

# CENTRALIZER AND LIFTABLE CENTRALIZER OF SPECIAL FLOWS OVER ROTATIONS

JEAN-PIERRE CONZE AND MARIUSZ LEMAŃCZYK

ABSTRACT. The liftable centralizer for special flows over irrational rotations is studied. It is shown that there are such flows under piecewise constant roof functions which are rigid and whose liftable centralizer is trivial.

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## Introduction

### Flows and special flows

Assume that  $(Z, \mathcal{D}, \rho)$  is a probability standard Borel space. In this paper, we will deal with measurable, measure-preserving<sup>1</sup>  $\mathbb{R}$ -actions, i.e., with flows  $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$  acting on  $(Z, \mathcal{D}, \rho)$  for which the map  $(z, t) \mapsto T_t z$  is measurable and  $\rho(T_t A) = \rho(A)$  for each  $A \in \mathcal{D}$  and  $t \in \mathbb{R}$ . It follows that the unitary representation in  $L^2(Z, \mathcal{D}, \rho)$  corresponding to  $\mathcal{T}$  is strongly (equivalently, weakly) continuous, i.e., the map  $t \mapsto T_t f$  is continuous for each  $f \in L^2(Z, \mathcal{D}, \rho)$ , where  $T_t f = f \circ T_t$ .

We constantly assume ergodicity of flows under consideration. According to Ambrose-Kakutani theorem [AmKa42] each such flow possesses a special representation, i.e., it can be represented as a special flow  $T^f = (T_t^f)_{t \in \mathbb{R}}$ , where  $T$  is an ergodic automorphism (often called a base) of a probability standard Borel space  $(X, \mathcal{B}, \mu)$ , and  $f : X \rightarrow \mathbb{R}^+$  is in  $L^1(X, \mathcal{B}, \mu)$  ( $f$  is often called a roof function).

Recall that  $T^f$  acts on  $(X^f, \mathcal{B}^f, \mu^f)$ , where  $X^f = \{(x, s) \in X \times \mathbb{R} : 0 \leq s < f(x)\}$  on which we consider the restriction of product  $\sigma$ -algebra and product measure (which is normalized:  $\mu^f(A) = (\mu \otimes \lambda_{\mathbb{R}})(A) / \int_X f d\mu$  for each  $A \in \mathcal{B}^f$ ). Then, for all  $t \in \mathbb{R}$  and  $(x, s) \in X^f$ , we have

$$T_t^f(x, s) = (T^n x, t + s - f^{(n)}(x)),$$

where  $n \in \mathbb{Z}$  is unique such that  $f^{(n)}(x) \leq t + s < f^{(n+1)}(x)$ . Here,

$$(1) \quad f^{(n)}(x) = f(x) + f(Tx) + \dots + f(T^{n-1}x) \text{ when } n > 0,$$

$f^{(0)}(x) = 0$  and the cocycle identity  $f^{(m+n)}(x) = f^{(n)}(x) + f^{(m)}(T^n x)$ , true for all integers  $m, n$  determines the values of  $f^{(m)}$  for negative integers.

Of course, in general, a flow has many special representations (with non-isomorphic bases). Originated by von Neumann [vNe32], it is a rather common and fruitful approach to study flows by choosing a suitable special representation. From that point of view a lot of attention has been devoted to study special flows over irrational rotations, or, more generally, over interval exchange transformations, as often they are natural special representations of interesting smooth, or smooth singular, flows on surfaces, see e.g. [Fa02], [FaKa16], [FrLe06], [FrLeLe07], [KhSi92], [Ko72], [Ko75], [Ko76], [Ko04], [Ku12], [Le00], [Sch09], [U107], [U111].

### Centralizer

A particular object of study in this paper is the *centralizer* of flows. We recall that given a flow  $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$  on  $(Z, \mathcal{D}, \rho)$ , its centralizer  $C(\mathcal{T})$  consists of all automorphisms  $W$  of  $(Z, \mathcal{D}, \rho)$  commuting with all  $T_t$ ,  $t \in \mathbb{R}$ . When  $C(\mathcal{T}) = \{T_t : t \in \mathbb{R}\}$ , then one says that  $\mathcal{T}$  has a *trivial centralizer*. In general,  $\{T_t : t \in \mathbb{R}\} \subset C(\mathcal{T})$  is a normal subgroup of  $C(\mathcal{T})$  and the quotient group  $C(\mathcal{T})/\{T_t : t \in \mathbb{R}\}$  is called the *essential centralizer* of  $\mathcal{T}$ . The essential centralizer can be quite big. Indeed, for example, it is uncountable when  $\mathcal{T}$  is rigid. Recall that rigidity means that for some sequence  $r_n \rightarrow \infty$ , we have  $T_{r_n} \rightarrow Id$  strongly in  $L^2(Z, \mathcal{D}, \rho)$ .<sup>2</sup> Prominent examples of rigid flows are given by the class of

<sup>1</sup>We tacitly assume that these  $\mathbb{R}$ -actions are *free*, i.e., for  $\rho$ -a.e.  $z \in Z$ , the map  $t \mapsto T_t z$  is 1-1.

<sup>2</sup>The fact that the essential centralizer is uncountable for rigid flows is folklore, see a proof of this fact, e.g. in [KaLe16], see also the proof of Proposition 9.1 below.

area-preserving smooth flows  $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$  without fixed points on  $\mathbb{T}^2$ , see [CoFoSi82], Chapter 16. Hence, such flows have uncountable essential centralizers.

### Centralizer for special flows. Liftable centralizer

First let us consider the continuous case:  $X$  is a compact metric space,  $f : X \rightarrow \mathbb{R}^+$  is continuous and  $W \in C(T^f)$  acting on  $X^f$  is also continuous, i.e., it belongs to  $C^{\text{top}}(T^f)$  (note that  $X^f$  has a natural metric making it a compact metric space, see Appendix).<sup>3</sup>

A result from [KeMaSe91] states that if  $X$  is a torus and  $T$  a minimal rotation on  $X$ , each element  $W$  of  $C^{\text{top}}(T^f)$  comes from an  $S \in C(T)$ , i.e.,  $Sx = x + \beta$  for some  $\beta \in X$  and a continuous  $g : X \rightarrow \mathbb{R}$  satisfying

$$(2) \quad f(Sx) - f(x) = g(Tx) - g(x) \text{ for all } x \in X.$$

To understand the meaning of the equation (2), consider it for the general setup of a special flow  $T^f$ :  $T$  is an ergodic automorphism of  $(X, \mathcal{B}, \mu)$ ,  $S$  is in  $C(T)$ ,  $g : X \rightarrow \mathbb{R}$  is measurable and

$$(3) \quad f(Sx) - f(x) = g(Tx) - g(x), \text{ for } \mu - \text{a.e. } x \in X.$$

Note that, up to natural identification,  $X^f$  is the space of orbits  $\{(T_f)^n(x, r) : n \in \mathbb{Z}\}$ ,  $(x, r) \in X \times \mathbb{R}$ , where  $T_f : X \times \mathbb{R} \rightarrow X \times \mathbb{R}$ ,

$$(4) \quad T_f(x, r) = (Tx, r + f(x)) \text{ for each } (x, r) \in X \times \mathbb{R}.$$

Now, the equation (3) means that  $T_f \circ S_g = S_g \circ T_f$ , where

$$(5) \quad S_g(x, r) = (Sx, r + g(x)).$$

So  $S_g$  also acts on  $X^f$  which is identified with  $X \times \mathbb{R} / \sim$  and it commutes with the quotient vertical action of  $\mathbb{R}$  which represents the special flow (in these “new coordinates”), see Section 1 for details. It follows that each measurable solution  $g$  of (3) yields an element of the centralizer of  $T^f$ . This part of the centralizer (which is clearly a subgroup), we will call the *liftable* centralizer of the special flow  $T^f$  and denote it by  $C^{\text{lift}}(T^f)$  (of course  $\{T_t^f : t \in \mathbb{R}\} \subset C^{\text{lift}}(T^f)$ ).

One can ask now whether the liftable centralizer is the whole centralizer of the flow under consideration. But the answer to such a question is clearly negative. For example if the base automorphism  $T$  has trivial centralizer, so must be the liftable centralizer of  $T^f$  for any roof function  $f$ .<sup>4</sup> Moreover, if  $C(T)$  is Abelian, then  $C^{\text{lift}}(T^f)$  is a nilpotent group of order at most 2 (see Section 1 for basic properties of  $C^{\text{lift}}(T^f)$ ). In fact, the essential liftable centralizer  $C^{\text{lift}}(T_f) / \{T_t^f : t \in \mathbb{R}\}$  is Abelian. Hence if  $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$  is an ergodic loosely Bernoulli flow, see [OrRuWe92], (on  $(Z, \mathcal{D}, \rho)$ ) whose centralizer is not a nilpotent group of order at most 2, then we cannot represent it over an ergodic  $T$  so that  $C(T)$  is Abelian (which is the case for irrational rotations) to have  $C^{\text{lift}}(T^f) = C(T^f)$ . This kind of general nonsense type arguments shows that, even for special flows over

<sup>3</sup>Clearly, under such assumptions, the special flow  $T^f$  is also a continuous flow. When  $T$  is uniquely ergodic with  $\mu$  the unique  $T$ -invariant measure, also  $T^f$  is uniquely ergodic; so each continuous  $W : X^f \rightarrow X^f$ ,  $T_t^f \circ W = W \circ T_t^f$  for all  $t \in \mathbb{R}$ , preserves the measure  $\mu^f$ .

<sup>4</sup>If  $\mathcal{T}$  is an arbitrary zero entropy and loosely Bernoulli flow then it will have a special representation  $T^f$  in which  $C(T) = \{T^n : n \in \mathbb{Z}\}$  [OrRuWe92].

irrational rotations, we cannot expect that the liftable centralizer is equal to the whole centralizer when  $f$  is arbitrary.<sup>5</sup>

### Liftable centralizer for special flows over irrational rotations

We now assume that  $X = \mathbb{T}$ , where  $\mathbb{T}$  stands for the additive circle represented as  $[0, 1[$  and  $Tx = R_\alpha x = x + \alpha \pmod{1}$ , where  $\alpha \in \mathbb{R}$  is irrational. Supported by the aforementioned topological result of [KeMaSe91], we may still ask whether  $C^{\text{lift}}(T^f) = C(T^f)$  when  $f$  is a “natural” function, meaning, more adapted to the topological or differentiable structure of the circle. As proved in [FrLe09], it is indeed the case whenever  $f$  is piecewise smooth with non-zero sum of jumps and  $Tx = x + \alpha$  with  $\alpha$  of bounded partial quotients.<sup>6</sup>

One can ask whether  $C^{\text{lift}}(T^f) = C(T^f)$  when  $f$  is smooth which, by [CoFoSi82], Chapter 16 is the case of smooth area-preserving flows without fixed points on  $\mathbb{T}^2$ . In this case  $T^f$  is rigid and hence the essential centralizer  $C(T^f)/\{T_t^f : t \in \mathbb{R}\}$  is uncountable. In fact, even  $C^{\text{lift}}(T^f)/\{T_t^f : t \in \mathbb{R}\}$  is uncountable, i.e., there is always an uncountable set of  $\beta \in \mathbb{T}$  for which we can indeed solve (3) (with  $Sx = x + \beta$ ) already when  $f$  is absolutely continuous – this result is also rather folklore, so we postpone the proof of this fact to Appendix. However, the answer to the question whether  $C^{\text{lift}}(T^f) = C(T^f)$  for  $f$  smooth is unknown. This phenomenon:  $T^f$  is rigid, the number of  $\beta$  for which (3) can be solved with  $Sx = x + \beta$  is uncountable, but the answer to the above question is unknown, still persists if we consider  $f = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$  and  $c_n = o(1/|n|)$ , see Proposition 9.5.

### Main result

In this paper we will study a relationship between  $C^{\text{lift}}(T^f)$  and  $C(T^f)$  in the class of step functions (for which the Fourier coefficients are clearly of order  $O(1/|n|)$ ). The main result is to show that they may give rigid flows whose liftable centralizer is trivial. More precisely, we will consider  $f = f_{a,b} : \mathbb{T} \rightarrow \mathbb{R}$  (with  $a, b > 0$ ) given by

$$(6) \quad f(x) = \begin{cases} a & \text{if } x \in [0, 1/2[, \\ b & \text{if } x \in [1/2, 1[. \end{cases}$$

Under the mild assumption  $a/b \notin \mathbb{Q} + \mathbb{Q}\alpha$  (which we assume to hold from now on), the special flow  $T^f$  is weakly mixing [FrLeLe07], [GuPa06].

Let  $\alpha \in [0, 1[$  be irrational with the partial quotients  $(a_n)_{n \geq 1}$ :  $\alpha = [0; a_1, a_2, \dots]$  and denominators  $q_n$ :  $q_0 = 1$ ,  $q_1 = a_1$  and  $q_{n+1} = a_{n+1}q_n + q_{n-1}$  for  $n \geq 1$ .

Let us now state the main theorem (where statement 1a) is taken from [KaLe16]) and [FrLeLe07]).

**Theorem 0.1.** *Let  $f = f_{a,b}$  with  $a - b = 2$ . For an irrational  $\alpha$ , let us consider the special flow  $T^f$  obtained from  $T = R_\alpha$  and  $f = f_{a,b}$ .*

1) *Suppose  $\alpha$  has bounded partial quotients.*

1a) *Then Ratner’s property is satisfied for  $T^f$  and therefore  $C(T^f)$  is at most countable*

<sup>5</sup>The situation does not change if additionally  $X$  is a compact metric space and we require  $f$  to be continuous. Indeed, each positive  $L^1$ -function is cohomologous to a positive continuous function [Ko72]. We emphasize that even if  $f$  is continuous we look for measurable solutions of (3).

<sup>6</sup>In fact, the essential centralizer is finite in this case [FrLe09].

modulo  $\{T_t^f : t \in \mathbb{R}\}$  (i.e., the essential centralizer is at most countable). In particular,  $T^f$  is not rigid.

1b)  $C^{\text{lift}}(T^f)$  is trivial modulo  $\{T_t^f : t \in \mathbb{R}\}$ .

2a) Suppose  $\alpha$  has unbounded partial quotients. Then the special flow  $T^f$  is rigid.

2b) If  $(q_{n_k})$  is even along a subsequence  $(n_k)$  such that  $a_{n_k+1} \uparrow \infty$ , then this subsequence is a rigidity sequence for  $T^f$  and  $C^{\text{lift}}(T^f)$  is uncountable modulo  $\{T_t^f : t \in \mathbb{R}\}$ .

2c) If there is  $n_0$  such that the denominators  $q_n$  of  $\alpha$  are odd for  $n \geq n_0$ , then the functional equation

$$(7) \quad f(x + \beta) - f(x) = g(x + \alpha) - g(x), \text{ for } \mu - a.e \ x \in \mathbb{T},$$

has no measurable solution  $g : \mathbb{T} \rightarrow \mathbb{R}$  for  $\beta \notin \mathbb{Z}\alpha + \mathbb{Z}$ . Equivalently, the liftable centralizer of  $T^f$  is trivial:  $C^{\text{lift}}(T^f) = \{T_t^f : t \in \mathbb{R}\}$ .

2d) More generally, if there is  $n_0$  such that  $(a_{n_k+1})$  is bounded along the sequence of all  $n_k$  such that  $q_{n_k}$  is even, then the conclusion is the same as in 2c).

It follows that the flows from Theorem 0.1 display a drastic change of ergodic properties of special flows under the same roof function when changing an irrational rotation as its base.

On one hand side, they seem to be interesting from the point of view of recent achievements in studying Ratner's property [Ra83], [Th95] in the class of special flows over irrational rotations and interval exchange transformations: [FaKa16], [FrLe06], [FrLeLe07], [Ka14], [Ka15], [KaKu15], [KaKuUl16]. Indeed (cf. Theorem 0.1, case 1)), when  $\alpha$  has bounded partial quotients then, as shown in [FrLeLe07],  $T^f$  enjoys (finite) Ratner's property. As proved recently in [KaLe16], flows with (finite) Ratner's property have at most countable (discrete) essential centralizer, in particular such flows cannot be rigid.

On the other hand (cf. Theorem 0.1, case 2a)), when  $\alpha$  has unbounded partial quotients,  $T^f$  is rigid, hence, it cannot possess the (finite) Ratner's property. Moreover, there are two different phenomena which imply rigidity of  $T^f$  in cases 2b) and 2c) in Theorem 0.1. To show that these phenomena are mutually exclusive, we will discuss them in Lemma 4.3, see also the second proof of Corollary 9.4.

The paper is organized as follows: in Section 1 we present some elementary properties of the liftable centralizer in a general setup. Section 2 is devoted to some reminders on cocycles and irrational rotations. In Section 3 we prove part 1 of Theorem 0.1. In Section 4 we prove part 2a and part 2b) of Theorem 0.1. In Section 5 we study the regularity of cocycles related to (7) and in Section 6 we prove the remaining part of Theorem 0.1. In Section 7 we show the non-regularity of the relevant cocycle for an exceptional set of values of  $\beta$ . Finally, in Appendix we study the centralizer for uniformly rigid flows and for smooth special flows over irrational rotations. We also show that the essential liftable centralizer is uncountable whenever the Fourier transform of  $f$  is of order  $o(1/|n|)$  and provide examples of Hölder continuous roof functions which yield special flows with trivial liftable centralizer.

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### 1. Lifiable centralizer of a special flow

Let  $T$  be an ergodic automorphism of a probability standard Borel space  $(X, \mathcal{B}, \mu)$ . It is not hard to see that, up to natural identification,  $(X^f, \mu^f)$  is the space of orbits  $\{(T_f)^n(x, r) : n \in \mathbb{Z}\}$ ,  $(x, r) \in X \times \mathbb{R}$  (considered with the quotient of the product measure  $\mu \otimes \lambda_{\mathbb{R}}$ ), where  $T_f : X \times \mathbb{R} \rightarrow X \times \mathbb{R}$  is given by (4). In these ‘‘coordinates’’ the special flow is the vertical action  $\sigma_t(x, r) = (x, r + t)$  on the quotient space  $X \times \mathbb{R} / \sim$ , where  $\sim$  is the equivalence relation given by the partition into orbits of  $T_f$ .

