U} = (U_t)_{t\in\mathbb{R}}$ is a (weakly) continuous unitary representation of $\mathbb{R}$ in $U(H)$. This will be also referred to as $(U_t)_{t\in\mathbb{R}}$ is a *unitary flow* in H. For each $g \in H$ one associates its *spectral measure* σ_g which is a finite positive Borel measure on $\mathbb{R}$ whose Fourier transform $(\widehat{\sigma}_g(t))_{t\in\mathbb{R}}$ is given by $\widehat{\sigma}_g(t) = \langle U_t g, g\rangle$, $t \in \mathbb{R}$. Each unitary flow on H is determined by two invariants: the *maximal spectral type*, that is, the equivalence class $\sigma_{\underline{U}}$ of the spectral measure σ_f for some $f \in H$ which dominates all other spectral measures σ_g, *i.e.* $\sigma_g \ll \sigma_f$ for each $g \in H$, and a measurable function $M_{\underline{U}} : \mathbb{R} = \widehat{\mathbb{R}} \to \mathbb{N} \cup \{\infty\}$ defined $\sigma_{\underline{U}}$-a.e., called the *multiplicity function*. The essential supremum $\mathcal{M}_{\underline{U}}$ of $M_{\underline{U}}$ is called the *maximal spectral multiplicity* of $\underline{U}$. For more about spectral theory of unitary flows we refer the reader to [5] and [21].

We will be mostly interested in Koopman representations, that is, we are given a measurable $\mathbb{R}$-representation $\mathcal{T} = (T_t)_{t\in\mathbb{R}}$ in the group $Aut(X,\mathcal{B},\mu)$ of measure-preserving automorphisms of a fixed probability standard Borel space $(X,\mathcal{B},\mu)$; measurability of such a representation means that for each $A, B \in \mathcal{B}$ that map $t \mapsto \mu(A \cap T_t B)$ is Borel. Such a measurable representation induces a unitary flow $\underline{U}_{\mathcal{T}} = (U_{T_t})_{t\in\mathbb{R}}$ (called a *Koopman representation*) on the space $L^2_0(X,\mathcal{B},\mu)$ of square integrable zero mean functions, here $U_{T_t}(f) = f\circ T_t$, $t \in \mathbb{R}$. We will write $\sigma_{\mathcal{T}}$ instead of $\sigma_{\underline{U}_{\mathcal{T}}}$ and call it the (reduced) maximal spectral type of $\mathcal{T}$ (while we consider $\mathbb{Z}$-actions, *i.e.* a single automorphism $T \in Aut(X,\mathcal{B},\mu)$, we write σ_T instead of $\sigma_{\{T^n:\, n\in\mathbb{Z}\}}$).

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Although classical, the spectral theory of dynamical systems, mainly in the context of $\mathbb{Z}$-actions, is still under intensive development, see *e.g.* the recent monograph [21] and the survey articles [14] and [25]. On one side the spectral theory provides natural invariants for objects considered in ergodic theory; on the other side, it also provides tools for constructing systems with unexpected dynamical properties. For example strong spectral properties of a system T may lead to constructions of other dynamics with some "exotic" properties and indeed such a spectral machinery have been presented in [18] (see also [13], Chapter 7 and [43]). The main role in this machinery is played by the property of *pairwise disjointness of convolutions* (PDC) of the maximal spectral type σ_T; in other words, constructions of interesting dynamics are done on the base of some $T \in Aut(X, \mathcal{B}, \mu)$ for which $\sigma_T^{*n} \perp \sigma_T^{*m}$ for all $m \neq n$.

The PDC property is clearly opposite to Kolmogorov's group property of the spectrum: the maximal spectral type is symmetric and dominates its convolution square. At a certain stage of development of ergodic theory, this latter property was conjectured to hold for all dynamical systems (see the report [36] and the appendix to the Russian translation of [16]). Historically, the first example of T without Kolmogorov's group property appeared in [20] in 1967. Then in 1980th, see [19] and [42], it turned out that it is the PDC property which is generic in the class of automorphisms of a probability standard Borel space. For the classical Chacon's transformation the PDC property has been proved in [35]. Clearly, such a property may hold only for systems with continuous singular spectra.

In the present paper, instead of the PDC property, we will consider a stronger property[1] which will be called the *simple convolution property* (SC property). In the context of flows, the SC property means that if we set $\sigma = \sigma_{\mathcal{T}}$ then for each $n \geq 1$ the conditional measures of the disintegration of $\sigma^{\otimes n}$ over σ^{*n} via the map $\mathbb{R}^n \ni (x_1, \ldots, x_n) \mapsto x_1 + \ldots + x_n \in \mathbb{R}$ are purely atomic with $n!$ atoms, and this definition can be easily adapted to $\mathbb{Z}-$ (and other group) actions. In [1], [2], [3] and [40] it has been shown that a "typical" automorphism with respect to the weak topology[2] of the automorphisms group $Aut(X, \mathcal{B}, \mu)$, Chacon's automorphism as well as some mixing automorphisms enjoy the SC property. The SC property is closely related to the theory of Gaussian systems (see [5] for basic properties of such systems): indeed, the SC property of $\mathcal{T}$ is in fact equivalent to the fact that the Gaussian $\mathbb{R}$-action uniquely determined by $\sigma_{\mathcal{T}}$ has simple spectrum, see e.g. [24] (the reader should notice however that no Gaussian system itself enjoy the SC property). This fact in turn is interesting from the point of view of harmonic analysis as the only known method of constructing Gaussian systems with simple spectra is via continuous measures supported on "small" Borel sets, namely, sets without rational relations (see [5]). It is a separate (and open) problem whether spectral measures of systems from [1], [2], [3], [40] as well as those given in the present work are concentrated on "small" Borel subsets. We also refer the reader to the recent works [24] and [26] where it has been shown that the SC property implies some strong joining property (we refer the reader to the monograph [13] for the join-

[1]It is not completely obvious that the SC property is stronger than the PDC property. This fact seems to be a folklore, see [24] for some stronger assertions.

[2]Recall that this topology is Polish and can be defined by the metric $d(S,T) = \sum_{n\geq 1} \frac{1}{2^n}\left(\mu(SA_n \triangle TA_n) + \mu(S^{-1}A_n \triangle T^{-1}A_n)\right)$, where $\{A_n : n \geq 1\}$ is a dense family in $\mathcal{B}$.

ing theory of dynamical systems) hence relating a purely spectral property with measure-theoretic properties of the underlying dynamical system.

The main aim of this paper is to give new natural examples of systems with the SC property, these are smooth or regular flows on $\mathbb{T}^2$. In fact we deal with examples of special flows $T^f = (T^f_t)_{t\in\mathbb{R}}$, where $Tx = x+\alpha$ and $f:\mathbb{T}\to\mathbb{R}^+$ (see Section 2 for formal definitions) giving rise to two classes of flows with the SC property:

(A) given $f\in C^\infty$, different from any trigonometric polynomial we will show that for "generic" $\alpha\in[0,1)$ the resulting special flow has the SC property;

(B) whenever f is piecewise absolutely continuous, with the sum of jumps different from zero, and α has unbounded partial quotients the SC property also holds true.

Recall that examples from the class (A) are given by smooth reparametrizations of the relevant linear flows. The class (B) was already studied by von Neumann in [31], where the weak mixing property of such special flows was shown. Many of them can be represented as smooth singular flows with finitely many singularities, see [11].

We will also note that the SC property is "typical" in the class of flows of a fixed probability standard Borel space $(X,\mathcal{B},\mu)$. Moreover, the SC property of a flow implies the SC property of all its non-zero time automorphisms.

In Proposition 3 below we will show that the SC property of a flow implies that for each $t\neq 0$ the maximal spectral multiplicity of the $\mathbb{Z}$-action T_t is equal to the maximal spectral multiplicity of the whole flow, i.e. $\mathcal{M}_{U_{T_t}} = \mathcal{M}_{\underline{U}_{\mathcal{T}}}$[3]. This provides a relationship with a problem raised by J.-P. Thouvenot in 1990th which we now describe.

Suppose that $(U_t)_{t\in\mathbb{R}}$ is a *unitary flow* in a separable Hilbert space H. Given $t\in\mathbb{R}\setminus\{0\}$ we have the corresponding $\mathbb{Z}$-representation $n\mapsto U_{nt}$. While it is obvious how the maximal spectral type of the $\mathbb{Z}$- representation is related to the maximal spectral type of $\underline{U}$[4], a possible relationship between the multiplicity functions remains unclear. This problem was already considered by Mathew and Nadkarni in [30], however no general result has been given there. Since it is rather clear that the map which to t associates the maximal spectral multiplicity is Borel, J.-P. Thouvenot asked, in the context of Koopman representations, what can be said about the Baire class of such maps. We recall here that by Lebesgue-Hausdorff theorem each Borel real-valued function on a metrizable space is Baire (e.g. [41], p. 91).

A classical example of the Koopman representation $\underline{U}_{\mathcal{T}} = (U_{T_t})_{t\in\mathbb{R}}$, where $T_t(x) = x+t \bmod 1$ on the additive circle $[0,1)$ in which the values $\mathcal{M}(t) := \mathcal{M}_{U_{T_{1/t}}}$ are either 1 (for t irrational) or ∞ (for t rational) gives an example of a Koopman representation for which the function $\mathcal{M}$ is indeed of second Baire class. If we fix $k\in\mathbb{Z}$ then for each $t\in\mathbb{R}$ for the above flow we have $e^{2\pi ik\cdot}\circ T_t = e^{2\pi ikt}\cdot e^{2\pi ik\cdot}$, so this example is a special case of a unitary flow with simple and discrete spectrum (the group of eigenvalues is equal to $\mathbb{Z}$), where the relation between $\mathcal{M}_{U_{T_t}}$ and $\mathcal{M}_{\underline{U}_{\mathcal{T}}}$ is well understood[5]. Since, the Baire category

[3]Note that in general $\mathcal{M}_{U_{T_t}} \geq \mathcal{M}_{\underline{U}_{\mathcal{T}}}$ and the inequality can be strict. In particular, for the flow $(T_t)_{t\in\mathbb{R}}$ with simple Lebesgue spectrum constructed recently in [34], we have $\mathcal{M}_{U_{T_t}} = \infty$ for each $t\in\mathbb{R}$ and $\mathcal{M}_{\underline{U}_{\mathcal{T}}} = 1$.

[4]It is the image of $\sigma_{\underline{U}}$ via the map $x\mapsto e^{2\pi itx}$.

[5]Indeed, let $(U_t)_{t\in\mathbb{R}}$ be a unitary flow in a separable Hilbert space H and suppose it has discrete and simple spectrum. Let $A\subset\mathbb{R}$ be the (countable) set of eigenvalues ($a\in\mathbb{R}$ is an

class "at most n" is closed under taking the function which is the maximum of two functions, the problem of Baire category class of the function $\mathcal{M}$ is reduced to the case of continuous maximal spectral type.

We will show here that whenever $\sigma_{\underline{U}}$ is continuous then the function $\mathcal{M}$ which to $t \in \mathbb{R}$ associates the maximal spectral multiplicity of $U_{1/t}$ is of the second Baire class, in particular the same result holds for arbitrary $\sigma_{\mathcal{T}}$. Since the above result is purely Hilbertian, it is done in Appendix.

To our knowledge the question of whether one can construct a Koopman representation $\underline{U}_{\mathcal{T}}$ of $\mathbb{R}$ with $\mathcal{T}$ weakly mixing and for which the function $\mathcal{M}$ is indeed of the second Baire class remains open. In fact, the mechanism described above of "producing" some extra multiplicity of time-t automorphisms in view of their non-ergodicity seems to be the only one which provides discontinuities of $\mathcal{M}$.

1 Integral operators in the weak closure of times of a flow. An analytic flow on $\mathbb{T}^2$ with the SC property

When a flow $\mathcal{T} = (T_t)_{t\in\mathbb{R}}$ acting on a probability standard Borel space $(X, \mathcal{B}, \mu)$ is given then its maximal reduced spectral type will be denoted by $\sigma = \sigma_{\mathcal{T}}$. We will always assume that the flows under consideration are weakly mixing (*i.e.* σ is continuous); this in particular implies that each hyperplane in $\mathbb{R}^n$ has zero $\sigma^{\otimes n} = \underbrace{\sigma \otimes \ldots \otimes \sigma}_{n}$-measure (in fact more algebraic varieties enjoy the same property, see e.g. Lemma 2 below). It follows that w.l.o.g. we can assume that the conditional measures $\sigma_c^{(n)}$ obtained from the disintegration

$$\sigma^{\otimes n} = \int_{\mathbb{R}} \sigma_c^{(n)}\, d\sigma^{*n}(c)$$

of $\sigma^{\otimes n}$ over $\sigma^{*n} = \underbrace{\sigma * \ldots * \sigma}_{n}$, which are concentrated on fibers of the map

$$C_n : (x_1, \ldots, x_n) \mapsto x_1 + \ldots + x_n \tag{1}$$

are in fact concentrated on n-tuples $(x_1, \ldots, x_n)$ such that $x_i \neq x_j$ whenever $i \neq j$. If, additionally, the flow under consideration has simple spectrum then

eigenvalue if for some non-zero $y \in H$, $U_t y = e^{2\pi i a t} y$ for all $t \in \mathbb{R}$). Then, for each $a \in A$ there is exactly one (up to a multiplicative constant of modulus one) $y_a \in H$, $\|y_a\| = 1$ such that $U_t(y_a) = e^{2\pi i a t} y_a$ for each $t \in \mathbb{R}$. It follows that $\{y_a\}$ is an orthonormal base of H and then if $x = \sum_{a\in A} c_a y_a$ and $U_{t_0} x = \lambda x$ (with $|\lambda| = 1$) then necessarily $e^{2\pi i a t_0} = \lambda$ whenever $c_a \neq 0$. If by $\sim_{t_0}$ we denote the equivalence relation on A given by

$$a \sim_{t_0} b \text{ if and only if } e^{2\pi i a t_0} = e^{2\pi i b t_0}$$

then it is easy to see that the maximal spectral multiplicity of U_{t_0} is equal to the maximal cardinality of cosets given by $\sim_{t_0}$. Moreover, the set $\{\frac{k}{a-b} : a, b \in A, a \neq b, \text{ and } k \in \mathbb{Z}\}$ is countable and for each $t \in \mathbb{R}$ belonging to the complement of this set the spectrum of U_t is simple. It follows that the function $\mathcal{M}$ is of at most second class of Baire, as it is a constant function on a cocountable set (indeed, any function f which is zero on a cocountable set $\mathbb{R} \setminus A$ is the pointwise sum of the series $\sum_{a\in A} f(a)\chi_{\{a\}}$; clearly $\chi_{\{a\}}$ is of the first Baire class).

the symmetric tensor product representation $U_{\mathcal{T}}^{\odot n} = \underbrace{U_{\mathcal{T}} \odot \ldots \odot U_{\mathcal{T}}}_{n}$ has simple spectrum if and only if the conditional measures $\sigma_c^{(n)}$ are purely atomic and have exactly $n!$ atoms in (σ^{*n}-a.e) fiber of the map (1) (see [5]).