Assume that  $S \in C(T)$  and the equation (3) is satisfied for some measurable  $g : X \rightarrow \mathbb{R}$ . For the map  $S_g$  defined by (5), it follows, that  $T_f \circ S_g = S_g \circ T_f$ , so  $S_g$  also acts on  $X^f$  (identified with  $X \times \mathbb{R} / \sim$ ). Moreover,  $\sigma_t \circ S_g = S_g \circ \sigma_t$  for all  $t \in \mathbb{R}$ . Finally,  $S_g$  determines an element  $\widetilde{S}_g \in C(T^f)$ . Let

$$(8) \quad C^{\text{lift}}(T^f) := \{\widetilde{S}_g : (S, g) \text{ satisfies (3)}\}$$

be the *liftable centralizer* of  $T^f$ . Note that  $C^{\text{lift}}(T^f)$  is a group as

$$(9) \quad \widetilde{S}_g \circ \widetilde{R}_h = \widetilde{S_g \circ R_h} = (S \circ R)_{g \circ R + h}, \quad \widetilde{S}_g^{-1} = (\widetilde{S_g})^{-1} = \widetilde{S_g^{-1}}.$$

Furthermore, for each  $t \in \mathbb{R}$ ,  $\widetilde{Id}_t = T_t^f$  as  $Id_t(x, r) = \sigma_t(x, r)$  (where we identify  $t$  with the constant function  $x \rightarrow t$ ). It follows that

$$(10) \quad \{T_t^f : t \in \mathbb{R}\} \subset C^{\text{lift}}(T^f).$$

By the same token, if  $g$  is a solution of (3) then so is  $\underline{g} + t$  (and these exhaust all measurable solutions because of ergodicity of  $T$ ). Hence  $\widetilde{S}_{g+t} = \widetilde{S_g} \circ T_t^f$ ,  $t \in \mathbb{R}$ . On the other hand, using (3), we have, for each  $k \in \mathbb{Z}$ ,

$$(11) \quad \widetilde{S}_g = S_g \circ \widetilde{(T_f)^k}.$$

**Proposition 1.1.** *a) The equality  $\widetilde{S}_g = \widetilde{R}_h$  holds if and only if  $S_g = R_h \circ (T_f)^k$  for some  $k \in \mathbb{Z}$ .*

*b) If  $\widetilde{S}_g \in C^{\text{lift}}(T^f)$  satisfies  $\widetilde{S}_g^s = Id$ , then there exists  $k \in \mathbb{Z}$  such that  $S^s = T^k$ . In other words, a finite order liftable element of  $C(T^f)$  must be a lift of a root of a power of  $T$ . Moreover, if  $C(T)$  is trivial, so is  $C^{\text{lift}}(T^f)$ .*

*Proof.* a) For  $\mu \otimes \lambda_{\mathbb{R}}$ -a.e.  $(x, r) \in X \times \mathbb{R}$ , we have

$$S_g(\{(T_f)^n(x, r) : n \in \mathbb{Z}\}) = R_h(\{(T_f)^n(x, r) : n \in \mathbb{Z}\}),$$

hence, with  $k = k(x, r)$ ,

$$(12) \quad S_g(x, r) = R_h \circ (T_f)^k(x, r).$$

Since the number of  $k$  is countable and, for a given  $k$ , the set of  $(x, r)$  for which (12) holds is measurable and  $T_f$ -invariant, by the ergodicity of  $T$ , we obtain that  $Sx = R \circ T^k x$  for  $\mu$ -a.e.  $x \in X$ , for some fixed  $k \in \mathbb{Z}$ .

b) The relation  $\widetilde{S}_g^s = Id$  is equivalent to: for a.e.  $(x, r)$ , there is  $k = k(x, r)$  such that:

$$(S^s x, r + \sum_{i=0}^{s-1} g(S^i x)) = (T^k x, r + \sum_{i=0}^{k-1} f(T^i x)).$$

As above, we obtain that this relation holds for some fixed  $k$  and so  $S^r = T^k$ .  $\square$

**Proposition 1.2.** *Assume that  $C(T)$  is an Abelian group. Then  $C^{\text{lift}}(T^f)$  is a nilpotent group of order at most 2.*

*Proof.* Let  $S, R$  be in  $C(T)$  such that  $S \circ R = R \circ S$  and (3) is satisfied for  $(S, g)$  and  $(R, h)$ , respectively. Then, using (9), we obtain for the commutator:

$$(S_g \circ R_h \circ S_g^{-1} \circ R_h^{-1})(x, r) = (x, r + g(S^{-1}x) - g(S^{-1}R^{-1}x) - h(R^{-1}x) + h(R^{-1}S^{-1}x)).$$

Using (3) for  $(S, g)$  and  $(R, h)$ , a simple calculation shows that  $g(S^{-1}x) - g(S^{-1}R^{-1}x) - h(R^{-1}x) + h(R^{-1}S^{-1}x)$  is  $T$ -invariant, hence a.e. equal to a constant  $t$ . It follows:

$$\widetilde{S}_g \circ \widetilde{R}_h \circ \widetilde{S}_g^{-1} \circ \widetilde{R}_h^{-1} = T_t^f$$

for some  $t \in \mathbb{R}$ . Since  $\{T_t^f : t \in \mathbb{R}\}$  is a subgroup of the center of  $C(T^f)$ , we have proved the following result.  $\square$

**Remark 1.3.** We would like to argue that, in general,  $C^{\text{lift}}(T^f)$  is neither a closed subgroup nor dense in  $C(T^f)$ . For this aim, consider any ergodic, rigid and loosely Bernoulli flow  $(R_t)$  on  $(Z, \mathcal{D}, \rho)$  (with the  $\mathbb{R}$ -action given by  $(R_t)$  free) for which

$$(13) \quad \overline{\{R_t : t \in \mathbb{R}\}} \neq C((R_t)_{t \in \mathbb{R}}).$$

Note that rigidity is equivalent to:

$$(14) \quad \{R_t : t \in \mathbb{R}\} \neq \overline{\{R_t : t \in \mathbb{R}\}}$$

(cf. the proof of Proposition 9.1). Clearly, properties (13) and (14) are invariants of isomorphism. Now, take a special representation  $T^f$  of the flow  $(R_t)$  in which  $C(T) = \{T^n : n \in \mathbb{Z}\}$ . Then  $C^{\text{lift}}(T^f) = \{T_t^f : t \in \mathbb{R}\}$ . But by (14) and (13),  $C^{\text{lift}}(T^f)$  is neither closed nor dense in  $C(T^f)$ .

## 2. Preliminaries

Let  $\beta$  be a real number in  $]0, 1[$ . With  $F := 1_{[0, \frac{1}{2}[} - 1_{[\frac{1}{2}, 0[}$ , we consider the cocycle generated over the rotation  $R_\alpha : x \rightarrow x + \alpha \pmod{1}$  by

$$(15) \quad \Phi_\beta := \frac{1}{2}F(\cdot - \beta) - \frac{1}{2}F.$$

Equation (7) (where  $f = 1_{[0, \frac{1}{2}[} - 1_{[\frac{1}{2}, 0[}$ ) reads  $\Phi_\beta = R_\alpha g - g$ . One of our goals of this and the following sections is to show that under the assumptions of Theorem 0.1, case 2c, on  $\alpha$ ,  $\Phi_\beta$  is not a coboundary, i.e., equation  $\Phi_\beta = R_\alpha g - g$  has no measurable solution  $g$  if  $\beta \notin \mathbb{Z}\alpha + \mathbb{Z}$ . As a matter of fact, we examine for  $\Phi_\beta$  the following three properties of increasing strength:

- (I)  $\Phi_\beta$  is not a coboundary,
- (II)  $\mathcal{E}(\Phi_\beta) \neq \{0\}$ ,
- (III)  $R_{\alpha, \Phi_\beta}$  is ergodic (as a skew product  $(x, r) \rightarrow (R_\alpha x, r + \Phi_\beta(x))$  on  $\mathbb{T} \times \mathbb{Z}$ ).

Clearly if  $\beta \in \mathbb{Z}\alpha + \mathbb{Z}$ , then  $\Phi_\beta$  is a coboundary. We exclude such values of  $\beta$  which will be called *trivial*.

Observe that, if the group  $\mathcal{E}(\Phi_\beta)$  of finite essential values of  $\Phi_\beta$  is not reduced to  $\{0\}$ , then  $\Phi_\beta$  is not a coboundary. We are going to show that, outside an exceptional set of values of  $\beta$ ,  $\mathcal{E}(\Phi_\beta) \neq \{0\}$ , which implies that  $\Phi_\beta$  is regular and is not a coboundary. It

remains an exceptional set of non trivial values for which  $\Phi_\beta$  is not a coboundary, but can be non regular, hence non ergodic (Theorem 7.3).

To summarize, we will show:

- for every non trivial value of  $\beta$ ,  $\Phi_\beta$  is not a coboundary,
- for most of the values of  $\beta$ , it is regular,
- for an exceptional set of non trivial values of  $\beta$ , it is non regular, hence non ergodic.

Let us first recall some facts about essential values and useful tools in the study of cocycles (cf. [Sc77], see also [Aa97], [CoRa09]).

### Reminders on cocycles

Let  $(\Phi^{(n)})$  be the cocycle (cf. (1)) over an ergodic dynamical system  $(X, \mu, T)$  generated by a measurable  $\Phi : X \rightarrow G$ ,<sup>7</sup> where  $G = \mathbb{Z}^d$  or  $\mathbb{R}^d$ . Denote by  $T_\Phi$  the corresponding skew product map  $T_\Phi(x, g) = (Tx, g + \Phi(x))$ ,  $(x, g) \in X \times G$ . An element  $a \in G \cup \{\infty\}$  is called an *essential value* of the cocycle  $(\Phi^{(n)})$  if, for every neighborhood  $V(a)$  of  $a$ , for every measurable subset  $B$  of positive measure,

$$(16) \quad \mu(B \cap T^{-n}B \cap \{x \in X : \Phi^{(n)}(x) \in V(a)\}) > 0, \text{ for some } n \in \mathbb{Z}.$$

We denote by  $\overline{\mathcal{E}}(\Phi)$  the *set of essential values* of the cocycle  $(\Phi^{(n)})$  and by  $\mathcal{E}(\Phi) = \overline{\mathcal{E}}(\Phi) \cap G$  the set of *finite essential values*.

A cocycle  $\Phi$  is called a *coboundary*, if there exists a measurable  $g : X \rightarrow G$  such that  $\Phi(x) = g(Tx) - g(x)$  for  $\mu$ -a.e.  $x \in X$ . Two cocycles with values in  $G$  are said to be *cohomologous* if their difference is a coboundary. Two cohomologous cocycles have the same set of essential values. If  $\Phi$  is not a coboundary, then  $\infty$  is an essential value of the cocycle generated by  $\Phi$ . Hence,  $\Phi$  is a coboundary if and only if  $\overline{\mathcal{E}}(\Phi) = \{0\}$ .

The set  $\mathcal{E}(\Phi)$  is a closed subgroup of  $G$  which coincides with the group of periods  $p$  of the measurable  $T_\Phi$ -invariant functions on  $X \times G$ , i.e., the elements  $p \in G$  such that, for every  $T_\Phi$ -invariant measurable  $H$ , we have  $H(x, y+p) = H(x, y)$ ,  $\mu \otimes m$ -a.e. ( $m = m_G$  stands for a Haar measure on  $G$ ). In particular,  $\mathcal{E}(\Phi) = G$  if and only if  $(X \times G, \mu \otimes m, T_\Phi)$  is ergodic.

The cocycle defined by  $\Phi$  is *regular*, if  $\Phi$  is cohomologous to a cocycle with values in a closed subgroup  $G_0$  of  $G$  and ergodic for the action on  $X \times G_0$ . More explicitly,  $\Phi$  is regular if there exists a measurable function  $\eta : X \rightarrow G$  such that  $\Phi := \Psi + \eta - \eta \circ T$   $\mu$ -a.e.,  $\Psi$  has its values in  $G_0$  and  $T_\Psi : (x, h) \rightarrow (Tx, h + \Psi(x))$  is ergodic for the product measure  $\mu \otimes m_{G_0}$  on  $X \times G_0$ . The group  $G_0$  in this definition is necessarily  $\mathcal{E}(\Phi)$ .

In the regular case there is a “nice” ergodic decomposition of the measure  $\mu \otimes m$  for the skew product map: any  $T_\Phi$ -invariant function can be written as  $V(y - \eta(x))$  for a function  $V$  which is invariant by the translations by elements of  $G_0$ . If the cocycle is non regular, then the ergodic decomposition of  $\mu \otimes m$  is based on a family of measures  $\mu_x$  ( $x \in X$ ) defined on  $X$ . Moreover, the measures  $\mu_x$  are infinite, singular with respect to the measure  $\mu$  and there are uncountably many of them pairwise mutually singular.

A way to prove the existence of essential values is to use the following lemma:

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<sup>7</sup>In what follows, often, we call  $\Phi$  itself a cocycle.



**Lemma 2.1.** ([LePaVo96]) *If  $(r_n)$  is a rigidity sequence for  $T$  and  $(\Phi^{(r_n)})_*\mu \rightarrow \nu$  weakly on  $G \cup \{\infty\}$ , then  $\text{supp}(\nu) \subset \overline{\mathcal{E}(\Phi)}$ .*

A form of this criterium adapted to cocycles with values in  $\mathbb{Z}$  is the following ([Co09]):

If  $a \in G \cup \{\infty\}$  is such that there exist  $\delta > 0$  and a rigidity sequence  $(r_n)_{n \geq 1}$  for  $T$  such that  $\mu(\{x \in X : \Phi^{(r_n)}(x) \in V(a)\}) \geq \delta$ , for every neighborhood  $V(a)$  of  $a$ , for  $n$  large enough, then  $a \in \overline{\mathcal{E}(\Phi)}$ . If such an element  $a$  exists and  $a \notin \{0, \infty\}$ , then  $\Phi$  is not a coboundary.

In particular, if there exist  $\delta > 0$  and a rigidity sequence  $(r_n)_{n \geq 1}$  for  $T$  such that  $\mu(\{x \in X : |\Phi^{(r_n)}(x)| \geq M\}) \geq \delta$ , for every  $M \geq 1$ , for  $n$  big enough, then  $\infty$  is an essential value and  $\Phi$  is not a coboundary.

We will also use implicitly the following remarks: Let  $f$  be a measurable  $\mathbb{Z}$ -valued function. Then if  $f$  is a  $T$ -coboundary in  $\mathbb{R}$ , it is a coboundary in  $\mathbb{Z}$ . Moreover, if  $T_f$  is ergodic for its action on  $X \times \mathbb{Z}$ , then the  $T_f$ -invariant functions on  $X \times \mathbb{R}$  are the 1-periodic functions depending only on the second coordinate.

### Reminders on continued fractions

For  $u \in \mathbb{R}$ ,  $\|u\|$  denotes its distance to the integers:  $\|u\| := \inf_{n \in \mathbb{Z}} |u - n| = \min(\{u\}, 1 - \{u\}) \in [0, \frac{1}{2}]$ . We will need the following inequalities:

$$(17) \quad 2|x| \leq |\sin \pi x| \leq \pi|x|, \text{ for } |x| \leq \frac{1}{2},$$

$$(18) \quad 2\|x\| \leq |\sin \pi x| \leq \pi\|x\|, \forall x \in \mathbb{R}.$$

Let  $\alpha \in [0, 1[$  be an irrational number. Then, for each  $n \geq 1$ , we write  $\alpha = \frac{p_n}{q_n} + \frac{\theta_n}{q_n}$ , where  $p_n$  and  $q_n$  are the numerators and denominators of  $\alpha$ . Recall that

$$(19) \quad \frac{1}{q_{n+1} + q_n} \leq \|q_n \alpha\| = |\theta_n| \leq \frac{1}{q_{n+1}} = \frac{1}{a_{n+1}q_n + q_{n-1}},$$

$$(20) \quad \frac{1}{a_{n+1} + 2} \leq q_n \|q_n \alpha\| = q_n |\theta_n| < \frac{q_n}{q_{n+1}} < \frac{1}{a_{n+1}},$$

$$(21) \quad \|k\alpha\| \geq \|q_{n-1}\alpha\| \geq \frac{1}{q_n + q_{n-1}} \geq \frac{1}{2q_n}, \text{ for } 1 \leq k < q_n.$$

We have also

$$(22) \quad (-1)^{n-1} p_n q_{n-1} = 1 + (-1)^{n-1} p_{n-1} q_n$$

and

$$(23) \quad \|q_n \alpha\| = (-1)^n (q_n \alpha - p_n), \theta_n = (-1)^n \|q_n \alpha\|, \alpha = \frac{p_n}{q_n} + (-1)^n \frac{\|q_n \alpha\|}{q_n}.$$

**Remark 2.2.** If the denominators  $q_n$  of  $\alpha$  are odd for  $n \geq n_0$ , then the partial quotients  $a_n$  are even for  $n \geq n_0 + 2$ . Conversely, if the partial quotients  $a_n$  of  $\alpha$  are even for  $n \geq n_0$ , then for  $n \geq n_0 - 1$  either all denominators are odd or are alternatively odd and even.

In the proof of Theorem 0.1 below, we will use the following lemma ([KrLi91], [Co09]):

**Lemma 2.3.** (*Kraaikamp and Liardet*) *If there exists  $n_0$  such that  $\|q_n\beta\| \leq \frac{1}{4}q_n\|q_n\alpha\|$  for  $n \geq n_0$ , then  $\beta \in \mathbb{Z}\alpha + \mathbb{Z}$ .*

The ratio

$$(24) \quad c_n(\beta) := \frac{\|q_n\beta\|}{q_n\|q_n\alpha\|}$$

will be important in the proof of Theorem 6.2. The previous lemma implies that, if  $c_n(\beta) \leq \frac{1}{4}, \forall n \geq n_0$  for some  $n_0 \geq 1$ , then  $\beta \in \mathbb{Z}\alpha + \mathbb{Z}$ .