Denote by $\mathcal{P}(\mathbb{R})$ the space of probability Borel measures on $\mathbb{R}$ (endowed with the weak-$*$-topology). Assume now that $P \in \mathcal{P}(\mathbb{R})$ and let $t_n \to \infty$ with

$$U_{T_{t_n}} \to \int_{\mathbb{R}} U_{T_t}\, dP(t)^6, \tag{2}$$

where the convergence takes place in the weak (operator) topology of $B(L^2(X,\mathcal{B},\mu))$ and the righthand operator is understood weakly. Consider any cyclic subspace of $U_{\mathcal{T}}$, say, generated by $\xi \in L_0^2(X,\mathcal{B},\mu)$. On such a subspace this representation is isomorphic to the representation $(V_t)_{t\in\mathbb{R}}$ acting on $L^2(\mathbb{R},\sigma_\xi)$, and given by the formula

$$V_t(f)(x) = e^{2\pi i t x} f(x), \tag{3}$$

where σ_ξ denotes the spectral measure of ξ. This cyclic space, as a closed subspace, is also weakly closed, so the convergence (2) takes place also on it. Denoting by $J = \int_{\mathbb{R}} V_t\, dP(t)$, for $f, g \in L^2(\mathbb{R},\sigma_\xi)$, we obtain that

$$\int_{\mathbb{R}} Jf \cdot \overline{g}\, d\sigma_\xi = \langle Jf, g\rangle_{L^2(\mathbb{R},\sigma_\xi)} = \int_{\mathbb{R}} \langle V_t f, g\rangle_{L^2(\mathbb{R},\sigma_\xi)}\, dP(t) =$$

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{2\pi i t s} f(s)\overline{g(s)}\, d\sigma_\xi(s) \right) dP(t) = \int_{\mathbb{R}} \widehat{P}(s) f(s) \overline{g(s)}\, d\sigma_\xi(s).$$

It follows that

$$Jf(s) = \widehat{P}(s) f(s). \tag{4}$$

Moreover, (2) means that $e^{2\pi i t_k(\cdot)} \to \widehat{P}(\cdot)$ weakly in $L^2(\mathbb{R},\sigma_\xi)$.

For any $n \geq 2$, $V_{t_k}^{\otimes n} \to J^{\otimes n}$ when $k \to \infty$ in the weak topology of $B(L^2(\mathbb{R}^n, \sigma_\xi^{\otimes n}))$. This is equivalent to saying that

$$e^{2\pi i t_k (x_1 + \ldots + x_n)} \to \widehat{P}(x_1) \cdot \ldots \cdot \widehat{P}(x_n) \tag{5}$$

in the weak topology of $L^2(\mathbb{R}^n, \sigma_\xi^{\otimes n})$. Denote $\mathcal{B}_{C_n}(\mathbb{R}^n) := C_n^{-1}(\mathcal{B}(\mathbb{R}))$. Let $L^2(\mathcal{B}_{C_n}(\mathbb{R}^n), \sigma_\xi^{\otimes n})$ be the subspace of L^2-functions measurable with respect to $\mathcal{B}_{C_n}(\mathbb{R}^n)$; it is a closed subspace of $L^2_{sym}(\mathbb{R}^n, \sigma_\xi^{\otimes n})$. Since the LHS elements in (5) belong to $L^2(\mathcal{B}_{C_n}(\mathbb{R}^n), \sigma_\xi^{\otimes n})$ which is weakly closed, the RHS limit belongs to the same subspace. In particular it follows that there exists a Borel function $F : \mathbb{R} \to \mathbb{C}$ such that

$$\widehat{P}(x_1) \cdot \ldots \cdot \widehat{P}(x_n) = F(x_1 + \ldots + x_n) \text{ for } \sigma_\xi^{\otimes n}\text{-a.e. } (x_1, \ldots, x_n). \tag{6}$$

Denote by $\mathcal{S}_n$ the group of all permutations π of $\{1, \ldots, n\}$. In order to show that the SC property holds for some flows we will constantly make use of the following simple lemma.

[6] The operator $J = \int_{\mathbb{R}} U_{T_t}\, dP(t)$ is a Markov operator of $L^2(X,\mathcal{B},\mu)$, *i.e.* $J1 = J^*1 = 1$ and $Jf \geq 0$ whenever $0 \leq f \in L^2(X,\mathcal{B},\mu)$; it also satisfies $J \circ U_{T_t} = U_{T_t} \circ J$ and therefore it corresponds to a self-joining of $\mathcal{T}$, see *e.g.* [21], [27], [39].

Lemma 1. *Let $\sigma \in \mathcal{P}(\mathbb{R})$ be continuous. Fix $n \geq 1$. Assume that $\mathcal{F} \subset CB(\mathbb{R}^n)$ (in particular $\mathcal{F} \subset L^2(\mathbb{R}^n, \sigma^{\otimes n})$) is a countable family of functions each of which element is $\sigma^{\otimes n}$-a.e. equal to a function measurable with respect to $\mathcal{B}_{C_n}(\mathbb{R}^n)$. Assume moreover that there exists $\tilde{A} \subset \mathbb{R}^n$, $\sigma^{\otimes n}(\tilde{A}) = 1$ such that for each $c \in \mathbb{R}$ if*

$$(x_1, \ldots, x_n), (x'_1, \ldots, x'_n) \in C_n^{-1}(c) \cap \tilde{A}$$

and

$$J(x_1, \ldots, x_n) = J(x'_1, \ldots, x'_n) \text{ for each } J \in \mathcal{F}$$

then $(x_1, \ldots, x_n) = (x'_{\pi(1)}, \ldots, x'_{\pi(n)})$ for some permutation $\pi \in \mathcal{S}_n$. Then for σ^{*n}-a.e. $c \in \mathbb{R}$ the conditional measure $\sigma_c^{(n)}$ is purely atomic concentrated on $n!$ atoms.

Proof. Let $\mathcal{F} = \{J_1, J_2, \ldots\}$. Then for each $m \geq 1$ let K_m be a Borel function (defined everywhere) which is $\mathcal{B}_{C_n}(\mathbb{R}^n)$-measurable and such that $J_m(x_1, \ldots, x_n) = K_m(x_1, \ldots, x_n)$ for each $(x_1, \ldots, x_n) \in \mathbb{R}^n \setminus \widetilde{B}_m$ where $\sigma^{\otimes n}(\widetilde{B}_m) = 0$. Set $\widetilde{C} = \widetilde{A} \setminus \bigcup_{m \geq 1} \widetilde{B}_m$. Then $\sigma^{\otimes n}(\widetilde{C}) = 1$ and the intersection of $\widetilde{C}$ with any variety $C_n^{-1}(c)$ is either empty or is equal to a set of the form $\{(x_{\pi(1)}, \ldots, x_{\pi(n)}) : \pi \in \mathcal{S}_n\}$. □

Remark 1. In our applications of Lemma 1, the set $\widetilde{A}$ will be obtained by discarding from $\mathbb{R}^n$ a countable union of some algebraic varieties. Then we will point out a certain family of functions which will "distinguish non-symmetric points" in the fibers of C_n (in the sense as in the above lemma). In order to be sure that the elements of the family are $\sigma^{\otimes n}$-a.e. equal to functions which are $\mathcal{B}_{C_n}(\mathbb{R}^n)$-measurable they are taken as some natural tensors whose components belong to the relevant weak closure of characters. For example, by (6) and (2), a countable family of tensors of the form $J = \widehat{P}(\cdot) \otimes \ldots \otimes \widehat{P}(\cdot)$ (with $P \in \mathcal{P}(\mathbb{R})$ satisfying (2)) is potentially a family for which the above method can work.

We will now concentrate on the weak closure of $\{U_{T_t} : t \in \mathbb{R}\}$ in case of a "typical" flow of a fixed probability standard Borel space $(X, \mathcal{B}, \mu)$. Denote by $Flow(X, \mathcal{B}, \mu)$ the space of all (measurable) flows acting on $(X, \mathcal{B}, \mu)$. Recall, see e.g. [38], that the Polish topology of $Flow(X, \mathcal{B}, \mu)$ is given by the metric

$$D(\mathcal{S}, \mathcal{T}) = \sup_{t \in [0,1]} d(S_t, T_t).$$

Assume that $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ acts on $(X, \mathcal{B}, \mu)$. Denote by $\mathcal{P}_{\mathcal{T}}(\mathbb{R})$ the subset of $P \in \mathcal{P}(\mathbb{R})$ such that the integral Markov operator $J = J_P(\mathcal{T}) := \int_{\mathbb{R}} U_{T_t}\, dP(t)$ belongs to the weak closure of the set $\{U_{T_t} : t \in \mathbb{R}\}$. Notice that $\mathcal{P}_{\mathcal{T}}(\mathbb{R})$ is closed in the weak topology of $\mathcal{P}(\mathbb{R})$. In [29] there has been constructed a weakly mixing flow $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ on $\mathbb{T}^2$ given by an analytic reparametrization of a linear flow such that $\mathcal{P}_{\mathcal{T}}(\mathbb{R}) = \mathcal{P}(\mathbb{R})$. Consequently, for each $P \in \mathcal{P}(\mathbb{R})$

$$\{\mathcal{T} \in Flow(X, \mathcal{B}, \mu) : \mathcal{T} \text{ is aperiodic and } P \in \mathcal{P}_{\mathcal{T}}(\mathbb{R})\} \neq \emptyset.$$

The class $\{J_P(\mathcal{T}) : P \in \mathcal{P}(\mathbb{R})\}$ of Markov operators has the following property: For each $S \in Aut\,(X, \mathcal{B}, \mu)$

$$S \circ J_P(\mathcal{T}) \circ S^{-1} = J_P(S \circ \mathcal{T} \circ S^{-1}) \tag{7}$$

where $S \circ \mathcal{T} \circ S^{-1} = (S \circ T_t \circ S^{-1})_{t\in\mathbb{R}}$. Assume that $\{A_i : i \in \mathbb{N}\}$ is a dense family of sets in $\mathcal{B}$ (considered with the pseudo-metric $\rho(A, B) = \mu(A \triangle B)$).

For $t \in \mathbb{R}$ and $\varepsilon > 0$ denote

$$\mathcal{W}(t, \varepsilon) = \{\mathcal{T} \in Flow(X, \mathcal{B}, \mu) : \sum_{i,j=1}^{\infty} \frac{1}{2^{i+j}} \left|\mu(T_t A_i \cap A_j) - \langle J_P(\mathcal{T}) 1_{A_i}, 1_{A_j}\rangle\right| < \varepsilon\}.$$

Note that $\mathcal{W}(t, \varepsilon)$ is open[7], and so is the set $\bigcup_{t\geq N} \mathcal{W}(t, \varepsilon)$ for each natural number $N \geq 1$. It follows that the set

$$\mathcal{W} := \bigcap_{\mathbb{Q}\ni\varepsilon>0} \bigcap_{N\geq 1} \bigcup_{t>N} \mathcal{W}(t, \varepsilon)$$

is G_δ. It is clear from the definition of $\mathcal{W}$ that

$$\mathcal{T} \in \mathcal{W} \;\Leftrightarrow\; (\exists t_n \to \infty)\ U_{T_{t_n}} \to J_P(\mathcal{T}) \text{ weakly},$$

in other words

$$\mathcal{W} = \{\mathcal{T} \in Flow(X, \mathcal{B}, \mu) : P \in \mathcal{P}_{\mathcal{T}}(\mathbb{R})\}. \tag{8}$$

It follows from (8), (7) and the fact that the conjugation class of an aperiodic flow is dense in $Flow(X, \mathcal{B}, \mu)$ (this is essentially proved in [15], see also [6]) that $\mathcal{W}$ is G_δ and dense.

By taking a countable a dense family of measures in $\mathcal{P}(\mathbb{R})$ and taking the corresponding intersection of G_δ and dense subsets of flows we obtain the following.

Corollary 1. *There exists a G_δ and dense family $\mathcal{D}$ of flows of a fixed probability standard Borel space $(X, \mathcal{B}, \mu)$ such that $\mathcal{P}_{\mathcal{T}}(\mathbb{R}) = \mathcal{P}(\mathbb{R})$ whenever $\mathcal{T} \in \mathcal{D}$.*

We now come back to Lemma 1 to show how it can be used in a concrete situation.

Example 1. Given $x \in [0, 1]$ and $y \in \mathbb{R}$ denote by $P_{x,y} := x\delta_0 + (1 - x)\delta_y$. Assume that $P_{x,y} \in \mathcal{P}_{\mathcal{T}}(\mathbb{R})$, where $x \in [0, 1], y \in \mathbb{R}$ run over some countable dense subsets of $[0, 1]$ and $\mathbb{R}$ respectively. Notice that it follows immediately that $P_{x,y} \in \mathcal{P}_{\mathcal{T}}(\mathbb{R})$ for each $x \in [0, 1]$ and each $y \in \mathbb{R}$. We have $\widehat{P}(t) = x+(1-x)e^{2\pi i t y}$. For arbitrary $n \geq 1$ set $\mathcal{F} = \mathcal{F}_n = \{\widehat{P}_{x,y}^{\otimes n}\}_{x,y}$ ($\mathcal{F} \subset L^2(\mathbb{R}^n, \sigma^{\otimes n})$, x and y run over arbitrary countable dense subsets of $[0, 1]$ and $\mathbb{R}$ respectively). Suppose that

$$\widehat{P}_{x,y}(t_1) \cdot \ldots \widehat{P}_{x,y}(t_n) = \widehat{P}_{x,y}(t_1') \cdot \ldots \widehat{P}_{x,y}(t_n'),$$

that is

$$\sum_{j=0}^{n} x^j (1 - x)^{n-j} \alpha_j(t_1, \ldots, t_n, y) = \sum_{j=0}^{n} x^j (1 - x)^{n-j} \alpha_j(t_1', \ldots, t_n', y).$$

[7]Note that the set of Markov operators of $L^2(X, \mathcal{B}, \mu)$ is a closed (hence compact) subset in the weak topology of the relevant unit ball; it is then metrizable and the formula $\tilde{d}(J_1, J_2) = \sum_{i,j=1}^{\infty} \frac{1}{2^{i+j}} \left|\langle J_1 1_{A_i}, 1_{A_j}\rangle - \langle J_2 1_{A_i}, 1_{A_j}\rangle\right|$ defines a metric compatible with the weak topology; in other words, $\mathcal{T} \in \mathcal{W}(t, \varepsilon)$ if and only if $\tilde{d}(U_{T_t}, J_{\mathcal{P}}(\mathcal{T})) < \varepsilon$.

Since the functions $x^j(1-x)^{n-j}$, $j=0,1,\ldots,n$ (defined on $[0,1]$) are linearly independent, $\alpha_j(t_1,\ldots,t_n,y)=\alpha_j(t_1',\ldots,t_n',y)$ for each j (and each y from the dense subset). Taking $j=n-1$ we obtain that $\alpha_j(t_1,\ldots,t_n,y)=\sum_{k=1}^n e^{2\pi i t_k y}$, so

$$\sum_{k=1}^n e^{2\pi i t_k y}=\sum_{k=1}^n e^{2\pi i t_k' y}$$

for each $y\in\mathbb{R}$. However the characters $x\mapsto e^{2\pi i s x}$, $s\in\mathbb{R}$, of the reals are linearly independent and hence $\{t_1,\ldots,t_n\}=\{t_1',\ldots,t_n'\}$. It follows that the flow $\mathcal{T}$ has the SC property.

If we now use Corollary 1 we obtain the following.

Corollary 2. *A "typical" flow of a fixed standard probability Borel space* $(X,\mathcal{B},\mu)$ *has the SC property.*

It is rather clear that the information given by the assumption in Example 1 is redundant. We now essentially weaken (*cf.* [1]) the assumption in the above example; however the SC property will still hold.

Example 2. Assume that $y_1,y_2\in\mathbb{R}\setminus\{0\}$ and

$$y_1/y_2\notin\mathbb{Q}. \tag{9}$$

Assume moreover that for $i=1,2$

$$\begin{array}{l}\text{the set of } x\in(0,1) \text{ such that}\\ P_{x,y_i}\in\mathcal{P}_{\mathcal{T}}(\mathbb{R}) \ \text{ is infinite.}\end{array} \tag{10}$$

Fix $n\geq 1$. Recall that the polynomials $\tau_0=1$ and

$$\tau_j(z_1,\ldots,z_n)=\sum_{1\leq k_1<\ldots<k_j\leq n} z_{k_1}\cdot\ldots\cdot z_{k_j},\quad j=1,\ldots,n$$

($z_i\in\mathbb{C}$) are called basic symmetric polynomials (in n variables) and whenever $z_i,z_i'\in\mathbb{C}$, $i=1,\ldots,n$, satisfy

$$\tau_j(z_1,\ldots,z_n)=\tau_j(z_1',\ldots,z_n') \ \text{ for } j=1,\ldots,n$$

then $\{z_1,\ldots,z_n\}=\{z_1',\ldots,z_n'\}$[8]. Since σ is continuous, the set

$$\widetilde{A}=\{(t_1,\ldots,t_n)\in\mathbb{R}^n:\ t_i-t_j\notin\frac{1}{y_1}\mathbb{Z}+\frac{1}{y_2}\mathbb{Z} \text{ whenever } i\neq j\}$$

has full $\sigma^{\otimes n}$–measure. Suppose now that $(t_1,\ldots,t_n),(t_1',\ldots,t_n')\in\widetilde{A}$ and that

$$\widehat{P}_{x,y_k}(t_1)\cdot\ldots\cdot\widehat{P}_{x,y_k}(t_n)=\widehat{P}_{x,y_k}(t_1')\cdot\ldots\cdot\widehat{P}_{x,y_k}(t_n')$$

[8]Indeed, for each $1\leq s\leq n$

$$\Pi_{i=1}^n(z_s-z_i')=\sum_{j=0}^n z_s^j\cdot(-1)^{n-j}\tau_{n-j}(z_1',\ldots,z_n')$$

$$=\sum_{j=0}^n z_s^j\cdot(-1)^{n-j}\tau_{n-j}(z_1,\ldots,z_n)=\Pi_{i=1}^n(z_s-z_i)=0.$$

for $P_{x,y_k} \in \mathcal{P}_{\mathcal{T}}(\mathbb{R})$. We have immediately

$$\sum_{j=0}^{n} x^j(1-x)^{n-j}\tau_{n-j}(e^{2\pi i t_1 y_k},\dots,e^{2\pi i t_n y_k})$$

$$=\sum_{j=0}^{n} x^j(1-x)^{n-j}\tau_{n-j}(e^{2\pi i t_1' y_k},\dots,e^{2\pi i t_n' y_k}),$$

or equivalently

$$\sum_{j=0}^{n}\left(\frac{x}{1-x}\right)^j\Big(\tau_{n-j}(e^{2\pi i t_1 y_k},\dots,e^{2\pi i t_n y_k})-\tau_{n-j}(e^{2\pi i t_1' y_k},\dots,e^{2\pi i t_n' y_k})\Big)=0.$$

In view of (10),

$$\tau_{n-j}(e^{2\pi i t_1 y_k},\dots,e^{2\pi i t_n y_k})=\tau_{n-j}(e^{2\pi i t_1' y_k},\dots,e^{2\pi i t_n' y_k})$$

for $j=0,1,\dots,n$ and therefore

$$\{e^{2\pi i t_1 y_k},\dots,e^{2\pi i t_n y_k}\}=\{e^{2\pi i t_1' y_k},\dots,e^{2\pi i t_n' y_k}\}.$$

It follows that

$$t_s y_k = t'_{j_s^{(k)}} y_k \pmod 1 \quad \text{for } s=1,\dots,n$$

(the map $s\mapsto j_s^{(k)}$ is $1-1$) but since $(t_1',\dots,t_n')\in\widetilde{A}$, $j_s^{(1)}=j_s^{(2)}$ for $s=1,\dots,n$. Applying (9) we obtain that $\{t_1,\dots,t_n\}=\{t_1',\dots,t_n'\}$.

It is now enough to consider

$$\mathcal{F}=\{(\widehat{P}_{x,y_k}(\cdot))^{\otimes n}:\ k=1,2,\ x \text{ runs over relevant countable sets}\}$$

as a family which separates non-symmetric points in the fibers of C_n to conclude that $\mathcal{T}$ has the SC property.

Example 3. Assume that for fixed $a,b\in(0,1)$ and $r_0\in\mathbb{R}\setminus\mathbb{Q}$ we have

$$P_{a,m},P_{b,mr_0}\in\mathcal{P}_{\mathcal{T}}(\mathbb{R}) \text{ for all } m\in\mathbb{Z}.$$

Set $M_s(x)=sx$, $(s,x\in\mathbb{R})$. Notice that $P_{a,s}=(M_s)_*P_{a,1}$. Moreover

$$\widehat{P}_{a,t}(s)=\widehat{P}_{a,s}(t) \text{ for each } s,t\in\mathbb{R}.$$

Since σ is continuous, the set

$$\widetilde{A}=\{(t_1,\dots,t_n)\in\mathbb{R}^n:\ \sum_{i=1}^{n} m_i t_i\notin\mathbb{Z}+r_0\mathbb{Z} \text{ for } m_i\in\mathbb{Z} \text{ and } \textstyle\sum_{i=1}^n m_i^2>0\}$$

has full $\sigma^{\otimes n}$-measure. Take $(t_1,\dots,t_n),(t_1',\dots,t_n')\in\widetilde{A}$ and suppose that for each $m\in\mathbb{Z}$

$$\widehat{P}_{a,m}(t_1)\cdot\ldots\cdot\widehat{P}_{a,m}(t_n)=\widehat{P}_{a,m}(t_1')\cdot\ldots\cdot\widehat{P}_{a,m}(t_n').$$

Equivalently

$$\widehat{P}_{a,t_1}(m)\cdot\ldots\cdot\widehat{P}_{a,t_n}(m)=\widehat{P}_{a,t_1'}(m)\cdot\ldots\cdot\widehat{P}_{a,t_n'}(m)$$

or else

$$(P_{a,t_1} * \ldots * P_{a,t_n})\widehat{\ }(m) = (P_{a,t'_1} * \ldots * P_{a,t'_n})\widehat{\ }(m).$$

The latter equality means that the images of the two measures $P_{a,t_1} * \ldots * P_{a,t_n}$ and $P_{a,t'_1} * \ldots * P_{a,t'_n}$ via the map $x \mapsto e^{2\pi i x}$ are equal and since they are purely atomic, the two sets

$$\left\{ \sum_{i=1}^{n} \varepsilon_i t_i \pmod 1 : \ \varepsilon_i = 0, 1, i = 1, \ldots, n \right\},$$

$$\left\{ \sum_{i=1}^{n} \varepsilon_i t'_i \pmod 1 : \ \varepsilon_i = 0, 1, i = 1, \ldots, n \right\}$$

are equal (note that the representation of a point in any of these sets is unique). It follows that modulo 1 we have

$$t_i = \sum_{j=1}^{n} \varepsilon_{ij} t'_i \ \text{ and } \ t'_i = \sum_{j=1}^{n} \varepsilon'_{ij} t_i$$

for $i = 1, \ldots, n$. Denote by $A = [\varepsilon_{ij}]$, $A' = [\varepsilon'_{ij}]$ the corresponding matrices. Since they are integer-valued

$$\begin{bmatrix} t_1 \\ \ldots \\ t_n \end{bmatrix} = AA' \begin{bmatrix} t_1 \\ \ldots \\ t_n \end{bmatrix} \pmod 1.$$

But $(t_1, \ldots, t_d) \in \widetilde{A}$, so $A'A = Id$ (and also $AA' = Id$). Since A, A^{-1} are matrices with frequencies $0, 1$, they are matrices of permutations, and therefore for each $i = 1, \ldots, n$

$$t_i = t'_{j_i} + m_i \tag{11}$$

for some unique j_i and $m_i \in \mathbb{Z}$.

We now repeat the same arguments for P_{b,mr_0} and we obtain that for each $m \in \mathbb{Z}$

$$(P_{a,t_1} * \ldots * P_{a,t_n})\widehat{\ }(mr_0) = (P_{a,t'_1} * \ldots * P_{a,t'_n})\widehat{\ }(mr_0).$$

This equality means that the images of the two measures $P_{a,t_1} * \ldots * P_{a,t_n}$ and $P_{a,t'_1} * \ldots * P_{a,t'_n}$ via the map $x \mapsto e^{2\pi i r_0 x}$ are the same and as before we obtain that for each $i = 1, \ldots, n$

$$t_i = t'_{k_i} + m'_i r_0. \tag{12}$$

If either $m_i \neq 0$ or $m'_i r_0 \neq 0$ we obtain a contradiction with the fact that we consider points from $\widetilde{A}$. Now, the SC property easily follows ($\mathcal{F} = \{\widehat{P}^{\otimes n}_{a,m}, \widehat{P}^{\otimes n}_{b,mr_0} : m \in \mathbb{Z}\}$).

Yet, we now will show that, in a sense, still the assumptions in Example 3 are redundant.

Example 4. Assume that for some $0 < a < 1$, $0 \neq \beta \in \mathbb{R}$ there exists a sequence $r_n \to \infty$ of real numbers such that for each $m \in \mathbb{N}$

$$U_{T_{mr_n}} \to aId + (1-a)U_{T_{m\beta}}. \tag{13}$$

(In particular, $P_{a,m\beta} \in \mathcal{P}_{\mathcal{T}}(\mathbb{R})$.) By considering $t_n = r_n/\beta$ and the flow $t \mapsto T_{t\beta}$ we can simply assume that $\beta = 1$ and instead of (13) we have

$$U_{T_{mt_n}} \to aId + (1-a)U_{T_m} = P_{a,m} \tag{14}$$

for each natural $m \geq 1$. By passing to a subsequence, and using the diagonalization procedure we can select a subsequence (t_{i_k}) of (t_n) so that

$$U_{T_{m(t_{i_{k+1}}-t_{i_k})}} \to P_{a,m}P^*_{a,m}. \tag{15}$$

If for $0 \leq b \leq 1$ we denote $Q_{b,m} = \frac{1-b}{2}U_{T_{-m}} + bId + \frac{1-b}{2}U_{T_m}$ then

$$P_{a,m}P^*_{a,m} = Q_{a^2+(1-a)^2,m} = Q^*_{a^2+(1-a)^2,m}.$$

By passing to a further subsequence, if necessary, we can assume that $\{t_{i_k}\} \to \gamma \in [0,1)$ and then since $U_{\{t_{i_k}\}} \to U_\gamma$ in the strong operator topology, we also have

$$U_{T_{m(n_{k+1}-n_k)}} \to P_{a,m}U_{T_{m\gamma}}(P_{a,m}U_{T_{m\gamma}})^* = Q_{a^2+(1-a)^2,m}, \tag{16}$$

where $n_k = [t_{i_k}]$ for $k \geq 1$. Note that $0 < a^2 + (1-a)^2 < 1$ and $2a(1-a) \leq a^2 + (1-a)^2$. We have obtained that for each $m \geq 1$ the operator $Q_{a^2+(1-a)^2,m}$ belongs to the weak closure of the set $\{U_{T_l} : l \in \mathbb{Z}\}$. Now, fix $m \geq 1$ and select (l_k) so that $U_{T_{l_k}} \to Q_{a^2+(1-a)^2,m}$. It follows that

$$U_{T_{l_k}} + U_{T_{-l_k}} \to 2Q_{a^2+(1-a)^2,m}.$$

Thus

$$Q_{a^2+(1-a)^2,l_k} \to (a^2 + (1-a)^2)Id + 2a(1-a)Q_{a^2+(1-a)^2,m}.$$

Therefore, in the weak closure of the set $\{U_{T_l} : l \in \mathbb{Z}\}$ we find the element

$$(a^2+(1-a)^2)Id + 2a(1-a)\left((a^2+(1-a)^2)Id + a(1-a)U_{T_{-m}} + a(1-a)U_{T_m}\right)$$

$$=: Q_{b_1,m}$$

and since $m \geq 1$ is arbitrary in this reasoning, we can iterate this procedure by replacing a by b_1 to obtain b_2 and so on. Note that $a < b_1 < b_2 < \ldots$ and $b_k \to 1$ (the latter follows from the fact that $b_kId + \frac{1-b_k}{2}(U_{T_s} + U_{T_{-s}})$ is in the weak closure of $\{U_{T_l} : l \in \mathbb{Z}\}$ and $b_k \geq 1 - b_k$). It follows that

(17) the flow (T_t) is rigid[9]

and

(18) the set of $0 < b < 1$ such that $bId + \frac{1-b}{2}(U_{T_{-m}} + U_{T_m})$ is in the weak closure of $\{U_{T_l} : l \in \mathbb{Z}\}$ is infinite.

---

[9]The assumption (16) is satisfied for special flows over irrational rotations by α with unbounded partial quotients and roof functions having the bounded variation property, see Remark 3. Moreover, it follows from [12] that there are many quasi-Hamiltonian flows on $\mathbb{T}^2$ for which some smooth change of time leads to (singular) flows satisfying the assumption (16).

(This part of the proof is due to V.V. Ryzhikov.)