We will use also the following lemma:

**Lemma 2.4.** *1) If  $q_n$  is odd and  $q_n\|q_n\alpha\| < 1/2$ , then  $F^{(q_n)} = \pm 1$ .*

*2) If  $q_n$  is even and  $q_n\|q_n\alpha\| < 1/2$ , then  $F^{(q_n)}(x) = \pm 2$  on a set  $I_n$  of measure  $\mu(I_n) \leq \frac{1}{a_{n+1}}$  and  $= 0$  elsewhere.*

*Proof.* The discontinuities of  $F^{(q_n)}(x) = \sum_{j=0}^{q_n-1} F(x + j\alpha)$  are  $t - j\alpha \pmod{1}$ , with  $j = 0, \dots, q_n - 1$ ,  $t = 0, \frac{1}{2}$  and the respective jumps are  $+2, -2$ .

1) Let us consider the case 1) where  $q_n$  is odd. The discontinuities are of the form  $\frac{r}{q_n} - j_1(r)\frac{\theta_n}{q_n}, \frac{r}{q_n} - j_2(r)\frac{\theta_n}{q_n} + \frac{1}{2q_n}$ , with jumps  $\pm 2$ , where  $0 \leq j_1(r), j_2(r) < q_n$ , for  $r = 0, \dots, q_n - 1$ . They belong respectively to  $[\frac{r}{q_n} - \frac{\delta_n}{q_n}, \frac{r}{q_n} + \frac{\delta_n}{q_n}]$  and  $[\frac{r}{q_n} + \frac{1}{2q_n} - \frac{\delta_n}{q_n}, \frac{r}{q_n} + \frac{1}{2q_n} + \frac{\delta_n}{q_n}]$ , where  $\delta_n := q_n\|q_n\alpha\| < 1/2$ .

As  $\frac{r}{q_n} + \frac{\delta_n}{q_n} < \frac{r}{q_n} + \frac{1}{2q_n} - \frac{\delta_n}{q_n}$ , the successive jumps of  $F^{(q_n)}$  are alternatively  $+2, -2$ , so that the values of  $F^{(q_n)}$  are  $u$  or  $u - 2$ , for a constant  $u$ . But  $F$  is antisymmetric:  $F(x + 1/2) = -F(x)$ , so also  $F^{(q_n)}$  is antisymmetric and non constant. Therefore the set of values is  $\{u, u - 2\} = \{-u, -u + 2\}$ , which implies  $u = 1$ .

2) Suppose  $q_n$  even. For  $r = 0, \dots, q_n - 1$ , there are now two discontinuities (with jump respectively  $+2, -2$  in any order) in  $[\frac{r}{q_n} - \frac{\delta_n}{q_n}, \frac{r}{q_n} + \frac{\delta_n}{q_n}]$ . It shows that the set of values of  $F^{(q_n)}$  belongs to  $\{u, u + 2, u - 2\}$ , for some integer constant  $u$ .

There is a constant  $u \in \mathbb{Z}$  such that, for each  $r = 0, \dots, q_n - 1$ , in restriction to the interval  $[\frac{r}{q_n} - \frac{1}{2q_n}, \frac{r}{q_n} + \frac{1}{2q_n}]$ , the function  $F^{(q_n)}$  takes the value  $u + 2$  or  $u - 2$  on a subinterval  $I_{n,r}$  of length  $\leq \frac{|j_1(r) - j_2(r)|}{q_n} |\theta_n| \leq |\theta_n|$  and the value  $u$  elsewhere.

Therefore, we have  $0 = \int_{\mathbb{T}} F^{(q_n)} d\mu = u + 2 \sum_r \pm \mu(I_{n,r})$ , hence  $|u| \leq 2 \sum_r \mu(I_{n,r}) \leq 2q_n |\theta_n| = 2\delta_n < 1$ . This implies  $u = 0$ . Let  $I_n = \cup_r I_{n,r}$ . As  $\sum_r \mu(I_{n,r}) \leq q_n |\theta_n| \leq \frac{1}{a_{n+1}}$ , the point 2) of the lemma is proved.  $\square$

### 3. Proof of the first part of Theorem 0.1

**Proof of 1a)** The fact that if  $\alpha$  has bounded partial quotients, then  $T^f$  has (finite) Ratner's property has been proved in [FrLeLe07]. Moreover, in [KaLe16] it has been proved that each flow satisfying (finite) Ratner's property has at most countable essential centralizer. On the other hand, we have already noticed that rigid flows have uncountable essential centralizer, whence our  $T^f$  cannot be rigid.

**Proof of 1b)** We want to show that if  $\alpha$  has bounded partial quotients, then, for a non-trivial  $\beta$ , equation (7) has no measurable solution. Our claim follows from the following result:

**Proposition 3.1.** *If  $\alpha$  has bounded partial quotients, then for  $\beta \notin \mathbb{Z}\alpha + \mathbb{Z}$  the cocycle  $\Phi_\beta$  is ergodic (as a  $\mathbb{Z}$ -valued cocycle).*

*Proof.* We use Lemma 2.3 and Proposition 3.8 in [CoPi14]. The lemma shows that the cocycle  $\Phi_\beta$  has “well separated discontinuities”, which implies ergodicity by the proposition.  $\square$

#### 4. Rigidity of a special flow, proof of parts 2a), 2b) of Theorem 0.1

Assume that  $(T_t)_{t \in \mathbb{R}}$  is a (measurable) measure-preserving flow on a probability standard Borel space  $(Z, \mathcal{D}, \rho)$ . We will consider  $T_t$  as a Markov operator<sup>8</sup> on  $L^2(Z, \mathcal{D}, \rho)$ :  $T_t f := f \circ T_t$ . The following result is essentially due to V. Ryzhikov (private communication).

**Lemma 4.1.** *Assume that  $(r_n)$  is a sequence of real numbers tending to  $\infty$ . Assume moreover that*

$$T_{mr_n} \xrightarrow[n \rightarrow \infty]{} \frac{1}{2}(T_{-m} + T_m) \text{ for all } m \in \mathbb{Z},$$

*weakly in the set of Markov operators. Then, the flow  $(T_t)_{t \in \mathbb{R}}$  is rigid.*

*Proof.* By assumption, for all  $m \geq 1$ , we have

$$(25) \quad T_{m([r_n] + \{r_n\})} \xrightarrow[n \rightarrow \infty]{} \frac{1}{2}(T_{-m} + T_m).$$

By passing to a subsequence, if necessary, we have  $T_{\{r_n\}} \xrightarrow[n \rightarrow \infty]{} T_r$  for some  $r \in [0, 1[$ , whence

$$(26) \quad T_{m\{r_n\}} \xrightarrow[n \rightarrow \infty]{} T_{mr} \text{ strongly, for all } m \in \mathbb{Z}.$$

Since the convergence in (26) is strong, by (25), we have

$$T_{m[r_n]} \xrightarrow[n \rightarrow \infty]{} \frac{1}{2}(T_{-m} + T_m) \circ T_{-mr} \text{ for all } m \in \mathbb{Z}.$$

Using basic properties of the weak operator topology,<sup>9</sup> we can now choose a sparse subsequence  $(r_{n_k})$  so that, for all  $m \in \mathbb{Z}$ ,

$$\begin{aligned} T_{m([r_{n_{k+1}}] - [r_{n_k}])} &\xrightarrow[k \rightarrow \infty]{} \frac{1}{2}(T_{-m} + T_m) \circ T_{-mr} \circ \left( \frac{1}{2}(T_{-m} + T_m) \circ T_{-mr} \right)^* \\ &= \frac{1}{2}(T_{-m} + T_m) \circ \frac{1}{2}(T_{-m} + T_m) = \frac{1}{4}T_{-2m} + \frac{1}{2}Id + \frac{1}{4}T_{2m}. \end{aligned}$$

<sup>8</sup>Recall that a linear contraction  $\Phi$  on  $L^2(Z, \mathcal{D}, \rho)$  is called Markov, if  $\Phi 1 = \Phi^* 1 = 1$  and  $\Phi h \geq 0$  whenever  $h \geq 0$ . The set of Markov operators is a convex set which is closed (hence compact) in the weak operator topology.

<sup>9</sup>If  $d$  is a metric compatible with the weak topology, then  $d(C_n \circ A, C_n \circ B) \xrightarrow[n \rightarrow \infty]{} d(C \circ A, C \circ B)$  for any linear contractions  $A, B$  and  $C_n \xrightarrow[n \rightarrow \infty]{} C$ . The subsequence  $(r_{n_k})$  is selected inductively, at the induction step,  $r_{n_{k+1}}$  is chosen so that  $T_{j[r_{n_{k+1}}]}$  is so close to  $D_j := \frac{1}{2}(T_{-j} + T_j) \circ T_{-jr}$  for  $j = 1, \dots, k$  to have  $T_{j([r_{n_{k+1}}] - [r_{n_k}])}$  is almost as close to  $D_j \circ D_j^*$  as  $T_{j[r_{n_k}]}$  is close to  $D_j$ , also  $T_{(k+1)[r_{n_{k+1}}]}$  is very close to  $D_{k+1}$ .

By passing to a further subsequence, if necessary, we can assume that  $[r_{n_k}]$  were chosen so that either they are all even or they are all odd. This yields

$$(27) \quad T_{2mj_k} \xrightarrow[k \rightarrow \infty]{} \frac{1}{4}T_{-2m} + \frac{1}{2}Id + \frac{1}{4}T_{2m}, \text{ for all } m \in \mathbb{Z},$$

where  $j_k = ([r_{n_{k+1}}] - [r_{n_k}])/2$ .

It follows that for each  $\ell \in \mathbb{Z}$  the operator  $\frac{1}{4}T_{-2\ell} + \frac{1}{2}Id + \frac{1}{4}T_{2\ell}$  is an accumulation point of the set  $\{T_n : n \in \mathbb{Z}\}$ . Fix  $m \geq 1$  and  $k \geq 1$ . By taking  $\ell = mj_k$ , it follows that the operator  $\frac{1}{4}T_{-2mj_k} + \frac{1}{2}Id + \frac{1}{4}T_{2mj_k}$  is an accumulation point of  $\{T_n : n \in \mathbb{Z}\}$ . Letting  $k \rightarrow \infty$  and using (27), we obtain that the operator

$$\frac{1}{4} \left( \frac{1}{4}T_{-2m} + \frac{1}{2}Id + \frac{1}{4}T_{2m} \right) + \frac{1}{2}Id + \frac{1}{4} \left( \frac{1}{4}T_{2m} + \frac{1}{2}Id + \frac{1}{4}T_{-2m} \right) = \frac{1}{8}T_{2m} + \frac{3}{4}Id + \frac{1}{8}T_{-2m}$$

is an accumulation point of the set  $\{T_n : n \in \mathbb{Z}\}$ . By iterating this procedure, we obtain that  $Id$  is an accumulation point of  $\{T_n : n \in \mathbb{Z}\}$  and the result follows.  $\square$

Consider now  $R_\alpha x = x + \alpha \bmod 1$  an irrational rotation on  $\mathbb{T}$ . Recall that  $(q_n)$  denote the sequence of denominators of  $\alpha$ . Recall also that, for any function  $\varphi$  on  $\mathbb{T}$  and a positive integer  $\ell$ , we denote by  $\varphi^{(\ell)}$  the ergodic sum  $\varphi^{(\ell)}(x) = \sum_{k=0}^{\ell-1} \varphi(x + k\alpha)$  (cf. (1)).

Let  $f : \mathbb{T} \rightarrow \mathbb{R}^+$  be of bounded variation. As noticed in [LePa07], we have

$$(28) \quad \|f^{(mq)} - mf_q\|_\infty \leq \frac{1}{2}m^2q\|q\alpha\|\text{Var}(f),$$

where  $q$  is a denominator of  $\alpha$  and  $f_\ell$  is the periodized function  $f_\ell(x) = \sum_{i=0}^{\ell-1} f(x + \frac{i}{\ell})$ .

Assume that  $\alpha$  has unbounded partial quotients and let  $q_{n_k}\|q_{n_k}\alpha\| \rightarrow 0$  along some subsequence  $(q_{n_k})$  of the sequence  $(q_n)$  of denominators of  $\alpha$ . Set  $c := \int_X f d\mu$  and  $F := f - c$ .

We can assume additionally that  $(F^{(q_{n_k})})_* \xrightarrow[k \rightarrow \infty]{} P$  in distribution ( $P$  is a probability measure concentrated on  $[-\text{Var}(f), \text{Var}(f)]$  by the Denjoy-Koksma inequality<sup>10</sup>). Denoting by  $mP$  the image of  $P$  via the map  $r \mapsto mr$ , it follows by (28) that

$$(29) \quad (F^{(mq_{n_k})})_* \xrightarrow[k \rightarrow \infty]{} mP$$

in distribution for each  $m \in \mathbb{Z}$ . By [FrLe04], we hence obtain the following weak convergence in the space of Markov operators:

$$(30) \quad T_{mcq_{n_k}}^f \xrightarrow[k \rightarrow \infty]{} \int_{\mathbb{R}} T_{-t}^f d(mP)(t).$$

Consider now our special case (cf. (6)) of  $f = f_{a,b}$  for which  $f(x) = a$  for  $x \in [0, \frac{1}{2})$  and  $f(x) = b$  for  $x \in [\frac{1}{2}, 1[$ . We assume that  $a, b > 0$ . Then  $c = \frac{1}{2}(a + b)$  and, if moreover we take  $a - b = 2$ ,  $F$  now becomes

$$F = 1_{[0, \frac{1}{2}[} - 1_{[\frac{1}{2}, 1[}.$$

Using Lemma 2.4 (Section 2), the following immediately follows:

<sup>10</sup>Recall that the Denjoy-Koksma inequality states  $|f^{(q_n)}(x)| \leq \text{Var}(f)$  for each zero mean, bounded variation  $f : \mathbb{T} \rightarrow \mathbb{R}$ ,  $n \geq 1$  and  $x \in \mathbb{T}$ .

**Lemma 4.2.** *Assume moreover that the denominators  $q_{n_k}$  above are all odd and  $a-b = 2$ . Then  $(F^{(q_{n_k})})_* \xrightarrow[k \rightarrow \infty]{} \frac{1}{2}(\delta_{-1} + \delta_1)$ .*

It follows from (29) that  $(F^{(mq_{n_k})})_* \xrightarrow[k \rightarrow \infty]{} \frac{1}{2}(\delta_{-m} + \delta_m)$  in distribution for each  $m \in \mathbb{Z}$  and then by (30) that

$$T_{mcq_{n_k}}^f \xrightarrow[k \rightarrow \infty]{} \frac{1}{2}(T_{-m}^f + T_m^f)$$

weakly in the set of Markov operators, for each  $m \in \mathbb{Z}$ .

### Proof of part 2a) and 2b) of Theorem 0.1

Assume that  $\alpha$  has unbounded partial quotients.

Then, either there is a subsequence  $(n_k)$  such that  $q_{n_k} \|q_{n_k} \alpha\| \xrightarrow[k \rightarrow \infty]{} 0$ , where each denominator  $q_{n_k}$ ,  $k \geq 1$ , is odd. If  $a-b = 2$ , then the special flow  $T^f$ , obtained by  $T = R_\alpha$  and  $f = f_{a,b}$ , is rigid. Indeed, the result follows from the previous discussion, using Lemma 4.1 (with  $r_n = cq_n$ ).

Or, there is a subsequence  $(n_k)$  such that  $q_{n_k} \|q_{n_k} \alpha\| \xrightarrow[k \rightarrow \infty]{} 0$ , where each denominator  $q_{n_k}$ ,  $k \geq 1$ , is even. Then, by the second part of Lemma 2.4, it implies that  $F^{(q_{n_k})} \rightarrow 0$  in  $L^2$ . The result follows by a folklore argument (cf. Proposition 9.6 in Appendix and the remark below).

This shows 2a). Part 2b) follows also from what precedes.  $\square$

### Remark on rigidity

Let us consider a special flow over the rotation  $R_\alpha$  by an irrational  $\alpha$ , under a roof function  $f$  in  $L^2$ . Assume that  $\int f d\mu = 1$ . Let  $f_0$  denote the centered function  $f - 1$ . It is not hard to see that the existence of a sequence  $(r_n)$  of integers tending to infinity such that

$$(31) \quad \|r_n \alpha\| \rightarrow 0, \quad \|f_0^{(r_n)}\|_2 \rightarrow 0$$

implies the rigidity of the special flow  $(T^f)$  (cf. proof of Proposition 9.4 in Appendix).

The following lemma shows that for  $f_0 = \varphi := 1_{[0, \frac{1}{2}[} - 1_{[\frac{1}{2}, 1[}$  there is no sequence  $(r_n)$  satisfying (31) if  $q_k$  is odd for  $k$  big enough. This implies that the method given by (31) cannot be used to prove rigidity in the framework of Theorem 0.1, case 2b).

**Lemma 4.3.** *There is  $\delta > 0$  such that, if  $q_k$  is odd for  $k \geq k_0$ , then for every integer  $s \geq k_0$ ,  $\|\varphi^{(s)}\|_2 \geq \delta$ .*

*Proof.* We have  $\varphi(x) = \sum_{r \in \mathbb{Z}} \frac{2}{\pi i(2r+1)} e^{2\pi i(2r+1)x}$ , hence (cf. (35) below)

$$\|\varphi^{(s)}\|_2^2 = \frac{4}{\pi^2} \sum_{r \in \mathbb{Z}} \frac{1}{(2r+1)^2} \left( \frac{\sin \pi s(2r+1)\alpha}{\sin \pi(2r+1)\alpha} \right)^2.$$

There is  $k$  such that  $q_k \leq s < q_{k+1}$ . Taking the term corresponding to  $2r+1 = q_k$ , in the above series and using the equality  $\|sq_k \alpha\| = s \|q_k \alpha\|$  valid since  $s < q_{k+1} \leq 1/\|q_k \alpha\|$ , we

get (up to a constant):

$$\|\varphi^{(s)}\|_2 \geq \frac{1}{q_k} \frac{s \|q_k \alpha\|}{\|q_k \alpha\|} = \frac{s}{q_k} \geq 1.$$

□

## 5. Centralizer and functional equation for $\Phi_\beta$

### 5.1. Regularity of a class of step cocycles.

In this subsection it is shown the existence of a large class of values of  $\beta$  such that Property (II) defined in Section 2 holds for  $\Phi_\beta$ . We start by a general result based on a Fourier computation.