Now, set $\sigma = \sigma_{\mathcal{T}}$ and let $\sigma_1 := \left(e^{2\pi i\cdot}\right)_* \sigma$. Then $\sigma_1 = \sigma_T$, where $T = T_1$. It follows from (18) that for infinitely many $0 < b < 1$ we can find a sequence $(n_k) = (n_k(b))$ such that

$$z^{n_k} \to b \cdot 1 + (1-b)Re(z) =: R_a(z) \text{ weakly in } L^2(\mathbb{S}^1, \sigma_1). \tag{19}$$

Let

$$\widetilde{A} = \{(z_1, \ldots, z_n) \in (\mathbb{S}^1)^n : z_1^{m_1} \cdot \ldots \cdot z_n^{m_n} \neq 1$$
$$\text{whenever } \sum_{j=1}^{n} m_n > 0,\ m_j \in \{0, 2\}, 1 \leq j \leq n\}.$$

Then, $\widetilde{A}$ has full $\sigma_1^{\otimes n}$–measure.

Suppose that $(z_1, \ldots, z_n), (z_1', \ldots, z_n') \in \widetilde{A}$ and that for all b under consideration

$$R_b(z_1) \cdot \ldots \cdot R_b(z_n) = R_b(z_1') \cdot \ldots \cdot R(z_n').$$

Hence, as in Example 2

$$\tau_j(Re(z_1), \ldots, Re(z_n)) = \tau_j(Re(z_1'), \ldots, Re(z_n')) \text{ for } j = 1, \ldots, n$$

and then $\{Re(z_1), \ldots, Re(z_n)\} = \{Re(z_1'), \ldots, Re(z_n')\}$. It follows that for each $j = 1, \ldots, n$, $z_j = z_{s_j}'^{\pm 1}$ and since the two points are in $\widetilde{A}$, $\{z_1, \ldots, z_n\} = \{z_1', \ldots, z_n'\}$.

We have proved that the automorphism $T = T_1$ has the SC property[10]. However, this implies that the (T_t) has the SC property, see Remark 4, Proposition 5 and 3.

2 Special flows over irrational rotations

Assume now that $Tx = x + \alpha$ is an irrational rotation on the additive circle $\mathbb{T} = [0, 1)$. Assume moreover that $f : \mathbb{T} \to \mathbb{R}$ is an L^1 zero mean function, and we will assume that $F = f + c > 0$ for some constant $c > 0$. We consider then the special flow T^F which is defined on the space

$$X^F = \{(x, t) \in \mathbb{T} \times \mathbb{R} : 0 \leq t < F(x)\}$$

with the natural (product) Borel and measure structure (see e.g. [5]). Under the action of the flow $T^F = (T_t^F)_{t \in \mathbb{R}}$, for $t > 0$, a point $(x, s) \in X^F$ goes vertically up with the unit speed until it reaches $(x, F(x))$ when it is identified with $(Tx, 0)$.

Suppose now that (n_t) is a rigidity sequence for T (*i.e.* $U_{T^{n_t}} \to Id$), $P \in \mathcal{P}(\mathbb{R})$ and assume moreover that

$$(f^{(n_t)})_*(\lambda_{\mathbb{T}}) \to P \text{ weakly, } i.e. \text{ in the topology of } \mathcal{P}(\mathbb{R}). \tag{20}$$

Under some additional assumptions on f (which will be satisfied for all classes of examples considered here) we have (see [9])

$$(T^F)_{ct_n} \to \int_{\mathbb{R}} T_{-t}^F \, dP(t). \tag{21}$$

[10]The fact that that infinitely many polynomials of degree 1 in the weak closure of powers of an automorphism implies the SC property was first proved in [1].

We refer the reader to [7], [9], [10], [19], [29] and [44] for concrete examples of special flows when we obtain a measure $P \in \mathcal{P}_{T^F}(\mathbb{R})$ which is not a Dirac measure.

3 Generic case for C^∞ roof function

We suppose now that f is a C^∞ function with infinitely many Fourier coefficients different from zero. Assume moreover that the Fourier coefficients of f are real, *i.e.* that $f(x) = f(1-x)$ for $x \in \mathbb{T}$. Take any constant $c > 0$ so that $F := f + c > 0$.

This section is devoted to a proof of the following.

Proposition 1. *There exists a G_δ dense set of irrational numbers α such that the special flow T^F $(Tx = x + \alpha)$ is weakly mixing and it has the simple convolution property.*

Proof. Denote $f_q(x) = f(x) + f(x + 1/q) + \ldots + f(x + (q-1)/q)$. We easily obtain that for $l \in \mathbb{Z}$

$$(22) \qquad \widehat{f_q}(lq) = q\widehat{f}(lq), \text{ and } \widehat{f_q}(n) = 0 \text{ when } q \text{ does not divide } n.$$

Because of C^∞ assumption, for each $n \geq 1$, $q^n \widehat{f}(q) \to 0$ when $q \to \infty$. Since infinitely many Fourier coefficients of f are different from zero we can select a subsequence (q_n) so that

$$(23) \qquad q^n |\widehat{f}(q)| \leq q_n^n |\widehat{f}(q_n)| \text{ for each } q \geq q_n.$$

Then, for some sequence (Q_n) of positive integers satisfying $Q_n > 2q_n$

$$(24) \qquad q_n Q_n |\widehat{f}(q_n)|^2 \to +\infty$$

(in fact $Q_n > q_n^j$ eventually for each $j \geq 1$). We will also assume that $\widehat{f}(q_n) > 0$ (the case where $\widehat{f}(q_n) < 0$ for the chosen subsequence is treated similarly). Then, for a generic set of α in $[0, 1)$ we see that $\|q_n\alpha\| \leq \frac{1}{2Q_n}$ for an infinite subsequence of (q_n). Now, we fix such an α and the corresponding subsequence of (q_n) will still be denoted by (q_n). Note that by the Légendre Theorem such a q_n will be a denominator of α.

Fix $\delta > 0$. We can choose a sequence (b_n) of positive integers such that

$$(25) \qquad b_n q_n \widehat{f}(q_n) \to \frac{\delta}{2}.$$

In view of (25), (24) and the fact that $\|q_n\alpha\| < \frac{1}{2Q_n}$ we have

$$b_n^2 q_n \|q_n\alpha\| = \mathrm{O}\left(\frac{(\delta^2/4)\|q_n\alpha\|}{q_n|\widehat{f}(q_n)|^2}\right) = \mathrm{O}\left(\frac{1}{Q_n q_n |\widehat{f}(q_n)|^2}\right) \to 0.$$

Now, since q_n is a denominator, for some $1 < p_n < q_n$ $(\gcd(p_n, q_n) = 1)$ we have $|\alpha - \frac{p_n}{q_n}| < 1/(2q_nQ_n)$ and therefore

$$|b_n f_{q_n}(x) - f^{(b_n q_n)}(x)| = \left|\sum_{j=1}^{b_n} \left(f_{q_n}(x) - f^{(q_n)}(x + (j-1)q_n\alpha)\right)\right| =$$

$$\left|\sum_{j=1}^{b_n}\left(\sum_{s=0}^{q_n-1}(f(x+s\frac{p_n}{q_n})-f(x+s\alpha+(j-1)q_n\alpha))\right)\right|\leq$$

$$\|f'\|_\infty\sum_{j=1}^{b_n}\sum_{s=0}^{q_n-1}\left(s\left|\frac{p_n}{q_n}-\alpha\right|+(j-1)\|q_n\alpha\|\right)\leq$$

$$\|f'\|_\infty\sum_{j=1}^{b_n}\sum_{s=0}^{q_n-1}j\|q_n\alpha\|\leq b_n^2q_n\|q_n\alpha\|\|f'\|_\infty\to 0.$$

It follows that $|b_nf_{q_n}(x)-f^{(b_nq_n)}(x)|\to 0$ uniformly and hence by (22)

$$f^{(b_nq_n)}(x)-b_nq_n\sum_{l=-\infty}^{\infty}\widehat{f}(lq_n)e^{2\pi ilq_nx}\to 0\ \text{ uniformly.}$$

But, in view of (23), $|\widehat{f}(lq_n)|\leq\frac{1}{l^n}|\widehat{f}(q_n)|$, whence (by (25))

$$b_nq_n\sum_{|l|\geq 2}|\widehat{f}(lq_n)|\leq b_nq_n\left|\widehat{f}(q_n)\right|\sum_{|l|\geq 2}\frac{1}{l^n}=\mathrm{O}\left(\sum_{|l|\geq 2}\frac{1}{l^n}\right)\to 0$$

when $n\to\infty$. It follows that

$$\begin{aligned}&f^{(b_nq_n)}(x)-b_nq_n\widehat{f}(q_n)2\mathrm{Re}(e^{2\pi iq_nx})\\&=f^{(b_nq_n)}(x)-2b_nq_n\widehat{f}(q_n)\cos(2\pi q_nx)\to 0\end{aligned}$$ [11]

still uniformly in $x\in\mathbb{T}$.

Since the map $x\mapsto q_nx$ (mod 1) preserves Lebesgue measure $\lambda_{\mathbb{T}}$, the function $2b_nq_n\widehat{f}(q_n)\cos(2\pi q_nx)$ has the same distribution as the function $\delta_n\cos(2\pi x)$ where $\delta_n=2b_nq_n\widehat{f}(q_n)$. It follows that one obtains as the limit the measure P_δ which is the distribution of the function $\delta\cos(2\pi x)$.

The Fourier transform of this distribution is the first Bessel function J_0:

$$\varphi_\delta(t)=\int_{\mathbb{T}}e^{it\delta\cos(2\pi x)}dx=\widehat{P}_\delta(t)=J_0(\delta t),$$

([4]). Note that J_0 is a real analytic (real-valued) function and that J_0 is even (the statements are seen directly from the facts that the distribution of the cos function is symmetric and supported on a bounded subset). Denote by $0<u_1<u_2<\ldots$ the sequence of positive zeros of J_0.

Recall that σ denotes the maximal (reduced) spectral type of T^F.

Let us notice that since σ is continuous,

$$\sigma^{\otimes n}\left(\{(t_1,\ldots,t_n)\in\mathbb{R}^n:\ \sum_{i=1}^n\varepsilon_it_i=0,(\varepsilon_1,\ldots,\varepsilon_n)\in\{0,1,2\}^n,\sum_{i=1}^n\varepsilon_i>0\}\right)=0$$

and moreover

$$\sigma^{\otimes n}\left(\{(t_1,\ldots,t_n)\in\mathbb{R}^n:\ |t_i|/|t_j|=u_k/u_l\text{ for }1\leq i\neq j\leq n\text{ and }k,l\geq 1\}\right)=0.$$

[11] Since the sequence $(b_nq_n\widehat{f}(q_n))$ is bounded, the sequennce $(\|f^{(b_nq_n)}\|_\infty)$ is also bounded; moreover, $b_nq_n\alpha\to 0$ modulo 1; therefore we can apply [9] and (21) holds with $t_n=b_nq_n$.

We now remove the two sets of $\sigma^{\otimes n}$-measure zero from $\mathbb{R}^n$ and take $(t_1, \ldots, t_n)$ in the remaining part of $\mathbb{R}^n$. We will show that once we know the sum $t_1+\ldots+t_n$ and moreover, for each $\delta > 0$, we know

$$\Phi(\delta) := \varphi_\delta(t_1) \cdot \ldots \cdot \varphi_\delta(t_n) = \varphi_\delta(|t_1|) \cdot \ldots \cdot \varphi_\delta(|t_n|)$$

then $t_1, \ldots, t_n$ are determined up to the order. To this end let us look at positive zeros of Φ. We can assume that $|t_1| < \ldots < |t_n|$. Then the first zero of Φ we obtain when $\delta = u_1/|t_n|$. It follows that

$$|t_n| = u_1/\text{first zero of } \Phi.$$

Notice that all points $u_k/|t_n|$ are zeros of Φ and moreover none of these points is of the form $u_1/|t_{n-1}|$. It follows that

$$|t_{n-1}| = u_1/\text{first zero of } \Phi \text{ different from } u_k/|t_n| \text{ for all } k \geq 1.$$

Similarly we obtain $|t_{n-2}|$ by dividing u_1 by the first zero of Φ different from $u_k/|t_n|$ and $u_k/|t_{n-1}|$ for all $k \geq 1$.

In this way, by an easy induction, we obtain that the numbers $|t_1|, \ldots, |t_n|$ are determined up to the order. However their sum is also given, and the equality $\sum_{i=1}^n t_i = \sum_{i=1}^n \varepsilon_i t_i$ with at least one $\varepsilon_i = -1$ has already been excluded, so the set $\{t_1, \ldots, t_n\}$ is determined.

Given $n \geq 1$, as the family $\mathcal{F}$ (see Lemma 1), it is enough to take

$$\mathcal{F} = \{\widehat{P}_{\delta_i}^{\otimes n} : \{\delta_i > 0 : i \geq 1\} \text{ is dense in } \mathbb{R}^+\}.$$

Indeed, if $\widehat{P}_{\delta_i}^{\otimes n}(t_1, \ldots, t_n) = \widehat{P}_{\delta_i}^{\otimes n}(t_1', \ldots, t_n')$ then (by continuity of J_0)

$$\varphi_\delta(t_1) \cdot \ldots \cdot \varphi_\delta(t_n) = \varphi_\delta(t_1') \cdot \ldots \cdot \varphi_\delta(t_n')$$

for each $\delta > 0$ and the result follows. □

Remark 2. It follows from [5] that all these smooth flows have simple spectrum.

4 The class of von Neumann

As previously we assume that $f : \mathbb{T} \to \mathbb{R}$, $\int_{\mathbb{T}} f \, d\lambda_{\mathbb{T}} = 0$ and we also assume that $F = f + c > 0$. Assume moreover that $f \in BV(\mathbb{T})$. Let $\alpha \in \mathbb{T}$ be irrational with the sequence (q_n) of its denominators.

Remark 3. For every $g \in BV(\mathbb{T})$ we have $\int_{\mathbb{T}} |g(x+t) - g(x)| dx \leq |t| Var\, g$. We use this inequality (instead of the L_∞-norm of the derivative of f we pass to $\int_{\mathbb{T}} |f(x+s\frac{p_n}{q_n}) - f(x+s\alpha+(j-1)q_n\alpha| \, dx)$ in the part of the proof of Proposition 1 in which we have estimated $|b_n f_{q_n}(x) - f^{(b_n q_n)}(x)|$ and we obtain an estimation for $\|b_n f_{q_n} - f^{(b_n q_n)}\|_{L^1}$ valid for all bounded variation functions.

By Remark 3, for each positive integer m and $q = q_n$,

$$\|f^{(mq)} - m f_q\|_1 \leq \frac{1}{2} m^2 q \|q\alpha\| Var(f). \tag{26}$$

From now on we assume that α has unbounded partial quotients. For simplicity of notation we assume that

$$q_n\|q_n\alpha\| \to 0. \tag{27}$$

By the Koksma inequality any weak limit of a subsequence of the distributions $(f^{(q_n)})_* = (f^{(q_n)})_*(\lambda_{\mathbb{T}})$. $n \geq 1$, is concentrated on a finite interval, and the same remains true for the sequence $((f^{(mq_n)})_*)_n$ (for any $m \geq 1$). By passing to a subsequence of (q_n) if necessary, we can assume that the sequence $(f_*^{(q_n)})$ converges weakly, and let $\nu = \lim_{s\to\infty}(f^{(q_n)})_*$, then for each $k \in \mathbb{Z}$ we have

$$\widehat{\nu}(k) = \lim_{s\to\infty}\int_{\mathbb{R}} e^{2\pi ikx}d(f^{(q_{n_s})})_*(x) = \lim_{s\to\infty}\int_{\mathbb{T}} e^{2\pi ikf^{(q_{n_s})}}\,d\lambda_{\mathbb{T}}. \tag{28}$$

Given $t \in \mathbb{R}$ set $\nu_t = (M_t)_*\nu$ (see Example 3). In view of (26) and (27), for each $m \in \mathbb{Z}$

$$f_*^{(mq_{n_s})} \to \nu_m \ \text{ weakly, when } s \to \infty \tag{29}$$

(indeed, for $m \in \mathbb{Z}$, $f^{(-mq_{n_s})}(x) = -f^{(mq_{n_s})}(x + mq_{n_s}\alpha))$. For $t \in \mathbb{R}$ set $\varphi(t) = \widehat{\nu}(t)$. We have

$$\widehat{\nu_m}(t) = \varphi(mt) = \widehat{\nu_t}(m) \quad \text{for each } m \in \mathbb{Z}. \tag{30}$$

Recall that σ denotes the maximal spectral type of T^F, where $Tx = x + \alpha$. Then by (29), (5) and (30)

$$e^{2\pi imq_n(t_1+\ldots+t_d)} \to \varphi(mt_1)\cdot\ldots\cdot\varphi(mt_d) \tag{31}$$

weakly in $L^2(\mathbb{R}^d, \sigma^{\otimes d})$ for each non-zero integer m and arbitrary integer $d \geq 1$.

Assume that f is piecewise absolutely continuous, with the sum of jumps $-S \neq 0$. We then have $f = g + h$, where h is absolutely continuous on $\mathbb{T}$, and g is piecewise linear with (a.e.) constant derivative equal to S (we assume that g is RHS continuous and that both functions g, h are of zero mean). By the Koksma inequality, $h^{(q_n)} \to 0$ uniformly (as well as for each $m \in \mathbb{Z}$, $h^{(mq_n)} \to 0$ uniformly). So in the computation of the limit distribution ν we can replace f by g. In particular we can replace f_{q_n} by g_{q_n} and then g_{q_n} by $\widetilde{g}_{q_n}$, where

$$\widetilde{g}_{q_n}(x) = g_{q_n}(x/q_n)$$

as the distributions of $\widetilde{g}_{q_n}$ and g_{q_n} are the same (indeed, the distributions of $j(\cdot)$ and $j(q_n\cdot)$ are the same; take $j(\cdot) = g_{q_n}(\cdot/q_n)$). The functions $\widetilde{g}_{q_n}$ are piecewise linear, with the constant derivative S on each interval of continuity. Moreover, the discontinuity points for $\widetilde{g}_{q_n}$ are of the form $q_n\beta$ with β a discontinuity point of g (indeed, the discontinuity points of g_{q_n} are of the form $\beta + i/q_n$) while the value of the jump at $q_n\beta$ is the sum of the values of the jumps at all discontinuity points γ of g for which $q_n\gamma = q_n\beta$. By passing to a subsequence, if necessary, we can assume that $(q_n\beta)$ converges for all discontinuity points β of g. Since there is only a finite set of possible values for jumps for all functions $\widetilde{g}_{q_n}$ and $\int_{\mathbb{T}}\widetilde{g}_{q_n}\,d\lambda_{\mathbb{T}} = 0$,

$$\widetilde{g}_{q_n} \to \widetilde{g} \ \text{ in } L^1(\mathbb{T}),$$

where the limit function has discontinuities at $\gamma_1, \ldots, \gamma_k$ (and k is not bigger than the number of discontinuity points of g) and moreover at each interval

$[\gamma_i, \gamma_{i+1})$ the function *widetildeg* has the constant derivative equal to S. The distribution of $\widetilde{g}$ is hence the limit distribution of the sequence $(f_*^{(q_n)})$, i.e. $\nu = \widetilde{g}_*$. It follows that

$$\widehat{\nu}(t) = \varphi(t) = \int_{\mathbb{T}} e^{2\pi it\widetilde{g}}\, d\lambda_{\mathbb{T}}. \tag{32}$$

Recall that the characteristic function of the distribution of the function ξ defined on an interval $[a, b)$ (considered with Lebesgue measure) by the formula $\xi(x) = C + S(x - a)$ is equal to $\frac{1}{2\pi itS}(e^{2\pi it\xi(b)} - e^{2\pi it\xi(a)})$, which implies that

$$\varphi(t) = \frac{1}{2\pi iSt} \sum_{j=1}^{2k} a_j e^{2\pi it\Delta_j}, \tag{33}$$

for some integers $a_1, \ldots, a_{2k}$ (equal to ± 1) and the numbers Δ_j are obtained as $\widetilde{g}(\gamma_i)$ and $\widetilde{g}(\gamma_i^-)$ ($\widetilde{g}(\gamma_i^-)$ stands for the LHS limit of $\widetilde{g}$ at γ_i). We first replace $\varphi(t)$ by $\psi(t) = 2\pi iSe^{-2\pi i\Delta t}\varphi(t)$, where $\Delta = \min\{\Delta_1, \ldots, \Delta_{2k}\}$ and we obtain that for each $m \in \mathbb{Z}$ there exists a measurable function $H_{d,m} : \mathbb{R} \to \mathbb{R}$ such that

$$\psi(mt_1) \cdot \ldots \cdot \psi(mt_d) = H_{m,d}(t_1 + \ldots + t_d) \tag{34}$$

for $\sigma^{\otimes d}$-a.e. $(t_1, \ldots, t_d) \in \mathbb{R}^d$. Setting $\Delta'_j = \Delta_j - \Delta$ we have $\Delta'_j \geq 0$, so by renaming points if necessary, we obtain that

$$0 = \Delta'_1 < \ldots < \Delta'_{2k}. \tag{35}$$

Set $\nu'_t = \frac{1}{t}\sum_{j=1}^{2k} a_j \delta_{\Delta'_j t}$. This is a real purely atomic measure with atoms at $\Delta'_j t$ and

$$\nu_t(\{\Delta'_j t\}) = a_j/t \ \text{ for } \ j = 1, \ldots, 2k. \tag{36}$$

In view of (34) we obtain that the map

$$(t_1, \ldots, t_d) \mapsto (\nu'_{t_1} * \ldots * \nu'_{t_d})\widehat{\ }(m)$$

is $\mathcal{B}_{C_d}(\mathbb{R}^d)$-measurable, which means that the map which to $(t_1, \ldots, t_d)$ associates the image of $\nu'_{t_1} * \ldots * \nu'_{t_d}$ via the map $x \mapsto e^{2\pi ix}$ depends only on the sum $t_1 + \ldots + t_d$ (for $\sigma^{\otimes d}$-a.a. points $(t_1, \ldots, t_d) \in \mathbb{R}^d$). Denote by J the subgroup of $\mathbb{R}$ generated by all Δ'_j's, and all $\Delta'_j\Delta'_{j'}$'s, $1 \leq j, j' \leq 2k$. This group is countable, so by continuity of σ the set

$$\widetilde{A}_d := \{(t_1, \ldots, t_d) \in \mathbb{R}^d : \ (\forall (b_1, \ldots, b_{2k}) \in J^{2k} \setminus \{\underline{0}\}) \ \sum_{j=1}^{2k} b_j t_j \notin \mathbb{Z} + J\}$$

has full $\sigma^{\otimes d}$-measure.

Assume $\underline{e} = (e_{i_1,\ldots,i_k})_{1 \leq i_1 < \ldots < i_k \leq d} \subset \mathbb{R}$ and $e' \in \mathbb{R}$. Set

$$(*) \quad X_{\underline{e},e'} = \{(t_1, \ldots, t_d) \in \mathbb{R}^d : \sum_{1 \leq i_1 < \ldots < i_k \leq d} e_{i_1,\ldots,i_k} t_{i_1} \ldots t_{i_k} = e'\}.$$

We need the following lemma about such algebraic varieties.

Lemma 2. *Assume that* $\sum_{1\le i_1<\ldots<i_k\le d} e^2_{i_1,\ldots,i_k}>0$. *Then for each continuous measure* σ

$$\sigma^{\otimes d}(X_{\underline{e},e'})=0.$$

Proof. The proof goes by induction on $d\ge 2$. For $d=2$ we consider the set of $(t_1,t_2)\in\mathbb{R}^2$ such that

$$e_{12}t_1t_2+e_1t_1+e_2t_2=e'.$$

We can assume that $e_{12}\neq 0$, otherwise we consider just a linear case. If we fix $t_2\in\mathbb{R}$, $t_2\neq -\frac{e_1}{e_{12}}$ then we have

$$(e_{12}t_2+e_1)t_1=e'-e_2t_2,$$

so there is only one $t_1\in\mathbb{R}$ satisfying this equation. By Fubini's theorem the set under consideration has $\sigma^{\otimes 2}$-measure zero.

Suppose the result being true for $d-1\ge 2$. With no loss of generality we can assume that there exist $i_2<\ldots<i_k$ such that $e_{1,i_2,\ldots,i_k}\neq 0$ (otherwise we already consider an algebraic variety of the form $(*)$ in $\mathbb{R}^{d-1}$). Then we consider Fubini's theorem with respect to the coordinates $(t_2,\ldots,t_d)$. In other words, we fix $(t_2,\ldots,t_d)$ and we look at the set of $t_1\in\mathbb{R}$ so that $(t_1,t_2,\ldots,t_d)\in X_{\underline{e},e'}$. Looking at the equation defining the variety we can see that the equation on t_1 is of the form

$$A(t_2,\ldots,t_d)t_1=B(t_2,\ldots,t_d).$$

Hence, either there is exactly one solution, or it is the whole $\mathbb{R}$. The latter case however holds only if $A(t_2,\ldots,t_d)=0$. Moreover the function $A(t_2,\ldots,t_d)$ is also of the algebraic form as in $(*)$, hence by the induction assumption the set of $(t_2,\ldots,t_d)$ satisfying $A(t_2,\ldots,t_d)=0$ is of $\sigma^{\otimes(d-1)}$-measure zero, and the lemma follows from Fubini's theorem. $\square$

Consider now the set

$$\widetilde{B}_d=\{(t_1,\ldots,t_d)\in\mathbb{R}^d:\ (\forall (n_1,\ldots,n_d)\in\mathbb{Z}^d\setminus\{\underline{0}\})\ \Pi_{i=1}^d t_i\neq \Pi_{i=1}^d (t_i+\frac{n_i}{\Delta_2'})\}.$$

In view of Lemma 2, $\sigma^{\otimes d}\left(\widetilde{B}_d^c\right)=0$. Hence the set $\widetilde{A}_d\cap\widetilde{B}_d$ has full $\sigma^{\otimes d}$ measure and in what follows we consider only $(t_1,\ldots,t_d)\in\widetilde{A}_d\cap\widetilde{B}_d$. The measure $\nu'_{t_1}*\ldots*\nu'_{t_d}$ is purely atomic with the set of atoms

$$\bigoplus_{j=1}^{2k}\{\Delta_1't_j,\Delta_2't_j,\ldots,\Delta_{2k}'t_j\}.$$

It follows that the image (on the additive circle) of this measure is also atomic with the set of atoms

$$\Omega(t_1,\ldots,t_d)=\bigoplus_{j=1}^{2k}\{\Delta_1't_j,\Delta_2't_j,\ldots,\Delta_{2k}'t_j\}\ (\mathrm{mod}\ 1).$$

Furthermore the map

$$e^{2\pi i\cdot}:(\mathbb{R},\nu'_{t_1}*\ldots*\nu'_{t_d})\to\left(\mathbb{T},\left(e^{2\pi i\cdot}\right)_*(\nu'_{t_1}*\ldots*\nu'_{t_d})\right)$$

is $1-1$ $\nu'_{t_1} * \ldots * \nu'_{t_d}$-a.e.

Since $(t_1,\ldots,t_d) \in \widetilde{A}_d$, it follows that

$$\Omega(t_1,\ldots,t_d) = \{\sum_{j=1}^{2k} \varepsilon_j \Delta'_{m_j} t_j \pmod 1 : \varepsilon_j \in \{0,1\}, 1 \le m_j \le 2k, j=1,\ldots,2k\}$$

and moreover the representation of a point in $\Omega(t_1,\ldots,t_d)$ is unique. Suppose that $\nu'_{t_1} * \ldots * \nu'_{t_d} = \nu'_{t'_1} * \ldots * \nu'_{t'_d}$, in particular

$$\Omega(t_1,\ldots,t_d) = \Omega(t'_1,\ldots,t'_d).$$

For each $1 \le l \le 2k$, $1 \le i \le d$ we have

$$\Delta'_l t'_i = \sum_{j=1}^{2k} \varepsilon_{l,i,j} \Delta'_{m_{l,i,j}} t_j \pmod 1$$

and similarly for $1 \le p \le 2k$, $1 \le j \le d$,

$$\Delta'_p t_j = \sum_{i=1}^{2k} \varepsilon'_{p,j,i} \Delta'_{m'_{p,j,i}} t'_i \pmod 1.$$

Thus

$$\Delta'_p t_j = \sum_{j'=1}^{2k} \left(\sum_{p'=1}^{2k} d^{(p,j)}_{p',j'} \Delta'_{p'} \right) t_{j'} \pmod 1,$$

where $d^{(p,j)}_{p',j'}$ are non-negative integers. Since $\sum_{p'=1}^{2k} d^{(p,j)}_{p',j'} \Delta'_{p'} \in J$, $\sum_{p'=1}^{2k} d^{(p,j)}_{p',j'} \Delta'_{p'} = 0$ for $j' \neq j$, and hence by (35), $d^{(p,j)}_{p',j'} = 0$ for $p' \ge 2$. For $j'=j$ we obtain that $\Delta'_p = \sum_{p'=1}^{2k} d^{(p,j)}_{p',j} \Delta'_{p'}$. It follows that $d^{(2,j)}_{2,j} = 1$ and $d^{(2,j)}_{p',j} = 0$ if $p' \ge 3$. This reasoning shows that

$$\Delta'_2 t_j = \Delta'_{m_j} t'_{i_j} \pmod 1 \text{ for all } j=1,\ldots,2k \tag{37}$$

where the map $j \mapsto i_j$ is univoque. But $\Delta'_2 t'_i = \sum_{j'=1}^{2k} \varepsilon_{2,i,j'} \Delta'_{m_{2,i,j'}} t_{j'} \pmod 1$, so

$$\begin{aligned}(\Delta'_2)^2 t_j &= \Delta'_2 (\Delta'_2 t_j) = \Delta'_2(\Delta'_{m_j} t'_{i_j} + s_j) = \Delta'_{m_j} \left(\Delta'_2 t'_{i_j}\right) + \Delta'_2 s_j \\ &= \sum_{j'=1}^{2k} \varepsilon_{2,i_j,j'} \Delta'_{m_j} \Delta'_{m_{2,i_j,j'}} t_{j'} + \left(\Delta'_{m_j} m_{i_j} + \Delta'_2 s_j\right)\end{aligned}$$