If  $\varphi$  is a centered BV function, we write  $\varphi(x) = \sum_{r \neq 0} \frac{\gamma_r(\varphi)}{r} e^{2\pi i r x}$  for its Fourier series and we have  $\sup_r |\gamma_r(\varphi)| < \infty$ .

**Theorem 5.1.** *Let  $\varphi$  be a centered BV real valued function. If there are a subsequence  $(q_{n_k})$  of denominators and a constant  $\delta > 0$  such that*

$$(32) \quad |\gamma_{q_{n_k}}(\varphi)| \geq \delta, \forall k \geq 1,$$

$$(33) \quad M := \sup_{k: a_{n_k+1}=1} a_{n_k} < \infty,$$

*then the cocycle generated by  $\varphi$  has a finite essential value  $\neq 0$  (hence  $\varphi$  is regular and is not a coboundary).*

*Proof.* We claim that there is a positive constant  $c$  such that  $\|\varphi^{(q_{n_k})}\|_2^2 \geq c, \forall k \geq 1$ .

By Lemma 2.1, since by the Denjoy-Koksma inequality  $\varphi^{(q_n)}$  is uniformly bounded by  $\text{Var}(\varphi)$ , this will imply that the cocycle generated by  $\varphi$  has a non zero essential value, hence is regular. Moreover, since  $\mathcal{E}(\varphi) \neq \{0\}$ ,  $\varphi$  is not a coboundary.

Now, we prove the claim. The ergodic sum of  $\varphi$  at time  $q$  and the square of its  $L^2$ -norm read:

$$(34) \quad \varphi^{(q)}(x) = \sum_{r \neq 0} \frac{\gamma_r(\varphi)}{r} \frac{e^{2\pi i q r \alpha} - 1}{e^{2\pi i r \alpha} - 1} e^{2\pi i r x},$$

$$(35) \quad \|\varphi^{(q)}\|_2^2 = \sum_{r \neq 0} \frac{|\gamma_r(\varphi)|^2}{r^2} \left( \frac{\sin \pi q r \alpha}{\sin \pi r \alpha} \right)^2.$$

Taking the term corresponding to  $r = q_n$ , we have, for  $n$  in the sequence  $\mathcal{S} = (n_k)$ :

$$(36) \quad \|\varphi^{(q_n)}\|_2^2 \geq \delta^2 \frac{1}{q_n^2} \left( \frac{\sin \pi q_n^2 \alpha}{\sin \pi q_n \alpha} \right)^2 = \delta^2 \frac{1}{q_n^2} \left( \frac{\sin \pi q_n \theta_n}{\sin \pi \theta_n} \right)^2,$$

with (see (20))

$$q_n |\theta_n| \leq \frac{q_n}{a_{n+1} q_n + q_{n-1}} \leq \frac{1}{a_{n+1}}.$$

If  $a_{n+1} \geq 2$ , then  $q_n |\theta_n| \leq \frac{1}{2}$  and it follows from (17) and (36) that:

$$(37) \quad \|\varphi^{(q_n)}\|_2^2 \geq \delta^2 \frac{1}{q_n^2} \left( \frac{2q_n \theta_n}{\pi \theta_n} \right)^2 = 4 \frac{\delta^2}{\pi^2}.$$

Now, for  $n$  in  $\mathcal{S} = (n_k)$ , suppose that  $a_{n+1} = 1$ , so that  $q_{n+1} = q_n + q_{n-1}$ . Considering still  $r = q_n$ , but in the Fourier series of  $\varphi^{(q_{n-1})}$ , we get the lower bound:

$$(38) \quad \|\varphi^{(q_{n-1})}\|_2^2 \geq \delta^2 \frac{1}{q_n^2} \left( \frac{\sin \pi q_{n-1} q_n \alpha}{\sin \pi q_n \alpha} \right)^2 = \delta^2 \frac{1}{q_n^2} \left( \frac{\sin \pi q_{n-1} \theta_n}{\sin \pi \theta_n} \right)^2.$$

By (19) we have  $q_{n-1} |\theta_n| \leq \frac{q_{n-1}}{q_{n+1}} = \frac{q_{n-1}}{q_n + q_{n-1}} \leq \frac{1}{2}$ , so we can use (17). From the hypothesis, for this value of  $n$ , we have  $a_n \leq M$ , so that

$$\frac{1}{q_n} \frac{|\sin \pi q_{n-1} \theta_n|}{|\sin \pi \theta_n|} \geq \frac{2}{\pi} \frac{q_{n-1}}{q_n} \geq \frac{2}{\pi} \frac{q_{n-1}}{a_n q_{n-1} + q_{n-2}} = \frac{2}{\pi} \frac{1}{a_n + q_{n-2}/q_{n-1}} \geq \frac{2}{\pi} \frac{1}{M+1}. \quad \square$$

*Remark:* When  $\varphi$  has values in  $\mathbb{Z}$  as in the examples below, we can give the following variant of the previous proof. By the Denjoy-Koksma inequality,  $\varphi^{(q_n)}$  takes a finite number of integral values in  $[-\text{Var}(\varphi), \text{Var}(\varphi)]$  and

$$\|\varphi^{(q_n)}\|_2^2 = \sum_{j: |j| \leq \text{Var}(\varphi)} j^2 \mu(\{\varphi^{(q_n)} = j\}).$$

By the claim in the previous proof,  $\|\varphi^{(q_{n_k})}\|_2^2 \geq c$  for a positive constant  $c$ . This implies that, on a subsequence of  $(n_k)$ ,  $\varphi^{(q_{n_k})}$  takes a fixed value  $j_0 \neq 0$  on sets whose measure is bounded away from 0. Therefore,  $j_0$  is a non zero essential value.

### 5.1.1. Examples, the step function $\Phi_\beta$ .

Now, we consider specific examples related to Theorem 0.1 and introduce some notation. The argument of the functions below are understood mod 1.

Let  $G(x) = \{x\} - \frac{1}{2}$ ,  $F = 1_{[0, \frac{1}{2}[} - 1_{[\frac{1}{2}, 0[}$ , as above, and  $\Phi_\beta := \frac{1}{2}F(\cdot - \beta) - \frac{1}{2}F$ , where  $\beta$  is a real number in  $]0, 1[$ .

If  $0 < \beta \leq \frac{1}{2}$ , then  $\Phi_\beta = -1_{[0, \beta[} + 1_{[\frac{1}{2}, \beta + \frac{1}{2}[}$ ; if  $\frac{1}{2} < \beta < 1$ , then  $\Phi_\beta = -1_{[\beta - \frac{1}{2}, \frac{1}{2}[} + 1_{[\beta, 1[}$ .

For  $\beta \neq \frac{1}{2}$ , the jumps of  $\Phi_\beta$  are respectively  $+1$  at  $\beta$ ,  $+1$  at  $\frac{1}{2}$ ,  $-1$  at  $\frac{1}{2} + \beta \bmod 1$ . The jump at 0 is  $\lim_{t \rightarrow 0^+} \Phi_\beta(t) - \lim_{t \rightarrow 1^-} \Phi_\beta(t) = -1$ .

Observe also that  $F = 2(R_{\frac{1}{2}} - I)G$ , so  $\Phi_\beta := \frac{1}{2}(R_{-\beta} - I)F = (R_{-\beta} - I)(R_{-\frac{1}{2}} - I)G$ .

More generally, let  $\beta_1, \dots, \beta_v$  be real numbers and set

$$(39) \quad \varphi_{\beta_1, \dots, \beta_v} := \prod_{j=1}^v (R_{-\beta_j} - I) G.$$

With this notation, the function  $\Phi_\beta$  considered before is  $\varphi_{\frac{1}{2}, \beta}$  and we have  $\varphi_{\beta, \gamma} = -(I - R_{-\gamma}) \zeta_\beta$ , with  $\zeta_\beta := 1_{[0, \beta[} - \beta$  since  $(R_{-\beta} - I)G = \zeta_\beta$ . The Fourier series of  $G$  and  $\varphi_{\beta_1, \dots, \beta_v}$  are respectively

$$G(x) = \frac{-1}{2\pi i} \sum_{r \neq 0} \frac{1}{r} e^{2\pi i r x}, \quad \varphi_{\beta_1, \dots, \beta_v}(x) = \frac{-1}{2\pi i} \sum_{r \neq 0} \frac{1}{r} \prod_{j=1}^v (e^{-2\pi i r \beta_j} - 1) e^{2\pi i r x},$$

and therefore

$$|\gamma_{q_n}(\varphi_{\beta_1, \dots, \beta_v})| = \frac{2^{v-1}}{\pi} \prod_{j=1}^v |\sin \pi q_n \beta_j|.$$

Immediately from Theorem 5.1, we obtain the following results:

**Corollary 5.2.** *If  $\limsup_n \prod_{j=1}^v \|q_n \beta_j\| > 0$  and  $\sup_{n: a_{n+1}=1} a_n < \infty$ , then the group of finite essential values of the cocycle  $\varphi_{\beta_1, \dots, \beta_v}$  is not reduced to 0.*

**Corollary 5.3.** *If there is subsequence  $(q_{n_k})_{k \geq 1}$  such that  $q_{n_k}$  is odd,  $\sup_{k: a_{n_k+1}=1} a_{n_k} < \infty$  and  $\limsup_k \|q_{n_k} \beta\| > 0$ , then  $\Phi_\beta = \varphi_{\frac{1}{2}, \beta}$  is regular and is not a coboundary.*

*In particular, this is true if  $q_n$  is odd for  $n$  big enough and  $\limsup_n \|q_n \beta\| > 0$ .*

*Proof.* The particular case follows from Remark 2.2, which shows that, if  $q_n$  is odd for  $n \geq n_0$ , then  $a_n$  is even, hence  $\geq 2$ , for  $n \geq n_0 + 1$ .  $\square$

## 6. Coboundary equation, end of the proof of Theorem 0.1

The aim of this section is to finish the proof of Theorem 0.1 by proving 2c) and 2d). We start by a preliminary discussion on the discontinuities of  $\varphi_\beta^{(q_n)}$  for a general  $\varphi$  and then of  $\Phi_\beta^{(q_n)}$ , which will be useful in the proof of Theorem 6.2.

Let  $\gamma$  be in  $[0, 1[$  and  $n \geq 1$ . Recall that  $\|q_n \gamma\| \leq \frac{1}{2}$ . We define  $t(\gamma, n) \in \mathbb{Z}$  and  $\varepsilon_n(\gamma) = \pm 1$  by

$$(40) \quad q_n \gamma = t(\gamma, n) + \varepsilon_n(\gamma) \|q_n \gamma\|.$$

So, we have:

$$(41) \quad \gamma = \frac{t(\gamma, n)}{q_n} + \varepsilon_n(\gamma) \frac{\|q_n \gamma\|}{q_n}.$$

Note that if  $\gamma = \alpha$ , then (41) reads (cf. the last equality in (23)):

$$(42) \quad \alpha = \frac{p_n}{q_n} + (-1)^n \frac{\|q_n \alpha\|}{q_n}.$$

If  $\gamma = \frac{1}{2}$ , then  $\|q_n \frac{1}{2}\| = 0$  or  $\frac{1}{2}$ , depending whether  $q_n$  is even or odd.

Suppose now that  $q_n$  is odd:  $q_n = 2q'_n + 1$ . Then, by (41), since  $\frac{1}{2} = \frac{q'_n}{q_n} + \frac{1}{2q_n}$ , we have:

$$(43) \quad \gamma + \frac{1}{2} = \frac{t(\gamma, n) + q'_n}{q_n} + \frac{1}{2q_n} + \varepsilon_n(\gamma) \frac{\|q_n \gamma\|}{q_n}.$$

In (40),  $t(\gamma, n)$  and  $\varepsilon_n(\gamma)$  are uniquely defined, excepted for  $\gamma = \frac{1}{2}$ . For this special value,  $\|q_n \frac{1}{2}\| = \frac{1}{2}$  and we have the representation  $\frac{1}{2} = \frac{q'_n}{q_n} + \frac{1}{2q_n}$ .

### *Location of the discontinuities of $\varphi^{(q_n)}$*

Let  $\varphi$  be a 1-periodic function. If  $\gamma$  is a discontinuity of  $\varphi$ , the discontinuities of  $\varphi^{(q_n)}$  corresponding to  $\gamma$  are located at  $\gamma - \ell \alpha \pmod{1}$ ,  $\ell = 0, 1, \dots, q_n - 1$ . We call them *discontinuities of type  $\gamma$* .

For a given denominator  $q_n$ , we consider the grid  $\{0, \frac{1}{q_n}, \frac{2}{q_n}, \dots, \frac{q_n-1}{q_n}\}$  and denote by  $I_{n,k} = I_k$  the interval  $[\frac{k}{q_n}, \frac{k+1}{q_n}[$ ,  $0 \leq k < q_n$ . In each interval  $I_k$ , there is one and only one discontinuity of type 0. For  $0 < \gamma < 1$ , there are 0, 1 or 2 discontinuities of type  $\gamma$  in each interval  $I_k$ , since  $\|\ell \alpha\| > \|q_n \alpha\| \geq \frac{1}{2q_n}$  for  $\ell = 1, \dots, q_n - 1$ , by (21).



For  $n \geq 1$  and  $\gamma \in [0, 1[$ , the map  $\ell \rightarrow -\ell p_n + t(\gamma, n) \pmod{q_n}$  defines a permutation of the set  $\{0, 1, \dots, q_n - 1\}$ . In view of (22), its inverse map is  $k \rightarrow u_n(k, \gamma)$ , where

$$(44) \quad u_n(k, \gamma) \in \{0, 1, \dots, q_n - 1\} \text{ and } u_n(k, \gamma) = (-1)^{n-1} q_{n-1}(-k + t(\gamma, n)) \pmod{q_n}.$$

We put  $A_n(\gamma) = \varepsilon_n(\gamma) \|q_n \gamma\|$ ,  $B_n(\gamma, k) = (-1)^{n-1} u_n(k, \gamma) \|q_n \alpha\|$ .

Using (41), (42) and the definition of  $u_n(k, \gamma)$ , for each discontinuity  $\gamma$  of  $\varphi$ , we can label the discontinuities  $\gamma - \ell \alpha \pmod{1}$ ,  $\ell = 0, \dots, q_n - 1$ , as  $\zeta(\gamma, k, n)$ ,  $k = 0, \dots, q_n - 1$ :

$$(45) \quad \zeta(\gamma, k, n) = \frac{k}{q_n} + \frac{\varepsilon_n(\gamma) \|q_n \gamma\|}{q_n} + \frac{(-1)^{n-1} u_n(k, \gamma) \|q_n \alpha\|}{q_n}$$

$$(46) \quad = \frac{k}{q_n} + \frac{A_n(\gamma) + B_n(\gamma, k)}{q_n}.$$

We have  $\|q_n \gamma\| \leq \frac{1}{2}$ ,  $u_n(k, \gamma) \|q_n \alpha\| < q_n \|q_n \alpha\| < 1/a_{n+1} \leq \frac{1}{2}$ , since  $a_{n+1}$  is even (because the  $q_n$ 's are odd). Hence  $|A_n(\gamma) + B_n(\gamma, k)| < 1$ .

Thus, equation (45) gives the position of the discontinuities of type  $\gamma$ :  $\zeta(\gamma, k, n)$  belongs to the interval  $I_k$  if  $A_n(\gamma) + B_n(\gamma, k) > 0$ , to the interval  $I_{k-1}$  if  $A_n(\gamma) + B_n(\gamma, k) < 0$ . (with the convention  $I_{-1} = I_{q_n-1}$ ). Moreover, for each  $\gamma$ , the sequence  $(\zeta(\gamma, k, n))$ ,  $k = 0, \dots, q_n - 1$  is increasing, since:

$$\zeta(\gamma, k+1, n) - \zeta(\gamma, k, n) = \frac{1}{q_n} + (-1)^{n-1} \frac{\|q_n \alpha\|}{q_n} (u_n(k+1, \gamma) - u_n(k, \gamma))$$

and  $\|q_n \alpha\| |u_n(k+1, \gamma) - u_n(k, \gamma)| \leq q_n \|q_n \alpha\| < 1$ .

*Discontinuities of  $\Phi_\beta^{(q_n)}$*

The discontinuities of  $\Phi_\beta^{(q_n)}$  are of type 0,  $\frac{1}{2}$ ,  $\beta$  and  $\beta + \frac{1}{2}$ . For the type 0,  $\frac{1}{2}$  and  $\beta$ , they read, for  $k = 0, 1, \dots, q_n - 1$ ,

$$(47) \quad \zeta(0, k, n) = \frac{k}{q_n} + (-1)^{n-1} \frac{u_n(k, 0) \|q_n \alpha\|}{q_n},$$

$$(48) \quad \zeta\left(\frac{1}{2}, k, n\right) = \frac{k}{q_n} + \frac{1}{2q_n} + (-1)^{n-1} \frac{u_n(k, \frac{1}{2}) \|q_n \alpha\|}{q_n},$$

$$(49) \quad \zeta(\beta, k, n) = \frac{k}{q_n} + \varepsilon_n(\beta) \frac{\|q_n \beta\|}{q_n} + (-1)^{n-1} \frac{u_n(k, \beta) \|q_n \alpha\|}{q_n}.$$

The discontinuities of type  $\beta + \frac{1}{2}$  can be written

$$(50) \quad \zeta_1\left(\beta + \frac{1}{2}, k, n\right) = \frac{k}{q_n} + \frac{1}{2q_n} + \varepsilon_n(\beta) \frac{\|q_n \beta\|}{q_n} + (-1)^{n-1} \frac{u'_n(k, \beta) \|q_n \alpha\|}{q_n},$$

where  $u'_n(k, \beta) = u_n(k, \beta) + (-1)^{n-1} q_{n-1} q'_n \pmod{q_n}$ . Indeed, using (49) and  $\frac{1}{2} = \frac{q'_n}{q_n} + \frac{1}{2q_n}$ , we have (mod 1)

$$\zeta(\beta, \ell, n) + \frac{1}{2} = \frac{\ell + q'_n}{q_n} + \frac{1}{2q_n} + \varepsilon_n(\beta) \frac{\|q_n \beta\|}{q_n} + (-1)^{n-1} \frac{u_n(\ell, \beta) \|q_n \alpha\|}{q_n}$$

and by taking  $\ell = k - q'_n$ , we get (50).