(with some $s_j, m_{i_j} \in \mathbb{Z}$) where the last summand belongs to J. It follows that $(\Delta'_2)^2 = \Delta'_{m_j} \Delta'_{m_{2,i_j,j}}$. Using once more (35) we have $m_j = m_{2,i_j,j} = 2$, and therefore

$$\Delta'_2 t_j = \Delta'_2 t'_{i_j} \pmod 1 \text{ for all } j=1,\ldots,d.$$

In other words there are $n_1,\ldots,n_d \in \mathbb{Z}$ such that

$$t_j + \frac{n_j}{\Delta'_2} = t'_{i_j} \text{ for } j=1,\ldots,d. \tag{38}$$

Now, the "masses" of the atoms of the circle image of $\nu'_{t_1} * \ldots * \nu'_{t_d}$ and of $\nu'_{t'_1} * \ldots * \nu'_{t'_d}$ must also be the same, so in view of (36), by looking at the mass of the atom at $0 \in \mathbb{R}$ (for $\nu'_{t_1} * \ldots * \nu'_{t_d}$) we find

$$\frac{a_1^d}{t_1 \cdot \ldots \cdot t_d} = \frac{a_1^d}{t'_1 \cdot \ldots \cdot t'_d},$$

hence $t_1 \cdot \ldots \cdot t_d = t'_1 \cdot \ldots \cdot t'_d$. By (38) we obtain $\Pi_{i=1}^d t_i = \Pi_{i=1}^d (t_i + \frac{n_i}{\Delta'_2})$. Since $(t_1, \ldots, t_d) \in \widetilde{B}_d$, the latter is possible only if $n_1 = \ldots = n_d = 0$ and we have $\{t_1, \ldots, t_k\} = \{t'_1, \ldots, t'_k\}$.

By taking as $\mathcal{F} = \mathcal{F}_d$ the family $\{\psi(m\cdot)^{\otimes d} : m \in \mathbb{Z}\}$ we have proved the following.

Proposition 2. *Assume that f is piecewise absolutely continuous with the sum of jumps different from zero. The for each α with unbounded partial quotients the special flow arising from T^F ($Tx = x + \alpha$, $F = f + c$) is weakly mixing and has the simple convolution property.*

5 Remarks

In Appendix we will answer the question raised by J.-P. Thouvenot and show that given a measurable flow $(T_t)_{t\in\mathbb{R}}$ the function

$$(v) \quad \mathbb{R} \ni t \mapsto \text{maximal spectral multiplicity of } U_{T_t}$$

is of the second Baire class (in fact this is a purely unitary representation theory result). The question whether the function (v) can be oscillatory on $\mathbb{R} \setminus \{0\}$ in case of weakly mixing flows seems to be open, and, it looks as if in all known examples of weakly mixing flows the function (v) is in fact constant on $\mathbb{R} \setminus \{0\}$. This latter phenomenon certainly holds for large classes of flows. The proposition below being our first sample of result in a more general theory.

Proposition 3. *For each flow (T_t) with the SC property, the function (v) is constant on $\mathbb{R} \setminus \{0\}$ equal to the maximal spectral multiplicity of the unitary flow $(U_{T_t})_{t\in\mathbb{R}}$.*

Proof. Denoting by σ the maximal spectral type of (T_t), all we need to show (see Appendix) is that the set

$$A(\sigma) = \{t \in \mathbb{R} : \sigma \not\perp \sigma * \delta_t\}$$

consists solely of 0. This is however already noticed in [27], see Corollary 4 therein. □

In order to say more about time-t automorphism we will need a stronger assertion.

Lemma 3. *Assume that (T_t) has the SC property. Then for each $n \geq 1$*

$$A(\sigma^{*n}) = \{0\}.$$

Proof. (cf. [24]) Recall first that for each $k \geq 2$ we can find $X_k \subset \mathbb{R}^k$ so that

$$\sigma^{\otimes k}(X_k) = 1 \ \text{ and } \ (x_1, \ldots, x_k) \in X_k \Rightarrow x_i \neq x_j \text{ for } i \neq j. \tag{39}$$

Denote

$$s_k : \mathbb{R}^k \to \mathbb{R}, \ s_k(x_1, \ldots, x_k) = x_1 + \ldots + x_k,$$
$$m_k : (\mathbb{S}^1)^k \to \mathbb{S}^1, \ m_k(z_1, \ldots, z_k) = z_1 \cdot \ldots \cdot z_k.$$

Denote

$$\sigma^{\otimes k} = \int_{\mathbb{R}} \delta_c \otimes \nu_c^{(k)} \, d\sigma^{*k}$$

the disintegration of $\sigma^{\otimes k}$ over σ^{*k}.

Suppose that for some $t \neq 0$ and $n \geq 2$, $\sigma^{*n} \not\perp \sigma^{*n} * \delta_t$. Then we can find Borel subsets $A, A' \subset \mathbb{R}$ such that

$$\sigma^{*n}(A) > 0, \ \sigma^{*n}(A + t) > 0, \ A \cap (A + t) = 0.$$

$$\sigma^{*n}(A') > 0, \ \sigma^{*n}(A' + t) > 0, \ A' \cap (A' + t) = 0.$$

Notice that

$$\sigma^{*2n}(A + A') = \sigma^{*n} * \sigma^{*n}(A + A') = (\sigma^{*n} \otimes \sigma^{*n})(A \times A').$$

Hence

$$\sigma^{*2n}((A + t) + (A' - t)) = \sigma^{*2n}(A + A') > 0.$$

It is not hard to see that if $(x_1, \ldots, x_n)$ is an atom of $\nu_c^{(n)}$ (in particular, $x_1 + \ldots + x_n = c$) and $(y_1, \ldots, y_n)$ is an atom of $\nu_d^{(n)}$ then for $\sigma^{*n} \otimes \sigma^{*n}$-a.e. $(c, d) \in \mathbb{R} \times \mathbb{R}$ we have $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ is an atom of $\nu_{c+d}^{(2n)}$, so, in particular, the $x_i \neq y_j$ for $i, j = 1, \ldots, n$.

It follows that for $\sigma^{*n} \otimes \sigma^{*n}$-a.e. $(c, d) \in A \times A'$ if $(x_1, \ldots, x_n)$ is an atom of $\nu_c^{(n)}$, $(y_1, \ldots, y_n)$ is an atom of $\nu_d^{(n)}$, $(x_1^{(t)}, \ldots, x_n^{(t)})$ is an atom of $\nu_{c+t}^{(n)}$, $(y_1^{(t)}, \ldots, y_n^{(t)})$ is an atom of $\nu_{d-t}^{(n)}$ then the sets $\{x_1, \ldots, x_n\}$, $\{x_1^{(t)}, \ldots, x_n^{(t)}\}$, $\{y_1, \ldots, y_n\}$, $\{y_1^{(t)}, \ldots, y_n^{(t)}\}$ are pairwise disjoint (with no loss of generality the numbers $c, d, c + t, d - t$ are different). It follows that

$$(x_1, \ldots, x_n, y_1, \ldots, y_n) \ \text{ and } (x_1^{(t)}, \ldots, x_n^{(t)}, y_1^{(t)}, \ldots, y_n^{(t)})$$

are now atoms of $\nu_{c+d}^{(2n)}$ and we obtain at least $2 \cdot (2n)!$ atoms of $\nu_{c+d}^{(2n)}$ which is a contradiction. □

Proposition 4. *Assume that (T_t) has the SC property. Then for each $t \neq 0$ the automorphism T_t has also the SC property (as $\mathbb{Z}$-action).*

Proof. Notice that the diagram

$$\begin{array}{ccc} \mathbb{R}^n & \stackrel{(e^{2\pi i t \cdot})^{\otimes n}}{\rightarrow} & (\mathbb{S}^1)^n \\ s_n \downarrow & & \downarrow m_n \\ \mathbb{R} & \stackrel{\rightarrow}{e^{2\pi i t \cdot}} & (\mathbb{S}^1)^n \end{array} \tag{40}$$

is commuting. In view of Lemma 3 we can choose Borel subsets $W_n \subset \mathbb{R}^n$ and $W^{(n)} \subset \mathbb{R}$ so that

$$\sigma^{\otimes n}(W_n) = 1,\ \sigma^{*n}(W^{(n)}) = 1$$

and the maps

$$W_n \ni (x_1, \ldots, x_n) \mapsto (e^{2\pi i t x_1}, \ldots, e^{2\pi i t x_n}),$$
$$W_n \ni y \mapsto e^{2\pi i t y}$$

are 1-1 $\sigma^{\otimes n}$- and σ^{*n}-a.e. respectively. Indeed,

$$(e^{2\pi i t x_1}, \ldots, e^{2\pi i t x_n}) = (e^{2\pi i t x'_1}, \ldots, e^{2\pi i t x'_n})$$

is equivalent to saying that

$$x_i = x'_i + l_i/t \text{ for some } l_i \in \mathbb{Z},$$

for $i = 1, \ldots, n$, so we use the fact that $\sigma \perp \sigma * \delta_{l_i/t}$ whenever $l_i \neq 0$ (select $W_{i,n} \subset \mathbb{R}$ so that $W_{i,n} \cap (W_{i,n} + l/t) = \emptyset$ for each $l \neq 0$ and set $W_n = W_{1,n} \times \ldots \times W_{n,n}$). For the second assertion we use the fact that $\sigma^{*n} \perp \sigma^{*n} * \delta_{l/t}$ for each $l \neq 0$.

It follows that in the diagram (40) the horizontal maps can be assumed to be 1-1 (a.e.), and the result immediately follows. □

Remark 4. Assume that $\sigma \in \mathcal{P}(\mathbb{R})$ is continuous and symmetric and denote by (G_t^σ) the corresponding the corresponding Gaussian flow. Assume moreover that (G_t^σ) has simple spectrum. It follows from the proof of Proposition 4 that for each $t \neq 0$ the Gaussian automorphism G_t^σ has also simple spectrum.

The other direction is trivial: if for some $t \neq 0$, G_t^σ has simple spectrum then the more the whole flow has simple spectrum. Note in passing that whenever (G_t^σ) has simple spectrum, then $A(\exp(\sigma)) = \{0\}$. Indeed, we have almost proved it except that we should show that whenever $n \neq m$

$$\sigma^{*n} \perp \sigma^{*m} * \delta_y \ \text{ for each } \ y \in \mathbb{R}.$$

We know this already for $y = 0$. Denote by $\tilde{\nu}$ the measure given by $\tilde{\nu}(A) = \nu(-A)$. Then if $\rho \ll \sigma^{*n}$ and $\rho \ll \sigma^{*m} * \delta_y$ then $\rho * \tilde{\rho} \ll \sigma^{*2n}$ (since σ is symmetric) and

$$\rho * \tilde{\rho} \ll \sigma^{*m} * \delta_y * \widetilde{\sigma^{*m} * \delta_y}$$
$$\ll \sigma^{*m} * \delta_y * \sigma^{*m} * \delta_{-y} = \sigma^{*2m}$$

and hence $\rho = 0$.

It follows that if $\mathcal{T}$ has the SC property then $A(\sigma_{\mathcal{T}}) = \{0\}$.

Corollary 3. *$\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ has the SC property if and only if for each $t \neq 0$, T_t has the SC property if and only if T_1 has the SC property and $1 \notin A(\sigma)$.*

Proof. Suppose that T_1 has the SC property and $1 \notin A(\sigma)$. Consider $L^2(\mathbb{R}, \sigma)$ and the flow $V_t f(x) = e^{2\pi i t x} f(x)$, $t \in \mathbb{R}$. Under the above assumption, the action of V_1 on that space is isomorphic to the unitary operator $W_{\sigma_1} : j(z) \mapsto zj(z)$ on $L^2(\mathbb{S}^1, \sigma_1)$, where $\sigma_1 = \left(e^{2\pi i \cdot}\right)_* (\sigma)$ (see Appendix). Moreover, the corresponding unitary operator $\bigoplus_{m=0}^\infty W_{\sigma_1}^{\odot m}$ acting on the symmetric Fock space $\bigoplus_{m=0}^\infty L^2(\mathbb{S}^1, \sigma_1)^{\odot m}$ is isomorphic to the unitary operatopr $\bigoplus_{m=0}^\infty V_1^{\odot m}$ acting on $\bigoplus_{m=0}^\infty L^2(\mathbb{R}, \sigma)^{\odot m}$. Since the former automorphism has simple spectrum, this is the same for the latter one and the whole Gaussian flow has the more simple spectrum. □

We will now try to derive more information about $A(\sigma_{\mathcal{T}})$ under the assumption of having a special operator in the weak closure of $\{U_{T_t} : t \in \mathbb{R}\}$. Our analysis is along the lines of [17] with an additional assumption that a function which is in the weak closure of characters has some additional topological properties.

Lemma 4. *Assume that* $\xi : \mathbb{R} \to \mathbb{C}$ *and set*

$$TL(\xi) = \{t \in \mathbb{R} : (\exists |c_t| = 1)(\forall x \in \mathbb{R})\ \xi(x+t) = c_t \xi(x)\}.$$

Then $TL(\xi)$ *is a subgroup of* $\mathbb{R}$*. Moreover,* $\xi|_{TL(\xi)}$ *is a multiple of a translation of an algebraic character of* $TL(\xi)$*.*

Proof. Without loss of generality we can assume that ξ is not a constant, in particular that ξ is not a zero function.

I. Assume additionally that $\xi(0) \neq 0$ and set

$$\tilde{\xi}(x) = \frac{\xi(x)}{\xi(0)} \quad \text{for } x \in \mathbb{R}.$$

Then

$$TL(\xi) = \{t \in \mathbb{R} : (\forall x \in \mathbb{R})\ \tilde{\xi}(x+t) = \tilde{\xi}(x)\tilde{\xi}(t)\}$$

(indeed, $c_t = \xi(t)/\xi(0)$). It is elementary to check that the RHS set is a subgroup of $\mathbb{R}$ on which $\tilde{\xi}(y_1 + y_2) = \tilde{\xi}(y_1)\tilde{\xi}(y_2)$.