Let us assume  $n$  odd (hence  $\alpha = \frac{p_n}{q_n} - \frac{\|q_n \alpha\|}{q_n}$ ). The discussion is analogous for  $n$  even.

For  $\gamma = 0$ , there is one and only one discontinuity, namely  $\zeta(0, k, n)$ , of type 0 of  $\Phi^{(q_n)}$  in  $I_k$ . By (47), it reads  $\frac{k}{q_n} + \frac{u_n(k, 0) \|q_n \alpha\|}{q_n}$ . Since  $u_n(k, 0) \|q_n \alpha\| \leq q_n \|q_n \alpha\| \leq \frac{1}{a_{n+1}}$ , this discontinuity is close to the left endpoint  $\frac{k}{q_n}$  of  $I_k$  if  $a_{n+1}$  is big.

By (48), the discontinuity of type  $\frac{1}{2}$  in  $I_k$  is  $\zeta(\frac{1}{2}, k, n) = \frac{k}{q_n} + \frac{1}{2q_n} + \frac{u_n(k, \frac{1}{2}) \|q_n \alpha\|}{q_n}$ , with  $0 \leq u_n(k, \frac{1}{2}) \|q_n \alpha\| \leq q_n \|q_n \alpha\| < \frac{1}{a_{n+1}}$ , hence located close to the middle of  $I_k$ . It is the only discontinuity of type  $\frac{1}{2}$  belonging to  $I_k$  if  $a_{n+1}$  is big.

By (49),  $\zeta(\beta, k, n)$ , discontinuity of type  $\beta$ , is close to  $\frac{k}{q_n}$  (hence close to  $\zeta(0, k, n)$ , either to the left or to the right of it), if  $\|q_n \beta\|$  is small and  $a_{n+1}$  is big. Furthermore, notice that the next discontinuity of type  $\beta$ ,  $\zeta(\beta, k+1, n)$ , may belong to  $I_k$ , but is close to the right endpoint  $\frac{k+1}{q_n}$  of  $I_k$ .

By (50),  $\zeta_1(\beta + \frac{1}{2}, k, n)$ , discontinuity of type  $\beta + \frac{1}{2}$ , is close to  $\frac{k}{q_n} + \frac{1}{2q_n}$  (hence to  $\zeta(\frac{1}{2}, k, n)$ , left or right), if  $\|q_n \beta\|$  is small and  $a_{n+1}$  is big.

We conclude these preliminaries by the following remark:

**Remark 6.1.** The set  $\{\beta \in \mathbb{T} : \Phi_\beta \text{ is an } R_\alpha - \text{coboundary}\}$  is an additive group, cf. (9).

**Theorem 6.2.** *Assume that the denominators  $q_n$  of  $\alpha$  are odd for  $n \geq n_0$ , for some  $n_0$ . Then, if  $\beta \notin \mathbb{Z}\alpha + \mathbb{Z}$ ,  $\Phi_\beta$  is not a coboundary, i.e., the functional equation*

$$(51) \quad \Phi_\beta(x) = g(x + \alpha) - g(x), \text{ for } \mu - \text{a.e. } x \in \mathbb{T},$$

has no measurable solution  $g$ .

*Proof.* Let us assume  $\beta \notin \mathbb{Z}\alpha + \mathbb{Z}$ . For  $\beta = \frac{1}{2}$ , we get  $\Phi_{\frac{1}{2}} = -F(x)$  which is not a coboundary for  $R_\alpha$ . So we can assume  $\beta \neq \frac{1}{2}$ .

For  $n \geq n_0$ , since  $q_n$  is odd, we have  $q_n \|q_n \alpha\| < a_{n+1}^{-1} \leq \frac{1}{2}$ . Therefore by Lemma 2.4,  $\sum_{j=0}^{q_n-1} F(x + j\alpha) = \pm 1$  for all  $x$ . It follows:

$$(52) \quad \Phi_\beta^{(q_n)}(x) = \frac{1}{2} \sum_{j=0}^{q_n-1} F(x - \beta + j\alpha) - \frac{1}{2} \sum_{j=0}^{q_n-1} F(x + j\alpha) = 1, -1 \text{ or } 0.$$

There are two cases depending on the behaviour of  $\|q_n \beta\|$ :

**A)**  $\limsup_n \|q_n \beta\| > 0$

In this case, we use Corollary 5.3 to conclude that  $\Phi_\beta$  is not a coboundary. A stronger conclusion is the following: there is a sequence  $(n_k)$  and  $\delta > 0$  such that  $\|\Phi_\beta^{(q_{n_k})}\|_2 \geq \delta$  which implies by (52) (cf. Theorem 5.1 and Corollary 5.3) that  $\Phi_\beta^{(q_{n_k})}(x) = \pm 1$  on sets whose measure is bounded away from 0. Hence 1 is an essential value of the cocycle and the skew map  $R_{\alpha, \Phi_\beta}$  is ergodic on  $X \times \mathbb{Z}$ . A fortiori,  $\Phi_\beta$  is not a coboundary.

**B)**  $\|q_n \beta\| \rightarrow 0$

We are going to show that in case B), for  $\beta \notin \mathbb{Z}\alpha + \mathbb{Z}$ , the cocycle  $\Phi_\beta$  is not a coboundary, which will conclude the proof of the theorem. But contrary to case A), ergodicity of the skew product may fail (see Remark 6.4 below).

We start by studying the support of  $\Phi_\beta^{(q_n)}$  deduced from the location of the discontinuities of the cocycle as studied above.

*Clusters of discontinuities and support of  $\Phi_\beta^{(q_n)}$*

Let  $n$  be such that  $\|q_n\beta\|$  is small and  $a_{n+1}$  is big. Then the picture is the following.

In the interval  $I_k = [\frac{k}{q_n}, \frac{k+1}{q_n}[$ , there is one and only one discontinuity of type 0,  $\zeta(0, k, n)$ , and one and only one of type  $\frac{1}{2}$ ,  $\zeta(\frac{1}{2}, k, n)$ , which are close respectively to the left endpoint  $\frac{k}{q_n}$  and the middle point  $\frac{k}{q_n} + \frac{1}{2q_n}$  of  $I_k$ .

There is a discontinuity of type  $\beta$ ,  $\zeta(\beta, k, n)$ , at left or at right of  $\zeta(0, k, n)$  and close to it. Likewise, there is a discontinuity  $\zeta(\beta + \frac{1}{2}, k, n)$  of type  $\beta + \frac{1}{2}$  at left or at right of  $\zeta(\frac{1}{2}, k, n)$  and close to it.

The discontinuity which is the nearest discontinuity to the discontinuity  $\zeta(0, k, n)$  of type 0 in  $I_k$  is  $\zeta(\beta, k, n)$  of type  $\beta$ . The nearest discontinuity close to a discontinuity of type  $\frac{1}{2}$  in  $I_k$  is  $\zeta(\beta + \frac{1}{2}, k, n)$  of type  $\beta + \frac{1}{2}$ .

This shows that the discontinuities of  $\Phi_\beta^{(q_n)}$  gather in well separated ‘‘clusters’’ (which here are groups of two discontinuities close together) (see Fig 1: graph of  $\Phi_\beta$ , with  $\alpha = \pi - 3$ ,  $\beta = 2 - \sqrt{2}$  and Fig. 2: graph of  $\Phi_\beta^{(q_1)}$ , with  $q_1 = 7$ , first denominator in the sequence of denominators of  $\alpha$ ).

This situation can be described as follows. Suppose for concreteness that  $\zeta(\beta, k, n)$  is located at the right to  $\zeta(0, k, n)$ . If a point  $x$  moves in  $I_k$  to the right, starting close to  $\zeta(0, k, n)$  at its left (hence close to  $\frac{k}{q_n}$ ), it crosses successively two discontinuities  $\zeta(0, k, n)$ ,  $\zeta(\beta, k, n)$ , with jumps respectively  $-1, +1$ . The cocycle  $\Phi_\beta^{(q_n)}(x)$  has a constant value  $v$  at the left of  $\zeta(0, k, n)$ , then again  $v$  after the discontinuities and keeps this value until it is close to  $\frac{k}{q_n} + \frac{1}{2q_n}$ . Therefore it keeps a constant value  $v$  on an interval of length close to  $\frac{1}{2q_n}$ . Then, still with  $x$  moving in  $I_k$ ,  $\Phi_\beta^{(q_n)}(x)$  takes again the value  $v$  after crossing the discontinuities  $\zeta(\frac{1}{2}, k, n)$  and  $\zeta(\beta + \frac{1}{2}, k, n)$ . It keeps this constant value on an interval again of length close to  $\frac{1}{2q_n}$ . On the whole interval  $[0, 1]$ ,  $\Phi_\beta^{(q_n)}$  takes the values  $v$  on a set of measure close to 1. Since the integral is 0, this implies  $v = 0$ .

By Lemma 2.3, we know that there is a subsequence  $(n_j)$  such that  $c_{n_j}(\beta) > \frac{1}{4}$ , where  $c_n$  is the ratio (24). Since  $\|q_{n_j}\beta\| \rightarrow 0$ , from this and (20), it follows:  $4\|q_{n_j}\beta\| \geq q_{n_j}\|q_{n_j}\alpha\| \geq q_{n_j}(q_{n_j+1} + q_{n_j})^{-1} \geq (a_{n_j+1} + 2)^{-1}$ . Therefore,  $a_{n_j+1} \rightarrow \infty$  since  $\|q_{n_j}\beta\| \rightarrow 0$ .

Since  $\|q_{n_j}(8\beta)\| = 8\|q_{n_j}\beta\|$  (because  $\|q_{n_j}\beta\|$  is close to 0), replacing  $\beta$  by  $\beta' = 8\beta$ , we get  $c_{n_j}(\beta') = c_{n_j}(8\beta) \geq 2$ . Observe that if (51) has a measurable solution for  $\beta$ , then the equation (51) corresponding to  $\beta'$  has a measurable solution, by Remark 6.1.

Therefore, for the proof of the non-existence of a measurable solution of (51), we can assume that, for a strictly increasing sequence  $\mathcal{S} = (n_j)$ , we have  $\|q_{n_j}\beta\| \rightarrow 0$  and  $c_{n_j}(\beta) \geq 2$ .

On the first half of the interval  $I_k$ ,  $\Phi_\beta^{(q_{n_j})}$  has its support on a small interval since  $\|q_{n_j}\beta\|$  is small. The idea of the proof is to consider  $\Phi_\beta^{(L_j q_{n_j})}$  the cocycle at time  $L_j q_{n_j}$  for a well chosen integer  $L_j$ . (see Fig. 3: graph of  $\Phi_\beta^{(3q_1)}$ ).

The idea of the proof is as follows. Suppose for concreteness that  $n_j$  is odd. The support of  $\Phi_\beta^{(L_j q_{n_j})}$  on the first half of  $I_k$  is a union of translates by multiples of  $\theta_{n_j} = -\|q_{n_j}\alpha\|$  of the support of  $\Phi_\beta^{(q_{n_j})}$ . Up to a certain amount of translates, there is no interference with the part of the support where  $\Phi_\beta^{(q_{n_j})}$  has an opposite sign. To cover a set of measure  $\geq \delta/q_{n_j}$  for some  $\delta > 0$  inside the interval  $I_{n_j}$ , we take  $L_j \sim \delta a_{n_j+1}$ . This is enough to get a big enough support; but nevertheless  $L_j q_{n_j}$  is still a sequence of rigidity times.

Now we make the argument more precise.

*Support of  $\Phi_\beta^{(L_j q_{n_j})}$*

Let  $n = n_j$  be in  $\mathcal{S}$ . If we assume for concreteness  $n$  odd and  $\varepsilon_n(\beta) = +1$ , using equations (47) to (50), we obtain that the value of  $\Phi_\beta^{(q_n)}$  is

$$\begin{aligned} -1 \text{ on } & I_{k,n}^0 := \left[ \frac{k}{q_n} + \frac{u_n(k, 0)\|q_n\alpha\|}{q_n}, \frac{k}{q_n} + \frac{\|q_n\beta\|}{q_n} + \frac{u_n(k, \beta)\|q_n\alpha\|}{q_n} \right], \\ 1 \text{ on } & I_{k,n}^1 := \left[ \frac{k}{q_n} + \frac{1}{2q_n} + \frac{u_n(k, \frac{1}{2})\|q_n\alpha\|}{q_n}, \frac{k}{q_n} + \frac{1}{2q_n} + \frac{\|q_n\beta\|}{q_n} + \frac{u'_n(k, \frac{1}{2} + \beta)\|q_n\alpha\|}{q_n} \right], \end{aligned}$$

and 0 elsewhere.

Since  $u_n(k, 0)\|q_n\alpha\|$  and  $u_n(k, \beta)\|q_n\alpha\|$  are both  $\leq q_n\|q_n\alpha\|$  and  $\|q_n\beta\| \geq c_n(\beta)q_n\|q_n\alpha\| \geq 2q_n\|q_n\alpha\|$  for  $n \in \mathcal{S}$ , we have  $\|q_n\beta\| + u_n(k, \beta)\|q_n\alpha\| - u_n(0, \beta)\|q_n\alpha\| > q_n\|q_n\alpha\|$ . Hence the length of  $I_{k,n}^0$  is bigger than  $\|q_n\alpha\| \geq \frac{1}{2}q_n^{-1}a_{n+1}^{-1}$ . Similarly, the length of  $I_{k,n}^1$  is bigger than  $\|q_n\alpha\|$ .

Let  $L_j = [\delta a_{n_j+1}]$ , where  $0 < \delta < 1/2$  be a constant. Therefore  $L_j\|q_{n_j}\alpha\| < \delta/q_{n_j}$  (so that  $R_\alpha^{L_j q_{n_j}} \rightarrow Id$  and  $\Phi_\beta^{(L_j q_{n_j})}$  tends to 0 in measure). Let us consider the sum  $\Phi_\beta^{(L_j q_{n_j})}$ .

The measure of the subset  $J^0(k, n_j)$  (resp.  $J^1(k, n_j)$ ) of  $[\frac{k}{q_{n_j}}, \frac{k+1}{q_{n_j}}[$  on which  $\Phi_\beta^{(L_j q_{n_j})} \leq -1$  (resp.  $\Phi_\beta^{(L_j q_{n_j})} \geq 1$ ) is the measure of the union  $\tilde{I}_{k,n_j}^0$  (resp.  $\tilde{I}_{k,n_j}^1$ ) of the intervals translated of  $I_{k,n_j}^0$  (resp. of  $I_{k,n_j}^1$ ) by  $u\theta_n$ ,  $u = 0, \dots, L_j - 1$ ; therefore is bigger than  $\delta a_{n_j+1}\|q_{n_j}\alpha\|$ .

Therefore, the measure of the union  $A_{n_j}^0 = \bigcup_{k=0}^{q_{n_j}-1} J^0(k, n_j)$  is bigger than

$$q_{n_j} L_j \|q_{n_j}\alpha\| = \delta a_{n_j+1} q_{n_j} \|q_{n_j}\alpha\| > \delta a_{n_j+1} q_{n_j} / (a_{n_j+1} q_{n_j} + q_{n_j-1}) \geq \frac{1}{2} \delta.$$

Hence, along the sequence  $\mathcal{S}$ ,  $\Phi_\beta^{(L_j q_n)}$  does not tend to 0 in measure and equation (51) has no measurable solution.  $\square$

**Remark 6.3.** Suppose that there are infinitely many even denominators, but that the following condition is satisfied:

$$(53) \quad M := \sup_{n: q_n \text{ even}} a_{q_n+1} < +\infty.$$

Then, if  $\lim_n \|q_n \beta\| = 0$ , the same proof as in case B) above applies. Indeed, it suffices to check that, for the subsequence  $(n_j)$  such that  $c_{n_j}(\beta) > \frac{1}{4}$  given by Lemma 2.3,  $q_{n_j}$  is odd for  $j$  big enough.

Suppose that  $q_{n_j}$  is even. Then, by (20) and (53), we have:

$$\|q_{n_j} \beta\| \geq \frac{1}{4} q_{n_j} \|q_{n_j} \alpha\| \geq \frac{1}{4} \frac{1}{a_{n_j+1} + 2} \geq \frac{1}{4} \frac{1}{M + 2}.$$

Therefore,  $q_{n_j}$  is odd, once  $j$  satisfies  $\|q_{n_j} \beta\| < \frac{1}{4} \frac{1}{M+2}$ .

**Remark 6.4.** In the next section the existence of values of  $\beta$  giving non-regularity will be shown. Regularity or non-regularity of the cocycle is related to the behavior of the sequence  $(c_{n_j}(\beta))_{j \geq 1}$ . (Recall that  $c_n(\beta)$  was defined in (24).)

Suppose that for a subsequence  $(n_{j_\ell})$  and a finite constant  $K$ , we have  $K^{-1} \leq c_{n_{j_\ell}}(\beta) \leq K$ , then the skew map  $R_{\alpha, \Phi_\beta}$  is regular. Indeed, the overlapping of the support occurs for at most  $K + 1$  translation by  $q_{n_\ell} \alpha$ . This implies that on sets with a measure bounded away from 0,  $\Phi_\beta^{(L_j q_{n_{j_\ell}})}$  takes a fixed non zero integer value, which therefore is an essential value of the cocycle. So we get that the cocycle is regular.

But  $\|q_{n_j} \beta\| / q_{n_j} = c_{n_j}(\beta) \|q_{n_j} \alpha\|$  can be much bigger than  $\|q_{n_j} \alpha\|$ , in which case there is a big overlapping of the translates of  $I$ . Therefore, if  $c_{n_j} \uparrow \infty$ , non-regularity can occur. In the next section we will see that this can be effectively the case.

### Proof of Theorem 0.1 part 2c)

This part now follows directly from Theorem 6.2.

### Proof of Theorem 0.1 part 2d)

This part will follow Theorem 6.9 below. We need some preliminary results.

Let  $J = \{0, \beta, \frac{1}{2}, \beta - \frac{1}{2}\}$ . For  $\beta \notin \mathbb{Z}/2$ , for every  $N \geq 1$ , the set of discontinuities of the ergodic sum  $\Phi_\beta^{(N)}$  consists of the distinct points  $t - j\alpha \bmod 1$ , where  $0 \leq j < N$  and  $t \in J$ , with jumps  $\pm 1$ . We write  $\{0 = \gamma_{N,1} < \dots < \gamma_{N,4N}\}$  for the elements in this set listed in natural order.

By minimality of the rotation  $R_\alpha$ , the following lemma holds:

**Lemma 6.5.** *Let  $\lambda > 0$ . For every  $\varepsilon > 0$ , there is  $L(\varepsilon)$  such that, for any  $L \geq L(\varepsilon)$ , we have  $\|b\alpha\| \leq \varepsilon$ , for some  $b \in [\lambda L, 2\lambda L]$ .*

**Proposition 6.6.** *Let  $\alpha$  be an irrational number such that the sequence  $(a_n)$  of its partial quotient does not tend to infinity. Then, if  $\beta \in (\mathbb{Q}\alpha + \mathbb{Q}) \setminus (\mathbb{Z}\alpha + \mathbb{Z})$ ,  $\Phi_\beta$  is not a measurable coboundary.*

*Proof.* We have  $\beta = \frac{\ell}{s}\alpha + \frac{r}{s}$ , with  $\ell, r, s$  integers and  $s \neq 0$ . By assumption, there is a fixed integer  $a$  and subsequence  $(n_k)$  such that  $a_{n_k+1} = a$ .

Let  $n$  be such that  $a_{n+1} = a$ . We have (cf. (21))  $\|k\alpha\| \geq \|q_n\alpha\|, \forall k \in [0, q_{n+1}[$ , and

$$\|q_n\alpha\| \geq \frac{1}{q_n + q_{n+1}} = \frac{1}{(a_{n+1} + 1)q_n + q_{n-1}} \geq \frac{1}{2 + a} \frac{1}{q_n}.$$

Put  $R = 2s$ . Let  $t \in J$ . By the previous inequalities, we have for an integer  $\ell(t)$ , for  $k \leq q_n/R$ :

$$(54) \quad \|k\alpha - t\| \geq \frac{1}{R} \|(Rk + \ell(t))\alpha\| \geq \frac{c}{q_n}, \text{ with } c := \frac{1}{R} \frac{1}{2 + a},$$

since  $Rk + \ell(t) < q_n + q_{n-1} \leq a_{n+1}q_n + q_{n-1} = q_{n+1}$ , for  $n$  big enough and  $k \leq q_n/R$ .

If  $(\varepsilon_i)$  is a sequence of positive numbers tending to 0, by Lemma 6.5 we can choose  $(k_i)$  and  $b_i \in [\frac{q_{n_k i}}{2R}, \frac{q_{n_k i}}{R}]$  such  $\|b_i\alpha\| \leq \varepsilon_i$ . So  $(b_i)$  is a sequence of rigidity times for the rotation by  $\alpha$ . This implies that, if  $\Phi_\beta$  is a measurable coboundary, the ergodic sums  $\Phi_\beta^{(b_i)}$  tends to 0 in measure.

On the other hand, for  $\ell = 1, \dots, 4b_i - 1$ ,  $\Phi_\beta^{(b_i)}$  is constant on the intervals  $]\gamma_{b_i, \ell}, \gamma_{b_i, \ell+1}[$  and we have  $\gamma_{b_i, \ell+1} - \gamma_{b_i, \ell} \geq \frac{c}{q_{n_k i}} \geq \frac{c}{2Rb_i}$  by (54). Therefore  $|\Phi_\beta^{(b_i)}| \geq 1$  on a set of measure bounded away from 0. This gives a contradiction.  $\square$

**Lemma 6.7.** *Let  $n \geq 1$  and  $\delta > 0$  be such that  $\|q_n\beta\| \geq \delta$ . For  $b \in [\frac{\delta}{4}q_n, \frac{\delta}{2}q_n]$ , we have*

$$(55) \quad \|\beta - j\alpha\| \geq \frac{\delta^2}{8} \frac{1}{b}, \text{ for } |j| < b.$$

*Proof.* From  $\|q_n(\beta - j\alpha)\| \geq \|q_n\beta\| - \|q_nj\alpha\| \geq \delta - |j|\|q_n\alpha\| \geq \delta - b/q_n \geq \frac{\delta}{2}$ , it follows:

$$\|\beta - j\alpha\| \geq q_n^{-1} \|q_n(\beta - j\alpha)\| \geq \frac{\delta}{2} q_n^{-1} \geq \frac{\delta}{2} \frac{\delta}{4} b^{-1} = \frac{\delta^2}{8} b^{-1}, \text{ for } 0 \leq j < b. \quad \square$$

**Proposition 6.8.** *Let  $\beta \in ]0, 1[$ . If there is a subsequence  $(q_{n_k})_{k \geq 1}$  of odd denominators of  $\alpha$  such that, for some  $\delta \in ]0, \frac{1}{2}[$ ,  $\|q_{n_k}\beta\| \geq \delta, \forall k \geq 1$ , then  $\Phi_\beta$  is not a measurable coboundary for the rotation  $R_\alpha$ .*

*Proof.* Let  $(\varepsilon_i)$  be a sequence of positive numbers tending to 0. By Lemma 6.5 we can choose  $(k_i)$  and  $b_i \in [\frac{\delta}{4}q_{n_{k_i}}, \frac{\delta}{2}q_{n_{k_i}}]$  such  $\|b_i\alpha\| \leq \varepsilon_i$ . Suppose that  $\Phi_\beta$  is a measurable coboundary. Then the ergodic sums  $\Phi_\beta^{(b_i)}$  tends to 0 in measure. We will show that this is not possible.

Since  $q_{n_k}$  is odd, we have  $\|q_{n_k}\frac{1}{2}\| = \frac{1}{2} \geq \delta$ . By Lemma 6.7, for the ergodic sum  $\Phi_\beta^{(b_i)}$  the discontinuities of type 0 are ‘‘well separated’’ from the discontinuities of type  $\beta$  and of type  $\frac{1}{2}$ , since we have, with  $c = \frac{\delta^2}{8}$ ,

$$\inf_{|\ell| < b_i} (\|\beta - \ell\alpha\|, \|\frac{1}{2} - \ell\alpha\|) \geq \frac{c}{b_i}.$$

Denote by  $\gamma_0$  any discontinuity of type 0. Let  $\gamma^-$  (resp.  $\gamma^+$ ) be the nearest discontinuity of type  $\beta$  or  $\frac{1}{2}$  at left (resp. at right) of  $\gamma_0$ .

The possible jumps between  $\gamma^-$  and  $\gamma_0$  (resp.  $\gamma_0$  and  $\gamma^+$ ) are only  $-1$ . Therefore,  $\Phi_\beta^{(b_i)}$  is non increasing on  $] \gamma^-, \gamma^+[$  and moreover its value is decreased by  $-1$  at the point  $\gamma_0$ . It follows that

$$\Phi_\beta^{(b_i)}(x) \geq 1, \text{ for } x \in ] \gamma^-, \gamma_0[, \text{ or } \Phi_\beta^{(b_i)}(x) \leq -1, \text{ for } x \in ] \gamma_0, \gamma^+[.$$

As we know that the distance between  $\gamma^-$  and  $\gamma_0$  (resp.  $\gamma_0$  and  $\gamma^+$ ) is  $\geq c/b_i$ , we conclude that, on the whole circle,  $|\Phi_\beta^{(b_i)}| \geq 1$  on a set of measure bounded away from 0. We get a contradiction.  $\square$

**Theorem 6.9.** *Let  $\alpha$  be an irrational number such that there are finitely many even denominators or  $\sup_{n: q_n \text{ even}} a_{n+1} < +\infty$ . Then, for  $\beta \notin \mathbb{Z}\alpha + \mathbb{Z}$ , equation*

$$(56) \quad \Phi_\beta(x) = g(x + \alpha) - g(x), \text{ for } \mu - a.e. \ x \in \mathbb{T},$$

has no measurable solution  $g$ .

*Proof.* If  $\limsup_{n: q_n \text{ odd}} \|q_n \beta\| > 0$ , the result follows from Proposition 6.8. Therefore we can assume  $\lim_{n: q_n \text{ odd}} \|q_n \beta\| = 0$ .

Observe that, if there is  $n_0$  such that all denominators  $q_n$  are odd for  $n \geq n_0$ , then the result follows from Theorem 6.2. Let us consider now the case where there are infinitely many even denominators and let denote by  $q_{n_1} < q_{n_2} < \dots$  their sequence.

Since the denominators  $q_{n_k+1}$  and  $q_{n_k-1}$  are odd, it holds  $\lim \|q_{n_k+1} \beta\| = \lim \|q_{n_k-1} \beta\| = 0$ . From the relation  $q_{n_k+1} \beta = a_{n_k+1} q_{n_k} \beta + q_{n_k-1} \beta$ , it follows  $\lim_k \|a_{n_k+1} q_{n_k} \beta\| = 0$ .

Let  $R$  denote the integer  $R = \prod_{j \in J} A_j$ , where  $\{A_j : j \in J\}$  is the finite set of values taken by the  $a_{n_k+1}$ 's. We get  $\lim_k \|q_{n_k} R \beta\| = 0$ . Therefore, for the whole sequence  $(q_n)$ , we have  $\lim_n \|q_n R \beta\| = 0$ .

If  $R\beta \notin \mathbb{Z}\alpha + \mathbb{Z}$ , then part B) of the proof of Theorem 6.2 applies to  $R\beta$  (see Remark 6.3). This shows that the function  $\Phi_{R\beta}$ , and so as well  $\Phi_\beta$ , is not a measurable coboundary.

Finally the remaining case  $\beta \in (\mathbb{Q}\alpha + \mathbb{Q}) \setminus (\mathbb{Z}\alpha + \mathbb{Z})$  is treated in Proposition 6.6.  $\square$

Remark that, if the hypothesis of the theorem is not satisfied, then the situation is that of Theorem 0.1 1b) and equation (56) has a solution for uncountably many  $\beta$ 's.

## 7. Ostrowski expansion and non-regularity for an exceptional set

As remarked above, the proof in case B) of Theorem 6.2 gives a result weaker than ergodicity. Actually, we will show that in that case there is a set of values of  $\beta$  for which ergodicity (as  $\mathbb{Z}$ -valued cocycle) fails and the cocycle  $\Phi_\beta$  is non-regular.

Denote by  $\mathcal{U}(\mathbb{T})$  the group of measurable functions from  $\mathbb{T} = [0, 1[$  to the group  $\mathcal{U}$  of complex numbers of modulus 1.

The non-regularity result is based on the following observation (cf. [Co09]): if  $g$  is cohomologous to  $g_1$  and to  $g_2$ , two functions with values respectively in closed subgroups with an intersection reduced to  $\{0\}$ , then  $\mathcal{E}(g) = \{0\}$ . This implies:

**Lemma 7.1.** *If  $\varphi$  is a  $\mathbb{Z}$ -valued cocycle such that there exists  $s \notin \mathbb{Q}$  for which the multiplicative equation  $e^{2\pi i s \varphi} = \psi \circ R_\alpha / \psi$  has a measurable solution  $\psi : \mathbb{T} \rightarrow \mathcal{U}$ , then  $\mathcal{E}(\varphi) = \{0\}$ . If  $\varphi$  is not a coboundary, then  $\overline{\mathcal{E}}(\varphi) = \{0, \infty\}$  and  $\varphi$  is non-regular.*

Let us consider the function  $\psi_{\beta,s} := e^{2\pi i s 1_{[0,\beta]}}$  on the circle and the multiplicative functional equation

$$(57) \quad e^{2\pi i s 1_{[0,\beta]}} = e^{2\pi i t} R_\alpha f / f, \text{ where } (\beta, s, t) \in [0, 1[ \times \mathbb{R} \times \mathbb{R} \text{ and } f \in \mathcal{U}(X),$$

This equation was studied by W. Veech in [Ve69], then by K. Merrill [Me85] who gave a sufficient condition on  $(\beta, s, t)$  for the existence of a solution, then by M. Guénaïs and F. Parreau [GuPa06] who gave a necessary and sufficient condition for (57) to have a measurable solution and extended it to more general step functions. The conditions are expressed in terms of the so-called Ostrowski expansion of a real  $\beta$ . For  $r \geq 1$ , we put

$$H_r(\alpha) := \left\{ \sum_{n \geq 0} b_n q_n \alpha \bmod 1, (b_n)_n \in \mathbb{Z}^{\mathbb{N}}, \text{ such that } \sum_{n \geq 0} \left( \frac{|b_n|}{a_{n+1}} \right)^r < +\infty \right\},$$

$$H_\infty(\alpha) := \left\{ \sum_{n \geq 0} b_n q_n \alpha \bmod 1, \frac{|b_n|}{a_{n+1}} \rightarrow 0 \right\}.$$

Recall the following characterization ([GuPa06]):

$$H_r(\alpha) = \left\{ \beta \in \mathbb{R} : \sum_{n \geq 0} \|q_n \beta\|^r < +\infty \right\}, \quad H_\infty(\alpha) = \left\{ \beta \in \mathbb{R} : \|q_n \beta\| \rightarrow 0 \right\}.$$

When  $\alpha$  is not of bounded type,  $H_\infty(\alpha)$  is an uncountable additive subgroup of  $\mathbb{R}$ .

**Theorem 7.2.** ([GuPa06]) *Equation (57) has a solution  $f \in \mathcal{U}(X)$  for the parameters  $(\beta, s, t)$  if and only if there is a sequence  $(b_n)$  in  $\mathbb{Z}$  such that:*

$$(58) \quad \beta = \sum_{n \geq 0} b_n q_n \alpha \bmod 1, \text{ with } \sum_{n \geq 0} \frac{|b_n|}{a_{n+1}} = C_1 < \infty,$$

$$(59) \quad \sum_{n \geq 0} \|b_n s\|^2 < \infty, \quad t = k\alpha - \sum_{n \geq 0} [b_n s] q_n \alpha \bmod 1, \text{ for an integer } k.$$

The size of  $c_n(\beta)$ , a key point in the proof of Theorem 6.2, is related to the  $b_n$ 's in the expansion of  $\beta$ . For a non trivial  $\beta$ , when the  $b_n$ 's are bounded, it can be shown by the method of Theorem 6.2 that  $\Phi_\beta$  is ergodic. At the opposite, a fast growth of the sequence  $(b_n)_{n \geq 1}$  implies the non-regularity of the cocycle:

**Theorem 7.3.** *Let  $R_\alpha$  be the rotation by  $\alpha$  with unbounded partial quotients. If  $\beta \notin \mathbb{Z}\alpha + \mathbb{Z}$  satisfies (58) with the lacunarity condition  $\sum_n (b_n/b_{n+1})^2 < \infty$ , then  $\Phi_\beta = \varphi_{\beta, \frac{1}{2}}$  defines a non-regular cocycle (and therefore the skew product  $R_{\alpha, \Phi_\beta}$  is not ergodic).*

*Proof.* By Theorem 7.2, if  $\beta$  satisfies (58), for  $s$  in the set  $\{s : \sum_{n \geq 0} \|b_n s\|^2 < \infty\}$ , there is a solution of (57). Moreover, the set of such  $s$  is uncountable if  $\sum_n (b_n/b_{n+1})^2 < \infty$ . There are thus  $\beta \notin \alpha\mathbb{Z} + \mathbb{Z}$ ,  $s \notin \mathbb{Q}$ ,  $t \in \mathbb{R}$  and  $\psi \in \mathcal{U}(X)$  of modulus 1 such that  $e^{2\pi i s 1_{[0,\beta]}} = e^{2\pi i t} \psi \circ R_\alpha / \psi$ .



For this choice of  $(\beta, s)$ ,  $e^{2\pi i s(1_{[0,\beta]} - 1_{[0,\beta]} \circ R_{\frac{1}{2}})}$  is a multiplicative coboundary. On the other hand, we have shown that  $1_{[0,\beta]} - 1_{[0,\beta]} \circ R_{\frac{1}{2}} = \varphi_{\beta, \frac{1}{2}} = \Phi_\beta$  is not an additive coboundary. Lemma 7.1 shows that  $\overline{\mathcal{E}}(\varphi_{\beta, \frac{1}{2}}) = \{0, \infty\}$ , which implies the non-regularity of  $\Phi_\beta$ .  $\square$

*Remark:* The previous result gives an explicit value of  $\gamma$ , namely  $\gamma = \frac{1}{2}$ , such that  $\varphi_{\beta, \gamma}$  is non regular. A generic result also holds (cf. [Co09]): if  $\beta$  satisfies (58) with the lacunarity condition  $\sum_n (b_n/b_{n+1})^2 < \infty$ , then, for a.e.  $\gamma$ ,  $\varphi_{\beta, \gamma}$  is a non-regular cocycle.

The previous result is for  $\alpha$  of Liouville type. At the opposite, if we take  $\alpha$  with bounded partial quotients, as we have seen (cf. Proposition 3.1), for  $\beta \notin \mathbb{Z}\alpha + \mathbb{Z}$ , the  $\mathbb{Z}$ -valued cocycle  $\Phi_\beta$  is ergodic.

**Remark 7.4.** Let us consider  $\varphi_{\beta, \gamma}$  (cf. Notation 39). For  $\beta, \gamma \in ]0, 1[$ , with  $\beta + \gamma < 1$ , this step function reads  $1_{[0,\beta]} - 1_{[\gamma, \beta + \gamma]}$ . Its Fourier coefficients of  $\varphi_{\beta, \gamma}$  are  $\frac{1}{2\pi i n}(e^{-2\pi i n \beta} - 1)(-e^{2\pi i n \gamma} - 1)$ .

The condition for  $\varphi_{\beta, \gamma}$  to be a coboundary with a transfer function in  $L^2(\mathbb{T})$ , i.e., such that the functional equation  $\varphi_{\beta, \gamma} = R_\alpha g - g$  has a solution  $g$  in  $L^2$ , is

$$(60) \quad \sum_{n \neq 0} \frac{1}{n^2} \frac{\|n\beta\|^2 \|n\gamma\|^2}{\|n\alpha\|^2} < \infty.$$

The following sufficient condition for the existence of an  $L^2$ -solution of the coboundary equation have been given is proved in [CoMa14]: If  $\beta, \gamma$  are in  $H_4(\alpha)$ , then (60) holds and there is  $g$  in  $L^2(\mathbb{T})$  solution of  $\varphi_{\beta, \gamma} = R_\alpha g - g$ .