II. If $\xi(0) = 0$ then fix $x_0 \in \mathbb{R}$ so that $\xi(x_0) \neq 0$. Then define $\xi_1(x) = \xi(x + x_0)$, notice that $TL(\xi) = TL(\xi_1)$ and use the first part of the proof. □

Proposition 5. *Assume that* σ *is a positive finite continuous Borel measure on* $\mathbb{R}$*. Assume moreover that for some sequence* $r_n \to \infty$ *and some analytic function* $\xi : \mathbb{R} \to \mathbb{C}$

$$e^{2\pi i r_n \cdot} \to \xi(\cdot) \quad \textit{weakly in } L^2(\mathbb{R}, \sigma). \tag{41}$$

(i) If ξ *is not a multiple of a character of* $\mathbb{R}$ *then* $A(\sigma)$ *is included in a cyclic subgroup of* $\mathbb{R}$*. In particular,* $H(\sigma)$ *(see Appendix) is cyclic.*

(ii) If additionally, the topological support of σ *is* $\mathbb{R}$*, and* ξ *is not the product of a character and a periodic analytic function, then* $A(\sigma)$=0*. In particular, the result holds if* σ *is the maximal spectral type of an aperiodic flow.*

Proof. Fix $t \in A(\sigma)$. Then find a positive finite Borel measure λ so that $\lambda \ll \sigma$, $\lambda * \delta_{-t} \ll \sigma$. In view of (41) for each $\rho \ll \sigma$

$$e^{2\pi i r_n \cdot} \to \xi(\cdot) \quad \text{weakly in } L^2(\mathbb{R}, \rho). \tag{42}$$

Passing to a subsequence if necessary, we can assume that $e^{2\pi r_n t} \to c$, $|c| = 1$. By (42) it follows that

$$\int e^{2\pi i r_n (x+t)}\, d\lambda(x) = \int e^{2\pi i r_n x}\, d(\lambda * \delta_{-t})(x) \to$$

$$\int \xi(x)\, d(\lambda * \delta_{-t})(x) = \int \xi(x + t_0)\, d\lambda(x).$$

On the other hand

$$\int e^{2\pi i r_n(x+t)}\, d\lambda(x) \to c\int \xi(x)\, d\lambda(x).$$

Since (by (42)) the above convergences also take place for each $\lambda_1 \ll \lambda$,

$$\xi(x+t) = c\xi(x) \ \text{ for } \lambda\text{-a.e. } x \in \mathbb{R}. \tag{43}$$

Since λ is continuous and ξ is analytic $\xi(x+t) = c\xi(x)$ for all $x \in \mathbb{R}$. In other words $t \in TL(\xi)$.

(i) Suppose now that $t_1, t_2 \in A(\sigma)$ and suppose that they are not in the same cyclic subgroup of $\mathbb{R}$. Since (by Lemma 4) $TL(\xi)$ is a subgroup of $\mathbb{R}$, it is dense in $\mathbb{R}$. Since (again by Lemma 4) for some complex κ and a real x_0, $\kappa\xi(\cdot + x_0)$ is a group homomorphism on $TL(\xi)$, $\kappa\xi(\cdot + x_0)$ is a continuous character on $\mathbb{R}$, whence ξ is a multiple of a character.

(ii) In view of (41), $|\xi(x)| \leq 1$ for σ-a.e. $x \in \mathbb{R}$ and hence by the assumption on the topological support, $|\xi| \leq 1$. We can also assume that $\xi(0) \neq 0$. Assume that $t \in A(\sigma)$. We have

$$\tilde{\xi}(x+t) = \tilde{\xi}(x)\tilde{\xi}(t) \ \text{ for all } x \in \mathbb{R}.$$

Moreover $|\tilde{\xi}| \leq M$ (where $M = 1/|\xi(0)|$). If $\tilde{\xi}(t) = 0$, then $\tilde{\xi} = 0$. Otherwise $\tilde{\xi}(-t) = \tilde{\xi}(t)^{-1}$ as clearly $\tilde{\xi}(0) = 1$. Suppose that $|\tilde{\xi}(t)| \neq 1$. Then by replacing t by $-t$ if necessary we have $|\tilde{\xi}| < 1$ and

$$|\tilde{\xi}(x+tn)| \leq M|\tilde{\xi}(t)|^n$$

which implies $\tilde{\xi} = 0$.

Now write $\xi(t) = e^{2\pi i\alpha}$ and consider the character $g(x) = e^{2\pi i\alpha x}$. We then clearly have

$$\tilde{\xi}(x+t)/g(x+t) = \tilde{\xi}(x)/g(x)$$

and the result follows. □

Looking at the proof of this proposition we obtain the following.

Corollary 4. *If additionally, for some probability measure P on $\mathbb{R}$,*

$$\xi(x) = \int e^{2\pi i t x}\, dP(t),$$

i.e. ξ is the Fourier transform of P then P is concentrated on a coset of a cyclic subgroup of $\mathbb{R}$.

Proof. Denote by $m \ll P$ the complex measure $f(s)dP(s)$ with $f(s) = c_t - e^{2\pi i t s}$. Now the equality (43) gives $\widehat{m}(x) = 0$ for all $x \in \mathbb{R}$ which means that m is the zero measure, or equivalently that

$$c_t = e^{2\pi i s t} \ \text{ for } P\text{-a.e. } s \in \mathbb{R}$$

and the result follows. □

Corollary 5. *Assume that σ is a positive finite continuous Borel measure on $\mathbb{R}$ with full topological support. Assume moreover that for some sequence $r_n \to \infty$ and some continuous function $\xi : \mathbb{R} \to \mathbb{C}$*

$$e^{2\pi i r_n \cdot} \to \xi(\cdot) \quad \textit{weakly in} \quad L^2(\mathbb{R}, \sigma). \tag{44}$$

(i) If ξ is not a multiple of a character of $\mathbb{R}$ then $H(\sigma)$ is cyclic.

(ii) If ξ is not the product of a character and a periodic continuous function, then $H(\sigma)$=0.

In particular the result holds if σ is the maximal spectral type of an aperiodic measurable flow.

Proof. We repeat the proof of Proposition 5 with $\lambda = \sigma$ and obtain (43) for σ-a.e. $x \in \mathbb{R}$. Since the topological support of σ is full and ξ is continuous we obtain $\xi(x+t) = c\xi(x)$ for all $x \in \mathbb{R}$ provided that $t \in H(\sigma)$. □

Consider now the family $\mathcal{K}$ of those $T \in Aut\,(X, \mathcal{B}, \mu)$ such that there exists a measurable flow $(T_t)_{t\in\mathbb{R}} \subset Aut\,(X, \mathcal{B}, \mu)$ such that $T_1 = T$ and such that EACH Markov operator $J : L^2(X, \mathcal{B}, \mu) \to L^2(X, \mathcal{B}, \mu)$ commuting with U_T is in the weak closure of $\{U_T^k : k \in \mathbb{Z}\}$. It follows from [38] and [22] that $\mathcal{K}$ is a residual subset of $Aut\,(X, \mathcal{B}, \mu)$.

Corollary 6. *Assume that T is an ergodic automorphism such that each Markov operator $J : L^2(X, \mathcal{B}, \mu) \to L^2(X, \mathcal{B}, \mu)$ commuting with U_T is in the weak closure of $\{U_T^k : k \in \mathbb{Z}\}$. Assume moreover that T is embeddable in a measurable flow $\mathcal{T} = (T_t)_{t\in\mathbb{R}}$, $T_1 = T$. Then $A(\sigma_{\mathcal{T}}) = 0$. In particular, the assertion of Proposition 3 holds.*

Proof. Take any continuous measure $P \in \mathcal{P}(\mathbb{R})$ with bounded topological support and define $J = \int_{\mathbb{R}} U_{T_t}\, dP$. Then J is a Markov operator commuting with U_T and hence by our standing assumption J is in the closure of $\{U_{T_t} : t \in \mathbb{R}\}$. Since $\widehat{P}$ is analytic, the result follows from Corollary 5. □

It follows from Proposition 3 that the same assertion follows for $\mathcal{T} = T^f$ where T^f is a special flows from the class (A). It is the same result for the von Neumann class of special flows T^f (see class (B)) where $Tx = x + \alpha$ with α arbitrary irrational. The reason is that, as shown in Corollary 6 in [28], each accumulation point $P \in \mathcal{P}(\mathbb{R})$ of $\{(f^{(q_n)})_* : n \geq 0\}$ is an absolutely continuous measure (here (q_n) stands for the sequence of denominators of α and there are no Diophantine restrictions on α). However, the results of [28] show actually more. The fact that $A(\sigma_{T^f}) = \{0\}$ is stable under small variation perturbations.

Corollary 7. *Assume that T^f is a von Neumann's flow. Then for each $g : \mathbb{T} \to \mathbb{R}$ of sufficiently small variation ($f + g > 0$) we have $A(\sigma_{T^{f+g}}) = 0$.*

Proof. The proof of Theorem 3 in [28] says that whenever g is of sufficiently small variation then

$$\limsup_{n\to\infty} \left| \int_{\mathbb{T}} e^{2\pi i k f^{(q_n)}}\, d\mu \right| \leq c < 1$$

for $k \in \mathbb{Z}$ large enough and therefore for each accumulation point P of $\{(f^{(q_n)})_* : n \geq 0\}$ the measure P cannot be discrete (see the proof of Proposition 12 in [28]). □

Another class of examples comes from smooth change of time of linear flows on $\mathbb{T}^2$ and is based on results from [8] and [33]. Given $k \geq 1$ take a linear flow $T_t(x,y) = (x + t\alpha, y + t)$ on $\mathbb{T}^2$ where

$$\liminf_{n\to\infty} q_n^{k+1}\|q_n\alpha\| = 0. \tag{45}$$

The flow (T_t) is given by the system of differential equations:

$$\frac{dx}{dt} = \alpha,\ \frac{dy}{dt} = 1.$$

Take $F \in C^{k-1}(\mathbb{T}^2)$, $F > 0$ and consider the flow $\mathcal{S} = (S_t)$ (which preserves $F d\lambda_{\mathbb{T}^2}$) coming from

$$\frac{dx}{dt} = \alpha/F(x,y),\ \frac{dy}{dt} = 1/F(x,y).$$

Assume that $\frac{\partial^k F}{\partial x^k}$ is piecewise absolutely continuous, which means that for some finite partitions into intervals on both coordinates x, y the function $\frac{\partial^k F}{\partial x^k}$ is absolutely continuous on all resulting closed rectangles, and

$$\int_0^1\int_0^1 \frac{\partial^{k+1}F}{\partial x^{k+1}}(x+s\alpha,s)\,dsdx = \int_{\mathbb{T}^2}\frac{\partial^{k+1}F}{\partial x^{k+1}}\,d\lambda_{\mathbb{T}^2} \neq 0. \tag{46}$$

Define $f(x) = \int_0^1 F(x+s\alpha,s)\,ds$. Then $f \in C^{k-1}(\mathbb{T})$ and (S_t) is represented as a special flow over the rotation by α and under f. Due to our assumptions on F, $D^{k-1}f$ is absolutely continuous and $D^k f$ is piecewise absolutely continuous with the sum of jumps different from zero: indeed, by (46)

$$\int_{\mathbb{T}} D^{k+1}f\,d\lambda_{\mathbb{T}} = \int_{\mathbb{T}^2}\frac{\partial^{k+1}F}{\partial x^{k+1}}\,d\lambda_{\mathbb{T}^2} \neq 0.$$

Then as an analysis in [9] shows in the weak closure of $\{U_{S_t} :\ t \in \mathbb{R}\}$ we will find an operator $\int_{\mathbb{R}} U_{S_t}\,dP(t)$ such that the measure P is not discrete.

Corollary 8. *Under the above assumptions on the smoothness of time change of the linear flow, $A(\sigma_{\mathcal{S}}) = \{0\}$ whenever (45) holds.*

Remark 5. We would like also to notice that in two classical cases, namely the Gaussian and the Poissonian case the function (v) is also constant. Indeed, it is enough to consider only the Gaussian case (as from the spectral point of view Poissonian systems form a subclass of Gaussian systems). If a Gaussian flow given by σ has simple spectrum then as we have already noticed $A(\exp\sigma) = \{0\}$.

If the spectrum is not simple then the maximal spectral multiplicity of the flow is infinite (and so it must be for each U_t, $t \neq 0$).

6 Appendix

6.1 Saturated subgroup of a measure

Let $\sigma \in \mathcal{P}(\mathbb{R})$. Set

$$H(\sigma) = \{t \in \mathbb{R} :\ \sigma * \delta_t \equiv \sigma\},\ \ A(\sigma) = \{t \in \mathbb{R} :\ \sigma * \delta_t \not\perp \sigma\}.$$

In [17], B. Host, J.-F. Méla and the second author of the paper have shown the following results:

(47) $H(\sigma)$ is Borel subgroup of $\mathbb{R}$; it has a natural metric (stronger than Euclidean metric) which makes it a Polish group; moreover this group is saturated;

$$\text{if } \sigma \text{ is } H(\sigma)\text{-ergodic then } A(\sigma) = H(\sigma), \tag{48}$$

(49) if σ is ergodic, then there exists a unique measure ν such that $\sigma \ll \nu$, ν is $H(\nu)$-ergodic and $H(\nu) = gp(A(\sigma))$

(D-ergodicity of σ means that the only set of positive σ-measure which is invariant under all translations by $d \in D$ equals $\mathbb{R}$, σ-a.e.);

(50) suppose that σ is singular, then $A(\sigma)$ is contained in a countable union of weak Dirichlet sets, and in particular $\rho(A(\sigma)) = 0$ for each full measure ρ.

Recall that a *weak Dirichlet set* E is a closed subset of $\mathbb{R}$ such that for each $\tau \in \mathcal{P}(\mathbb{R})$ concentrated on E we have $\widehat{\tau}(n_t) \to 1$ for some increasing sequence (n_t) of integers; $\rho \in \mathcal{P}(\mathbb{R})$ is *full* if for each probability $\rho' \ll \rho$ we have $\limsup_{t\to\infty} |\widehat{\rho'}(t)| < 1$.

6.2 Spectral multiplicity at instance t

The analysis given here is not original, and can be found in [30]; we recall it for completeness. Since any spectral decomposition of H under $\underline{U}$ is invariant under each unitary operator U_{t_0} in order to determine the Baire category class of the function $\mathcal{M}$ we need to consider only simple spectrum case.