Therefore, if  $\alpha$  has unbounded partial quotients, there is an uncountable set of pairs of real numbers  $\beta$  and  $\gamma$  such that  $\varphi_{\beta, \gamma}$  is a coboundary  $R_\alpha g - g$  for  $R_\alpha$  with  $g$  in  $L^2$ .

## 8. Questions

**Question 1.** Is there a special measure-theoretic property that permits to single out the elements  $W = \widetilde{S}_g$  from the  $C^{\text{lift}}(T^f)$ ? For example, is it true that if  $S \circ T^k$  has entropy zero for each  $k \in \mathbb{Z}$ , then so is the entropy of  $W$ ?

**Question 2.** (cf. Remark 1.3) Given a flow  $(R_t)$  on  $(Z, \mathcal{D}, \rho)$ , for each measurable subgroup  $G \subset C((R_t)_{t \in \mathbb{R}})$ , can we find a special representation  $T^f$  of  $(R_t)$  such that  $C^{\text{lift}}(T^f)$  “realizes”  $G$ ? (I.e., a measure-theoretic isomorphism  $I$  between the flow and its special representation yields  $I(G) = C^{\text{lift}}(T^f)$ .)

In particular, does there exist a flow  $(R_t)_{t \in \mathbb{R}}$  such that for no special representation  $T^f$  of it we have  $C(T^f) = C^{\text{lift}}(T^f)$ ?

**Question 3.** (cf. Remark 1.3 and Question 2) Can we find  $\alpha$  and  $f$  regular for which  $C^{\text{lift}}(T^f)$  is not closed?

**Question 4.** Assume that  $Tx = x + \alpha$  and  $f : \mathbb{T} \rightarrow \mathbb{R}^+$  is smooth (we recall that then  $T^f$  is rigid). Is it true that  $C^{\text{lift}}(T^f) = C(T^f)$ ?

## 9. Appendix. Centralizer for uniformly rigid special flows

### 9.1. Continuous centralizer of uniformly rigid flows.

Let  $(X, d)$  be a compact metric space and let  $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$  be a continuous flow on it, i.e., it is a one-parameter group of homeomorphisms of  $X$ :  $T_t \in \text{Homeo}(X)$  for  $t \in \mathbb{R}$  and

$$(61) \quad \text{the map } (x, t) \mapsto T_t x \text{ is continuous.}^{11}$$

We then have

$$(62) \quad \text{the map } t \mapsto T_t \text{ is continuous,}$$

where on  $\text{Homeo}(X)$  we consider the uniform topology:  $\rho(V, W) := \sup_{x \in X} (d(Vx, Wx) + d(V^{-1}x, W^{-1}x))$  whenever  $V, W \in \text{Homeo}(X)$  (with this topology  $\text{Homeo}(X)$  becomes a Polish group). Indeed, we only need to show that, whenever  $\epsilon > 0$ , we have  $d(x, T_t x) < \epsilon$  for all  $x \in X$  and  $|t| < \delta$  for some  $\delta > 0$  which results immediately from the uniform continuity of the map  $(x, t) \mapsto T_t x$  on  $X \times [-1, 1]$ .

A flow  $\mathcal{T}$  is called *uniformly rigid* if for some sequence  $s_n \rightarrow \infty$ , we have  $T_{s_n} \rightarrow \text{Id}$  uniformly. We can now repeat the ‘‘measurable’’ proof from [KaLe16] in the continuous setting.

**Proposition 9.1.** *Assume that a flow  $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$  is uniformly rigid. Then the essential topological centralizer  $C^{\text{top}}(\mathcal{T}) / \{T_t : t \in \mathbb{R}\}$  is uncountable.*

*Proof.* Consider

$$H := \{T_t : t \in \mathbb{R}\} \subset \overline{\{T_t : t \in \mathbb{R}\}} := G \subset \text{Homeo}(X),$$

where  $G$  is a Polish group. If  $H$  is a proper subgroup, then it must be a set of first category, and hence, it cannot have only countably many cosets (as  $G$  is Polish without isolated points). If  $H = G$ , then  $H$  itself is Polish, and by (62) the map  $t \mapsto T_t$  is continuous. Since this map is 1-1, by the open map theorem for topological groups, the map  $t \mapsto T_t$  has to be a homeomorphism, and the continuity of the inverse yields a contradiction with the uniform rigidity of the special flow  $\mathcal{T}$ .  $\square$

### Continuous special flows

Let  $(X, d_X)$  be a compact metric space and  $f : X \rightarrow \mathbb{R}^+$  continuous. In particular, for some  $\eta > 0$  we have  $f(x) \geq \eta$  for each  $x \in X$ . Set  $\overline{X}^f = \{(x, r) \in X \times \mathbb{R} : 0 \leq r \leq f(x)\}$ . Then  $\overline{X}^f$  is a compact metric space with the product metric  $d$  (the product of  $d_X$  and the Euclidean metric  $d_{\mathbb{R}}$  on  $\mathbb{R}$ ). Let  $T : X \rightarrow X$  be a homeomorphism. Define the equivalence relation  $\sim$  on  $\overline{X}^f$  with the only non-trivial gluing  $(x, f(x)) \sim (Tx, 0)$ . The resulting space denoted by  $X^f$  is Hausdorff and compact (and we could identify it with  $\{(x, r) : x \in X, 0 \leq r < f(x)\}$ ). Let  $D$  be the quotient metric defined by  $D((x, r), (x', r')) :=$

$$(63) \quad \begin{cases} \inf \{d((x, r), (x_1, r_1)) + d((x'_1, r'_1), (x_2, r_2)) + \dots + d((x'_n, r'_n), (x', r'))\}, \\ \text{where } (x_i, r_i) \sim (x'_i, r'_i), i = 1, \dots, n \}. \end{cases}$$

Then  $T^f$  becomes a continuous flow on the compact metric space  $X^f$ .

### Uniform rigidity of special flows

**Proposition 9.2.** *Let  $T$  be uniformly rigid, that is, for some increasing sequence  $(q_n) \subset \mathbb{N}$  we have  $T^{q_n} \rightarrow \text{Id}$  uniformly. If there exists  $(s_n) \subset \mathbb{R}$  such that  $f^{(q_n)}(\cdot) - s_n \rightarrow 0$  uniformly, then  $T_{s_n}^f \rightarrow \text{Id}$  uniformly.*

*Proof.* For each  $(x, r) \in X^f$ , we have

$$\begin{aligned} D(T_{s_n}^f(x, r), (x, r)) &= D(T_{s_n - f^{(q_n)}(x)}^f T_r^f T_{f^{(q_n)}(x)}^f(x, 0), (x, r)) = \\ D(T_{s_n - f^{(q_n)}(x)}^f T_r^f(T^{q_n}x, 0), (x, r)) &= D(T_{s_n - f^{(q_n)}(x)}^f(T^{q_n}x, r), (x, r)) \leq \\ D(T_{s_n - f^{(q_n)}(x)}^f(T^{q_n}x, r), (T^{q_n}x, r)) &+ D((T^{q_n}x, r), (x, r)) \end{aligned}$$

and the two last summands are small by (62) (if  $n$  is sufficiently large) and the definition of  $D$ .  $\square$

Directly from Proposition 9.1, we obtain the following.

**Corollary 9.3.** *Under the assumptions of Proposition 9.2, the essential (topological) centralizer  $C^{\text{top}}(X^f, T^f)/\{T_t^f : t \in \mathbb{R}\}$  is uncountable.*

## 9.2. Smooth special flows over irrational rotations.

Let us come back to special flows over irrational rotations ( $X = \mathbb{T}$ ,  $Tx = x + \alpha$ ). Let  $f : \mathbb{T} \rightarrow \mathbb{R}^+$ . For simplicity, we assume that  $\int_{\mathbb{T}} f d\mu = 1$  and set  $f_0 := f - 1$ . Then, it follows from [He89] that if  $f$  is absolutely continuous (AC), then  $f^{(q_n)}(\cdot) - q_n \rightarrow 0$  uniformly. Hence  $T^f$  is uniformly rigid.

**Corollary 9.4.** *Let  $Tx = x + \alpha$  and  $f : X \rightarrow \mathbb{R}^+$  be AC. Then the essential topological centralizer of  $T^f$  is uncountable. Moreover, there exists an uncountable set of  $\beta \in \mathbb{T}$  such that the functional equation*

$$(64) \quad f(x + \beta) - f(x) = g(x + \alpha) - g(x)$$

has a solution in continuous functions  $g : X \rightarrow \mathbb{R}$ .

*Proof.* The first part follows from the uniform rigidity and Proposition 9.1, the second one is a consequence of the first one and of the result from [KeMaSe91] on the form of homeomorphisms commuting with  $T^f$ .  $\square$

We will now show a different (direct) proof (cf. [LeMa94]) of the fact that whenever  $f$  is AC then we can solve (64) for uncountably many  $\beta$ .

For this aim select a subsequence  $(q_{n_k})_{k \geq 1}$  of denominators of  $\alpha$  so that

$$(65) \quad \sum_{k \geq 1} \|f_0^{(q_{n_k})}\|_{C(\mathbb{T})} < +\infty \text{ and } \sum_{k \geq 1} \|q_{n_k} \alpha\| < +\infty$$

(remembering that  $f_0^{(q_n)} \rightarrow 0$  uniformly and  $\|q_n \alpha\| \rightarrow 0$ ). We have, for each  $x \in \mathbb{T}$  and  $k \geq 1$ ,

$$f_0^{(q_{n_k})}(x + \alpha) - f_0^{(q_{n_k})}(x) = f_0(x + q_{n_k} \alpha) - f_0(x).$$

By replacing  $x$  by  $x + \sum_{j < k} q_{n_j} \alpha$ , we obtain

$$f_0^{(q_{n_k})}\left(x + \sum_{j=0}^{k-1} q_{n_j} \alpha + \alpha\right) - f_0^{(q_{n_k})}\left(x + \sum_{j=0}^{k-1} q_{n_j} \alpha\right) = f_0\left(x + \sum_{j=0}^{k-1} q_{n_j} \alpha + q_{n_k} \alpha\right) - f_0\left(x + \sum_{j=0}^{k-1} q_{n_j} \alpha\right).$$

Now, the RHS of the above equality is telescopic, and when we sum it up, by (65), we obtain  $f_0(x + \beta) - f_0(x)$  with  $\sum_{k \geq 1} q_{n_k} \alpha = \beta$ , while for the LHS the series

$$\sum_{k \geq 1} f_0^{(q_{n_k})}(x + \beta_k), \text{ where } \beta_k = \sum_{j=0}^{k-1} q_{n_j} \alpha,$$

converges uniformly as it converges absolutely by (65). By (65), we have  $\sum_{k \geq 1} f_0^{(q_{n_k})}(\cdot + \beta_k) = g$ . Hence we obtain (64). Note finally that if in the above reasoning we replace  $q_{n_k}$  by  $\epsilon_k q_{n_k}$ , with  $\epsilon \in \{0, 1\}^{\mathbb{N}}$  (with infinitely many  $k$  for which  $\epsilon_k = 1$ ), using a unicity argument in the Ostrowski expansion of  $\beta$ , we obtain an uncountable set of  $\beta \in \mathbb{T}$  for which we can solve (64).

The above method can be also applied when the roof function  $f = \sum_{n \in \mathbb{Z}} a_n e^{2\pi n x}$  satisfies  $a_n = o(1/|n|)$ . Indeed, as proved in [LeMa94], under this assumption,  $f_0^{(q_n)} \rightarrow 0$  in  $L^2(\mathbb{T})$ . It follows that the corresponding special flow is rigid, whence its essential centralizer is uncountable. But by repeating the above proof, we obtain:

**Proposition 9.5.** *Let  $f = \sum_{n \in \mathbb{Z}} c_n e^{2\pi i n x}$  satisfy  $c_n = o(1/|n|)$ . Then for each irrational  $\alpha$  the set of  $\beta$  for which we can solve (51) with  $g \in L^2(\mathbb{T})$  is uncountable. Equivalently, the essential liftable centralizer is uncountable.*

**Proposition 9.6.** *Let  $f$  be in  $L^2(\mathbb{T})$  such that, for an irrational  $\alpha$  and a strictly increasing sequence  $(r_n)_{n \geq 1}$ ,*

$$\|r_n \alpha\| \rightarrow 0, \quad \|f^{(r_n)}\|_2 \rightarrow 0.$$

*Then the set of  $\beta$  for which we can solve  $f(x + \beta) - f(x) = g(x + \alpha) - g(x)$  with  $g \in L^2(\mathbb{T})$  is uncountable.*

**Remark 9.7.** For the smooth case  $C^2$ , A. Kanigowski gave a Fourier analysis type argument showing that the set of  $\beta$  for which (64) can be solved is residual.

### 9.3. Special flow with Hölderian roof function and trivial liftable centralizer.

The aim of this section is to show the following result (to be compared with Corollary 9.4).

**Proposition 9.8.** *For each  $\alpha$  with bounded partial quotients, there is  $F$  which is Hölder continuous with any Hölder exponent  $0 < \kappa < 1$  and such that the functional equation*

$$(66) \quad F(x + \beta) - F(x) = g(x + \alpha) - g(x)$$

*has a measurable solution  $g$  only for  $\beta \in \mathbb{Z}\alpha + \mathbb{Z}$ . In other words, the liftable centralizer of the special flow  $R_\alpha^F$  is trivial.*

To prove Proposition 9.8, given  $\alpha$  with bounded partial quotients, we will construct below a class of ergodic continuous cocycles  $F$  such that the functional equation (66) has a measurable solution  $g$  only for  $\beta \in \mathbb{Z}\alpha + \mathbb{Z}$ . Our construction is similar to the constructions of ergodic cocycles using lacunary Fourier series, see Volný [Vo03], Brémont [Br10]. We start with two remarks.

1) Recall that a sequence  $\Lambda = (n_k)$  of positive integers is called lacunary if  $\inf_k \frac{n_{k+1}}{n_k} > 1$ . We say that  $f \in L^1(\mathbb{T})$  is a *lacunary* if  $f(x) = \sum_{n \in \Lambda} c_n(f) e^{2\pi i n x}$ , where  $\Lambda$  is a lacunary sequence.

Recall that if  $f$  is lacunary, then, as  $f(x + \beta) - f(x)$  is also lacunary, by a result of M. Herman (edited in [He04]), the cocycle  $f(x + \beta) - f(x)$  is a measurable coboundary if and only if it is a coboundary in  $L^2$ .

Therefore, if  $F$  is lacunary, a measurable solution  $g$  of (66) exists if and only if

$$(67) \quad \sum_{n \neq 0} |c_n(F)|^2 \frac{|\sin(\pi n \beta)|^2}{|\sin(\pi n \alpha)|^2} < \infty.$$

2) Let  $\alpha$  be an irrational with bounded partial quotients. Then, the sequence  $(q_n)$  of denominators of  $\alpha$  is lacunary. Indeed, setting  $A := \max_n a_n$ , for all  $n \geq 3$ , we have:  $q_{n-1} \leq Aq_{n-2} + q_{n-3} \leq (A+1)q_{n-2}$ ; whence

$$(68) \quad q_n \geq q_{n-1} + q_{n-2} \geq \left(1 + \frac{1}{A+1}\right) q_{n-1}.$$

Moreover, see (19), we have:  $\frac{q_k}{q_{k+1} + q_k} \leq q_k \|q_k \alpha\| \leq \frac{q_k}{a_{k+1} q_k + q_{k-1}} \leq \frac{1}{a_{k+1}}$ , so

$$(69) \quad \frac{1}{A+1} \leq q_k \|q_k \alpha\| \leq 1.$$

**Lemma 9.9.** *For each irrational  $\alpha$  and  $n \geq 1$ , we have:*

$$(70) \quad q_1 + q_2 + \dots + q_n \leq 2q_{n+1},$$

$$(71) \quad \frac{1}{q_{n+1}} + \dots + \frac{1}{q_{n+k}} + \dots \leq \frac{C}{q_{n+1}},$$

where  $C = 5 + 2\sqrt{5}$ .

*Proof.* 1) Inequality (70) is clearly satisfied for  $n = 0, 1$ . If we assume that the inequality is true for  $n-1$  and  $n$ , then:  $q_1 + q_2 + \dots + q_{n-1} + q_n + q_{n+1} \leq 2q_n + q_n + q_{n+1} \leq 2(q_n + q_{n+1}) \leq 2q_{n+2}$ , so (70) holds.

2) For  $n \geq 1$  fixed, set  $r_0 = q_n$ ,  $r_1 = q_{n+1}$ ,  $r_{k+1} = r_k + r_{k-1}$ , for  $k \geq 1$ . It follows immediately by induction that  $q_{n+k} \geq r_k$ ,  $\forall k \geq 0$ .

Denote  $c = \frac{1}{\sqrt{5}}$  and let  $\lambda_1 = \frac{\sqrt{5}}{2} + \frac{1}{2}$ ,  $\lambda_2 = -\frac{\sqrt{5}}{2} + \frac{1}{2}$  be the two roots of the polynomial  $\lambda^2 - \lambda - 1$ . Since  $\lambda_j^{\ell+1} = \lambda_j^\ell + \lambda_j^{\ell-1}$  for each  $j = 1, 2$  and  $\ell \geq 1$ , we obtain by induction that:

$$(72) \quad q_{n+k} \geq r_k = c \lambda_1^k (q_{n+1} - \lambda_2 q_n) - c \lambda_2^k (q_{n+1} - \lambda_1 q_n), \quad k \geq 0, n \geq 1.$$

Take  $k \geq 1$ . Since  $\lambda_2 < 0$  and  $\left|1 - \frac{\lambda_1 q_n}{q_{n+1}}\right| < \lambda_1$  (as  $\lambda_1 > 1$ ), from (72), we obtain

$$q_{n+k} \geq c \lambda_1^k \left(1 - \lambda_1 \left(\frac{|\lambda_2|}{\lambda_1}\right)^k\right) q_{n+1} = c \lambda_1^k q_{n+1} \left(1 - |\lambda_2| \left(\frac{|\lambda_2|}{\lambda_1}\right)^{k-1}\right) \geq c \lambda_1^k q_{n+1} (1 - |\lambda_2|).$$

It follows that for  $k \geq 1$ , we have  $q_{n+k} \geq c_1 \lambda_1^k q_{n+1}$ , with  $c_1 := \frac{3\sqrt{5}-5}{10}$ . Finally, we obtain

$$q_{n+1} \sum_{k \geq 1} q_{n+k}^{-1} \leq c_1^{-1} \sum_{k \geq 1} \lambda_1^{-k} = c_1^{-1} (\lambda_1 - 1)^{-1} = 5 + 2\sqrt{5}. \quad \square$$

Let  $s = (m_k)$  be an increasing sequence of positive integers and  $\delta > 0$ . We set

$$(73) \quad F_1(x) = \sum_{k \geq 1} \frac{\sin(2\pi q_k x)}{q_k}, \quad F_s(x) = \sum_{k \geq 1} \frac{\sin(2\pi q_{m_k} x)}{q_{m_k}}, \quad F = F_s + \delta F_1.$$

**Proposition 9.10.** *Let  $\alpha$  be such that the sequence  $(q_n)$  is lacunary (in particular, we can take  $\alpha$  with bounded partial quotients). If  $\beta$  is such that equation (66) for  $F$  has a measurable solution, then  $\beta \in \mathbb{Z}\alpha + \mathbb{Z}$ . The function  $F$  satisfies the regularity condition:*

$$(74) \quad |F(x+h) - F(x)| \leq C|h| \log\left(\frac{1}{|h|}\right).$$

In particular,  $F$  is Hölderian with any exponent  $0 < \kappa < 1$ .