From now on we suppose that $\underline{U} = (U_t)_{t\in\mathbb{R}}$ has simple spectrum, so by using Spectral Theorem we can assume that

$$U_t : L^2(\mathbb{R}, \sigma) \to L^2(\mathbb{R}, \sigma), \;\; (U_t f)(x) = e^{2\pi i t x} f(x).$$

Let $s \in \mathbb{R} \setminus \{0\}$. Given $k \in \mathbb{Z}$ we denote

$$H_{s,k} = L^2\left(\mathbb{R}, \sigma|_{\left[\frac{k}{s}, \frac{k+1}{s}\right)}\right) = \left\{ f \in L^2(\mathbb{R}, \sigma) : \; f = 0 \text{ on } \left[\frac{k}{s}, \frac{k+1}{s}\right)^c \right\}.$$

Clearly each subspace $H_{s,k}$ is closed and U_s-invariant. We also have

$$L^2(\mathbb{R}, \sigma) = \bigoplus_{k\in\mathbb{Z}} H_{s,k}. \tag{51}$$

If we set $p_{s,k} : (\mathbb{R}, \sigma|_{\left[\frac{k}{s}, \frac{k+1}{s}\right)}) \to (\mathbb{T}, \sigma_{s,k})$, $p_{s,k}(x) = e^{2\pi i s x}$ and

$$\sigma_{s,k} = (p_{s,k})_* \left(\sigma|_{\left[\frac{k}{s}, \frac{k+1}{s}\right)} \right),$$

we obtain that $p_{s,k}$ is 1-1 a.e. and moreover we have:

Lemma 5. *For each $k \in \mathbb{Z}$, the spectrum of U_s on $H_{s,k}$ is simple.*

Proof. Let V denote the unitary operator on $L^2(\mathbb{T}, \sigma_{k,s})$ given by $(Vf)(z) = zf(z)$ (where we have the correspondence $z \leftrightarrow e^{2\pi isx}$). Then, let $W : L^2(\mathbb{T}, \sigma_{k,s}) \to H_{s,k}$, $(Wf)(e^{2\pi isx}) = f(x)$. Since $p_{s,k}$ is injective, W is unitary. Now for every $f \in L^2(\mathbb{T}, \sigma_{s,k})$, we have

$$W(U_s f)(e^{2\pi isx}) = U_s f(x) = e^{2\pi isx} f(x)$$

while

$$V(Wf)(e^{2\pi isx}) = e^{2\pi isx}(Wf)(e^{2\pi isx}) = e^{2\pi isx} f(x),$$

so $U_s Wf = WVf$ and the result follows. □

In view of (51) and Lemma 5, for each $s \neq 0$, we have:

(52) the maximal spectral type of $U_{1/s}$ is equal to $\sigma_{1/s} := (e^{2\pi i \frac{1}{s}(\cdot)})_* \sigma$ and the multiplicity function $\mathcal{M}(U_{1/s}) = \mathcal{M}(s) = \mathcal{M}(s,z)$ is given by the formula $\mathcal{M}(s,z) = \sum_{k\in\mathbb{Z}} \frac{d\sigma_{1/s,k}}{d\sigma_{1/s}}(z)$.

Remark 6. Let us notice that Lemma 5 can be rephrased in the following way: For each $s \neq 0$

(53) $U_{1/s}$ on $L^2(\mathbb{R}, \sigma|_{ks,(k+1)s)}$ has simple spectrum and is isomorphic to $U_{1/s}$ on $L^2\left(\mathbb{R}, \sigma * \delta_{-ks}|_{[0,s)}\right)$ for each $k \in \mathbb{Z}$.

Indeed, the isomorphism is given by

$$W : L^2\left(\mathbb{R}, \sigma|_{[ks,(k+1)s)}\right) \to L^2\left(\mathbb{R}, \sigma * \delta_{-ks}|_{[0,s)}\right),\ (Wf)(x) = f(ks + x).$$

It follows from (53) that the maximal spectral multiplicity of $U_{1/s}$ is at least m if and only if there exist $\nu \in \mathcal{P}(\mathbb{R})$ concentrated on $[0,s)$ and integers $k_1 < \ldots < k_m$ such that

$$\nu \ll \sigma * \delta_{-k_i s}|_{[0,s)},\ i = 1, \ldots, m. \tag{54}$$

By replacing in (54) the measure ν by $\nu * \delta_{k_1 s}$ and k_i by $k_i - k_1$ we can assume that $k_1 = 0 < k_2 < \ldots < k_m$ and $\nu \ll \sigma * \delta_{-k_i s}$, for $i = 1, \ldots, m$, $k_1 = 0$. Suppose now that ν is a probability measure on $\mathbb{R}$ such that

$$\nu \ll \sigma * \delta_{n_i s},\ \ i = 1, \ldots, m,\ n_1 = 0 < n_2 < \ldots < n_m.$$

With no loss of generality ν is concentrated on $[ls, (l+1)s)$. Then

$$\nu' := \nu * \delta_{-ls} \ll \sigma * \delta_{(n_i - l)s}$$

and ν' is concentrated on $[0,s)$. Thus

$$\nu' \ll \sigma * \delta_{(n_i - l)s}|_{[0,s)},\ \ i = 1, \ldots, m.$$

It follows that $U_{1/s}$ has the maximal spectral multiplicity at least m if and only if there exist $\nu \in \mathcal{P}(\mathbb{R})$ and integers $0 = n_1 < n_2 < \ldots < n_m$

$$\nu \ll \sigma * \delta_{-n_i s},\ i = 1, \ldots, s. \tag{55}$$

We now have the following two direct corollaries.

(56) If $s \in H(\sigma)$, then $U_{1/s}$ has the uniform infinite multiplicity (and its spectral type equals $(e^{2\pi i \frac{1}{s}(\cdot)})_*\sigma|_{[0,s)}$);

and

(57) $U_{1/s}$ has no simple spectrum if and only if $s \in \bigcup_{p\neq 0} \frac{1}{p}A(\sigma)$.

It follows from (50):

Corollary 9. *If σ is singular, then for each full measure $\rho \in \mathcal{P}(\mathbb{R})$, U_t has simple spectrum for ρ-a.e. $t \in \mathbb{R}$.* □

6.3 Continuous measures on the real line

Let $\sigma \in \mathcal{P}(\mathbb{R})$. Assume additionally that σ is continuous. Denote $F_\sigma : \mathbb{R} \to [0,1]$ the distribution function of σ, i.e. $F_\sigma(x) = \sigma((-\infty, x])$. Denote by λ Lebesgue measure on $[0,1]$. Since $\sigma((-\infty,x]) = \lambda((0, F_\sigma(x)])$,

(58) F_σ is continuous, strictly increasing and it establishes an isomorphism between $(\mathbb{R},\sigma)$ and $([0,1],\lambda)$.

Notice also that F_σ preserves the Lebesgue decomposition: if $\nu \in \mathcal{P}(\mathbb{R})$ and $\nu = \nu^a + \nu^s$ is the Lebesgue decomposition of ν with respect to σ, then $(F_\sigma)_*\nu = (F_\sigma)_*\nu^a + (F_\sigma)_*\nu^s$ is the Lebesgue decomposition of $(F_\sigma)_*\nu$ with respect to λ.

Recall (see [37]) that a family $\Omega \subset 2^{\mathbb{R}}$ of open sets is called an *essential family for σ* if:
(i) there exists $\beta > 0$ such that for each $E \in \Omega$ one can find an interval P containing E such that $\sigma(P) \leq \beta\sigma(E)$;
(ii) for each $x \in \mathbb{R}$, for each $\delta > 0$, there exists $E \in \Omega$ such that $\operatorname{diam} E < \delta$ and $x \in E$.

Note that the image of an essential family for σ via F_σ is an essential family of λ (and F_σ^{-1} enjoys the same property with the roles of σ and λ interchanged). This allows us to carry over the classical results on derivation well-known in case of Lebesgue measure (see [37]) to σ. Indeed, let $\nu \in \mathcal{P}(\mathbb{R})$. If we set

$$\overline{\Delta}_r(x) = \sup\left\{\frac{\nu(E)}{\sigma(E)} : x \in E, E \in \Omega, \operatorname{diam} E < r\right\},$$

$$\underline{\Delta}_r(x) = \inf\left\{\frac{\nu(E)}{\sigma(E)} : x \in E, E \in \Omega, \operatorname{diam} E < r\right\},$$

then for σ-a.e. $x \in \mathbb{R}$

$$\lim_{0<r\to 0} \overline{\Delta}_r(x) = \lim_{0<r\to 0} \underline{\Delta}_r(x) =: D_{\nu,\sigma}(x).$$

Indeed, $\frac{\nu(E)}{\sigma(E)} = \frac{(F_\sigma)_*\nu(F_\sigma E)}{\lambda(F_\sigma E)}$, so we simply apply a relevant result for Lebesgue measure. Similarly we obtain that for each Borel $B \subset \mathbb{R}$,

$$\nu(B) = \nu^s(B) + \int_B D_{\nu,\sigma}\, d\sigma,$$

in other words $D_{\nu,\sigma} = \frac{d\nu}{d\sigma}$ for σ-a.e. $x \in \mathbb{R}$. In particular we have proved the following.

Corollary 10. *If σ is continuous, then for each $\nu \in \mathcal{P}(\mathbb{R})$ we have*

$$\frac{d\nu}{d\sigma}(x) = \lim_{n\to\infty} \frac{\nu((x-\frac{1}{n}, x+\frac{1}{n}))}{\sigma((x-\frac{1}{n}, x+\frac{1}{n}))}$$

for σ-a.e. $x \in \mathbb{R}$. □

6.4 Maximal spectral multiplicity as a function of time

Assume that $\underline{U} = (U_t)_{t\in\mathbb{R}}$ is a flow with simple spectrum whose maximal spectral type is σ. We constantly assume that σ is continuous. Given $s \in \mathbb{R}$ set $\sigma_s = \sigma * \delta_s$. If $s_i \to s$ in $\mathbb{R}$, then $\sigma_{s_i} \to \sigma_s$ weakly and since all measures under consideration are continuous,

$$\sigma_{s_i}\left(\left(x-\frac{1}{k}, x+\frac{1}{k}\right)\right) \to \sigma_s\left(\left(x-\frac{1}{k}, x+\frac{1}{k}\right)\right) \text{ when } i \to \infty \tag{59}$$

for each $x \in \mathbb{R}$ and $k \geq 1$.

Fix $t \in \mathbb{R} \setminus \{0\}$. Let $m \in \mathbb{N}$. Then (see Remark 6) the maximal spectral multiplicity of $U_{1/t}$ is at least m if and only if there exist $n_1, \ldots, n_m \in \mathbb{Z}$ such that one can find $\nu \in \mathcal{P}(\mathbb{R})$ absolutely continuous with respect to each measure σ_{n_it}, $i = 1, \ldots, m$; in other words

$$\sigma\left(\left\{x \in \mathbb{R} : \frac{d\sigma_{n_1t}}{d\sigma}(x) \cdot \ldots \cdot \frac{d\sigma_{n_mt}}{d\sigma}(x) \neq 0\right\}\right) > 0. \tag{60}$$

By Corollary 10, the condition (60) is equivalent to saying that

$$\sigma\left(\left\{x \in \mathbb{R} : \lim_{k\to\infty} \Pi_{j=1}^m \frac{\sigma_{n_jt}\left((x-\frac{1}{k}, x+\frac{1}{k})\right)}{\sigma\left((x-\frac{1}{k}, x+\frac{1}{k})\right)} \neq 0\right\}\right) > 0. \tag{61}$$

If we denote $G_{k,t}(x) = \min\left(1, \Pi_{j=1}^m \frac{\sigma_{n_jt}\left((x-\frac{1}{k}, x+\frac{1}{k})\right)}{\sigma\left((x-\frac{1}{k}, x+\frac{1}{k})\right)}\right)$, then we obtain that the sequence $(G_{k,t})_{k\geq 1}$ is convergent σ-a.e. and we have (61) held if and only if

$$\sigma\left(\left\{x \in \mathbb{R} : \lim_{k\to\infty} G_{k,t}(x) > 0\right\}\right) > 0$$

or, which is the same, $\int \lim_{k\to\infty} G_{k,t}\, d\sigma > 0$. Since this integral is equal to $\lim_{k\to\infty} \int G_{k,t}\, d\sigma$, the inequality (60) is equivalent to saying that $\lim_{k\to\infty} J_k(t) > 0$, where $J_k(t) = \int G_{k,t}\, d\sigma$. It follows from (59) that the map $t \mapsto J_k(t)$ is continuous.

Proposition 6. *If $(U_t)_{t\in\mathbb{R}}$ is a unitary flow with simple continuous spectrum, then the function $\mathcal{M} = \mathcal{M}(t)$ which to $t \in \mathbb{R}$ associates the maximal spectral multiplicity of $U_{1/t}$ is of second Baire class.*

Proof. If we set $J(t) = J_{n_1,\dots,n_m}(t) = \lim_{k\to\infty} G_k(t)$, then the set $\{t \in \mathbb{R} : J_{n_1,\dots,n_m}(t) > 0\}$ is of type $\mathcal{F}_\sigma$ (it is equal to $\bigcup_{p,q}\bigcap_{r=q}^{\infty}[J_r \geq \frac{1}{p}]$). If we take the union (which will be countable) with respect to all choices $n_1 < \ldots < n_m$ we obtain that the set A_m of those $t \in \mathbb{R}$ for which the maximal spectral multiplicity of $U_{1/t}$ is at least m is always of type $\mathcal{F}_\sigma$. For $t \neq 0$ we have

$$\mathcal{M}(t) = \sum_{m\geq 1} \chi_{A_m}(t)$$

and $A_m = \cup_{l\geq 1} F_{l,m}$, where each $F_{l,m}$ is closed. By the Urysohn lemma, every indicator function $\chi_{F_{l,m}}$ is a function of the first Baire class. If one fixes m, for each l large enough

$$\chi_{F_{l,1}} + \ldots + \chi_{F_{l,l}} \geq \chi_{F_{l,1}} + \ldots + \chi_{F_{l,m}}.$$

The sums of the RHS of the above inequality converge pointwise to the multiplicity $\mathcal{M}(\cdot)$ when $l \to \infty$. On the other hand, the limit of the sums of the LHS of the above inequality exists. Therefore,

$$\lim_{l\to\infty} \left(\chi_{F_{l,1}}(\cdot) + \ldots + \chi_{F_{l,l}}(\cdot)\right) \geq \mathcal{M}(\cdot). \tag{62}$$

But

$$\chi_{F_{l,1}}(t) + \ldots + \chi_{F_{l,l}}(t) \leq \chi_{A_1}(t) + \ldots + \chi_{A_l}(t) \leq \mathcal{M}(t).$$

Hence we obtain the equality in (62) and $\mathcal{M}(\cdot)$ is of the second Baire class which completes the proof. □

Remark 7. It follows from the proof that A_2 is of type $\mathcal{F}_\sigma$, so the set of t for which $U_{1/t}$ has simple spectrum is of type $\mathcal{G}_\delta$. If σ is singular, then this set is also dense by (50) and by (57).

Remark 8. If σ is singular and ergodic, then $H(\sigma)$ is a dense subgroup of $\mathbb{R}$. Therefore the function $\mathcal{M} = \mathcal{M}(\cdot)$ has no continuity points. It follows that in this case $\mathcal{M}(\cdot)$ is not of the first Baire class.

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