Moreover, if  $\alpha$  has bounded partial quotients, then the sequence  $s$  and  $\delta$  can be chosen so that the extension map  $R_{\alpha, F}$  on  $\mathbb{T} \times \mathbb{R} : (x, y) \rightarrow (x + \alpha, y + F(x))$  is ergodic.

*Proof.* 1) Since, by assumption, the sequence  $(q_k)$  of denominators of  $\alpha$  is lacunary, the function  $F_s + \delta F_1$  is lacunary and so is the function  $(R_\beta - I)(F_s + \delta F_1)$ . It follows by [He04] that if  $(R_\beta - I)(F_s + \delta F_1)$  is a measurable coboundary, then equation (66) can be solved in  $L^2$ , which (by (67)) implies:

$$\delta^2 \sum_{j \notin s} \frac{1}{q_j^2} \frac{\|q_j \beta\|^2}{\|q_j \alpha\|^2} + (1 + \delta)^2 \sum_k \frac{1}{q_{m_k}^2} \frac{\|q_{m_k} \beta\|^2}{\|q_{m_k} \alpha\|^2} < \infty.$$

It follows that  $\sum_k \frac{1}{q_k^2} \frac{\|q_k \beta\|^2}{\|q_k \alpha\|^2} < \infty$ , which implies that there is  $k_0$  such that  $\|q_k \beta\| \leq \frac{1}{4} q_k \|q_k \alpha\|$ , for  $k \geq k_0$  and therefore, by Lemma 2.3,  $\beta \in \mathbb{Z}\alpha + \mathbb{Z}$ .

2) By Lemma 9.9, for any  $L \geq 1$ , we have:  $|F(x+h) - F(x)|$

$$\leq C' \sum_{k=1}^{L-1} \frac{|\sin(2\pi q_k(x+h)) - \sin(2\pi q_k x)|}{q_k} + 2 \sum_{k \geq L} \frac{1}{q_k} \leq C'|h|L + \frac{C'}{q_L}$$

for a constant  $C' > 0$ . Recall that  $q_n \geq C' \lambda_1^n$ , with  $\lambda_1 > 1$  (cf. the proof of Lemma 9.9). It suffices to show (74) for  $h \in ]0, 1/\lambda_1]$ . Since  $h \leq \lambda_1^{-1}$ , there exists  $y = y(h) \geq 1$  such that  $hy = \lambda_1^{-y}$ . We have  $\lambda_1^{-y} \geq h$ , whence  $y \leq \frac{1}{\ln \lambda_1} \ln \frac{1}{h}$ .

For  $0 \leq h \leq e^{-\frac{1}{2}}$ , we have  $1 \leq 2 \ln \frac{1}{h}$ . Let us take  $L = [y] + 1$  (that is,  $L = L(h)$ ). We have

$$\begin{aligned} hL + \frac{1}{q_L} &= O(hy + \lambda_1^{-y}) = O(hy) = \\ &= O\left(h \frac{1}{\ln \lambda_1} \ln \frac{1}{|h|}\right) = O\left(h \ln \frac{1}{|h|}\right) \end{aligned}$$

for  $0 \leq h \leq e^{-\frac{1}{2}}$ , hence (74) holds.

Now, for  $0 < \kappa < 1$  and  $0 < h \leq 1$ , we have  $h \ln \frac{1}{h} \leq \frac{1}{1-\kappa} h^\kappa$ .<sup>12</sup> Therefore  $F$  is Hölderian with exponent  $\kappa$ .

3) Let  $G(x) = \sum_{k \geq 1} \frac{1}{u_k} \sin(2\pi v_k x)$ , where  $(v_k)$  is a sequence of integers and  $(u_k)$  is a sequence of positive numbers such that  $\sum 1/u_k < \infty$ . If  $(t_k)$  is an increasing sequence

<sup>12</sup>This inequality is equivalent to  $\ln \frac{1}{h^{1-\kappa}} \leq \frac{1}{h^{1-\kappa}}$ .

of integers, the ergodic sums of  $G$  at time  $t_n$  reads:

$$(75) \quad G^{(t_n)}(x) = \sum_{k \geq 1} \frac{1}{u_k} \frac{\sin(\pi v_k t_n \alpha)}{\sin(\pi v_k \alpha)} \sin(\pi v_k (2x + (t_n - 1)\alpha)) = A_n + B_n + C_n,$$

where in (75)  $A_n, B_n, C_n$  are respectively the partial sums  $\sum_{k < n}, \sum_{k=n}, \sum_{k > n}$ .

Now, take  $G = F_s$  given by (73) and consider  $t_n = q_{m_n}$ . The decomposition (75) yields (for some constant  $c' > 0$ ):

$$(76) \quad |A_n| \leq \sum_{k \leq n-1} \frac{1}{q_{m_k}} \frac{|\sin(\pi q_{m_k} q_{m_n} \alpha)|}{|\sin(\pi q_{m_k} \alpha)|} \leq c' \frac{1}{q_{m_{n+1}}} \sum_{k \leq n-1} q_{m_k};$$

using (69),

$$(77) \quad |B_n| = \frac{1}{q_{m_n}} \frac{|\sin(\pi q_{m_n} q_{m_n} \alpha)|}{|\sin(\pi q_{m_n} \alpha)|} \asymp 1;$$

$$(78) \quad |C_n| \leq \sum_{k \geq n+1} \frac{1}{q_{m_k}} \frac{|\sin(\pi q_{m_k} q_{m_n} \alpha)|}{|\sin(\pi q_{m_k} \alpha)|} \leq c' q_{m_n} \sum_{k \geq n+1} \frac{1}{q_{m_k}}.$$

By (70) and (71), we have

$$|A_n| \leq 2 \frac{q_{m_{n-1}+1}}{q_{m_{n+1}}}, \quad |C_n| \leq C \frac{q_{m_n}}{q_{m_{n+1}}}.$$

It follows that we can select the sequence  $s = (m_k)$  such that the terms  $A_n, C_n$  cannot cancel the contribution of  $B_n$ . That is,  $B_n$  is bounded away from zero (and is clearly bounded) and the behavior of  $F_s^{(q_{m_n})}$  is similar to the behavior of  $B_n \sin(2\pi q_{m_n} x)$ . This, by Lemma 2.1 yields an uncountable set of essential values of  $F_s$ .

Remark that we have boundedness of the ergodic sums of  $F_1$  at time  $q_n$ :  $\|F_1^{(q_n)}\|_2 \leq 2\pi$ . It follows that, for  $\delta > 0$  small enough, the above property of the existence of an uncountable set of essential values for  $F_s$  is still satisfied for  $F_s + \delta F_1$  (in other words, we obtain a stability of ergodicity of  $F_s$  by some perturbations).  $\square$

## REFERENCES

- [Aa97] J. Aaronson, *An Introduction to Infinite Ergodic Theory*, Mathematical Surveys and Monographs **50**, AMS, Providence, 1997.
- [AmKa42] W. Ambrose, S. Kakutani, *Structure and continuity of measurable flows*, Duke Math. J. **9** (1942), 25-42.
- [Br10] J. Brémont, *Ergodic non-abelian smooth extensions of an irrational rotation*. J. Lond. Math. Soc. (2) **81** (2010), no. 2, 457-476.
- [Co76] J.-P. Conze, *Équipartition et ergodicité de transformations cylindriques*, Publications des Séminaires de mathématiques et informatique de Rennes, PSMIR, 1976, fasc. 2, sémin. de proba. I, available at <http://www.numdam.org>
- [Co09] J.-P. Conze, *Recurrence, ergodicity and invariant measures for cocycles over a rotation*, pp. 45-70 in *Contemporary Mathematics* **485**, AMS, Providence, RI, 2009.
- [CoP14] J.-P. Conze and A. Pieckniewska, *On multiple ergodicity of affine cocycles over irrational rotations*, Israel J. Math., 201(2): 543-584, 2014.

- [CoMa14] J.-P. Conze and J. Marco, *Remarks on step over rotations, centralizers and coboundaries*, In *Ergodic theory and dynamical systems*, De Gruyter, Proc. Math., pp. 93-115. De Gruyter, Berlin, 2014.
- [CoRa09] J.-P. Conze, A. Raugi, *On the ergodic decomposition for a cocycle*, Colloquium Mathematicum, vol. 117, 2009 No.1.
- [CoFoSi82] I.P. Cornfeld, S.V. Fomin, Y.G. Sinai, *Ergodic Theory*, Springer-Verlag, New York, 1982.
- [Fa02] B. Fayad, *Analytic mixing reparametrizations of irrational flows*, Ergodic Theory Dynam. Systems 22 (2002), 437-468.
- [FaKa16] B. Fayad, A. Kanigowski, *Multiple mixing for a class of conservative surface flows*, Inventiones Math. **203** (2016), 555-614.
- [FrLe04] K. Frączek, M. Lemańczyk, *A class of special flows over irrational rotations which is disjoint from mixing flows*, Ergodic Theory Dynam. Systems 24 (2004), 1083-1095.
- [FrLe06] K. Frączek, M. Lemańczyk, *On mild mixing of special flows over irrational rotations under piecewise smooth maps*, Ergodic Theory Dynam. Systems **26** (2006), 719-738.
- [FrLe09] K. Frączek, M. Lemańczyk, *Smooth singular flows in dimension 2 with the minimal self-joining property*, Monatshefte Math. **156** (2009), 11-45.
- [FrLeLe07] K. Frączek, M. Lemańczyk, E. Lesigne, *Mild mixing property for special flows under piecewise constant functions*, Discrete Contin. Dynam. Systems **19** (2007), 691-710.
- [Gl03] E. Glasner, *Ergodic Theory via Joinings*, vol. 101 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2003.
- [GuPa06] M. Guenais, F. Parreau, *Valeurs propres de transformations liées aux rotations irrationnelles et aux fonctions en escalier*, 2006.
- [He89] M.R. Herman, *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations*, Thèse de Doctorat d'Etat, Université Paris-Sud, 1976.
- [He04] M.R. Herman,  *$L^2$ -regularity of measurable solutions of a finite-difference equation of the circle*, Ergodic Theory and Dynamical Systems, vol. 24, no 05 (2004), p. 1277-1281.
- [Ka14] A. Kanigowski, *Ratner's property for special flows over irrational rotations under functions of bounded variation.II*, Colloq. Math. **136** (2014), 125-147.
- [Ka15] A. Kanigowski, *Ratner's property for special flows over irrational rotations under functions of bounded variation*, Ergodic Theory Dynam. Systems **35** (2015), 915-934.
- [KaKu15] A. Kanigowski, J. Kułaga-Przymus, *Ratner's property and mild mixing for smooth flows on surfaces*, Ergodic Theory Dynam. Systems **36** (2016), 2512-2537.
- [KaKuUl16] A. Kanigowski, J. Kułaga-Przymus, C. Ulcigrai, *Multiple mixing and parabolic divergence in smooth area-preserving flows on higher genus surfaces*, arXiv:1606.09189.
- [KaLe16] A. Kanigowski, M. Lemańczyk, *Flows with Ratner's property have discrete essential centralizer*, Studia Math. **237** (2017), 185-194
- [KeMaSe91] H. Keynes, N. Markley, M. Sears, *The structure of automorphisms of real suspension flows*, Ergodic Theory Dynam. Systems **11** (1991), 349-364.
- [KhSi92] K.M. Khanin, Ya.G. Sinai, *Mixing of some classes of special flows over rotations of the circle*, Funct. Anal. Appl. **26** (1992), 155-169.
- [Ko72] A.V. Kočergin, *On the absence of mixing in special flows over the rotation of a circle and in flows on a two-dimensional torus*, (Russian) Dokl. Akad. Nauk SSSR 205 (1972), 949-952.
- [Ko75] A.V. Kočergin, *Mixing in special flows over a rearrangement of segments and in smooth flows on surfaces*, Mat. USSR Sbornik 25 (1975), 471-502.



- [Ko76] A.V. Kochergin, *Non-degenerated saddles and absence of mixing*, Mat. Zametki 19 (1976), 453-468.
- [Ko04] A.V. Kochergin, *Nondegenerate fixed points and mixing in flows on a two-dimensional torus. II.*, (Russian) Mat. Sb. 195 (2004), 15-46.
- [KrLi91] C. Kraaikamp, P. Liardet, *Good approximations and continued fractions*, Proc. Amer. Math. Soc. **112** (1991), 303-309.
- [Ku12] J. Kulaga, *On the self-similarity problem for smooth flows on orientable surfaces*, Ergodic Theory Dynam. Systems **32** (2012), 1615-1660.
- [Le00] M. Lemańczyk, *Sur l'absence de mélange pour des flots spéciaux au dessus d'une rotation irrationnelle*, Coll. Math. **84/85** (2000), 29-41.
- [LeMa94] M. Lemańczyk, C. Mauduit, *Ergodicity of a class of cocycles over irrational rotations*, Journal London Math. Soc. **49** (1994), 124-132.
- [LePa07] M. Lemańczyk, F. Parreau, *Special flows over irrational rotations with the simple convolutions property*, unpublished preprint available at <http://www-users.mat.umk.pl/~mlem/publications.php>.
- [LePaVo96] M. Lemańczyk, F. Parreau, D. Volný, *Ergodic properties of real cocycles and pseudo-homogeneous Banach spaces*, Trans. Amer. Math. Soc. **348** (1996), 4919-4938.
- [Me85] K. Merrill, *Cohomology of steps functions under irrational rotations*, Israel J. Math., **52** (1985), 93-134.
- [vNe32] J. von Neumann, *Zur Operatorenmethode in der Klassischen Mechanik*, Annals Math. **33** (1932), 587-642.
- [OrRuWe92] D. Ornstein, D. Rudolph, B. Weiss, *Equivalence of measure-preserving transformations*, Memoirs AMS (1982), vol. 37, no 262.
- [Ra83] M. Ratner, *Horocycle flows, joinings and rigidity of products*, Annals Math. **118** (1983), 277-313.
- [Sch09] D. Scheglov, *Absence of mixing for smooth flows on genus two surfaces*, J. Mod. Dyn. **3** (2009), 13-34.
- [Sc77] K. Schmidt, *Lectures on Cocycles of Ergodic Transformations Groups*, Lect. Notes in Math. vol. 1, Mac Millan Co. of India (1977).
- [Th95] J.-P. Thouvenot, *Some properties and applications of joinings in ergodic theory*, Ergodic Th. and its Connections with Harmonic Anal., London Math. Soc. 1995, 207-235.
- [Ul07] C. Ulcigrai, *Mixing for suspension flows over interval exchange transformations*, Ergodic Theory Dynam. Systems **27** (2007), 991-1035.
- [Ul11] C. Ulcigrai, *Absence of mixing in area-preserving flows on surfaces*, Annals Math. **173** (2) (2011), 1743- 1778.
- [Ve69] W.A. Veech, *Strict ergodicity in zero dimensional dynamical systems and the Kronecker-Weyl theorem modulo 2*. Trans. A.M.S., 140:1-33, 1969.
- [Vo03] D. Volný, *Completely squashable smooth ergodic cocycles over irrational rotations*, Topol. Methods Nonlinear Anal. 22 (2003), no. 2, 331-344.

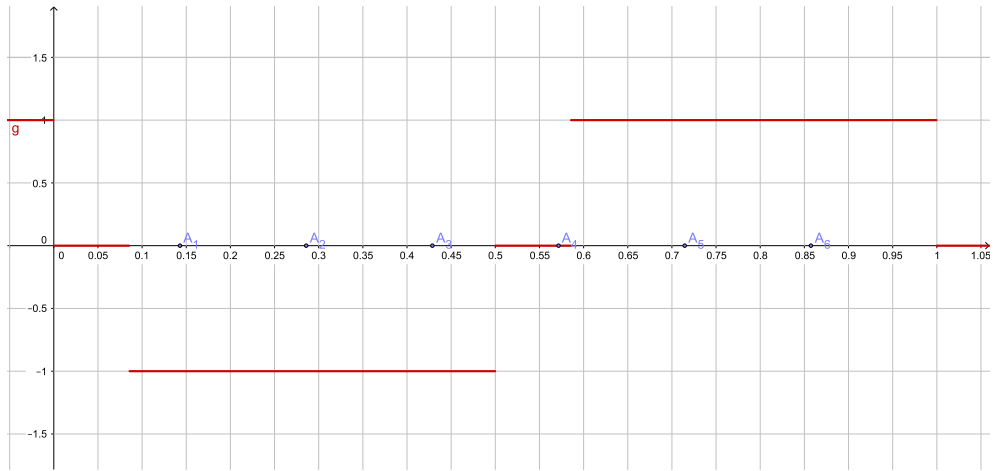


Fig.1 rotation  $\alpha = \pi - 3$ ,  $\beta = 2 - \sqrt{2}$ ,  $\Phi = \Phi_\beta$ ,  $A_k = \frac{k}{7}$ , graph of  $\Phi_\beta$

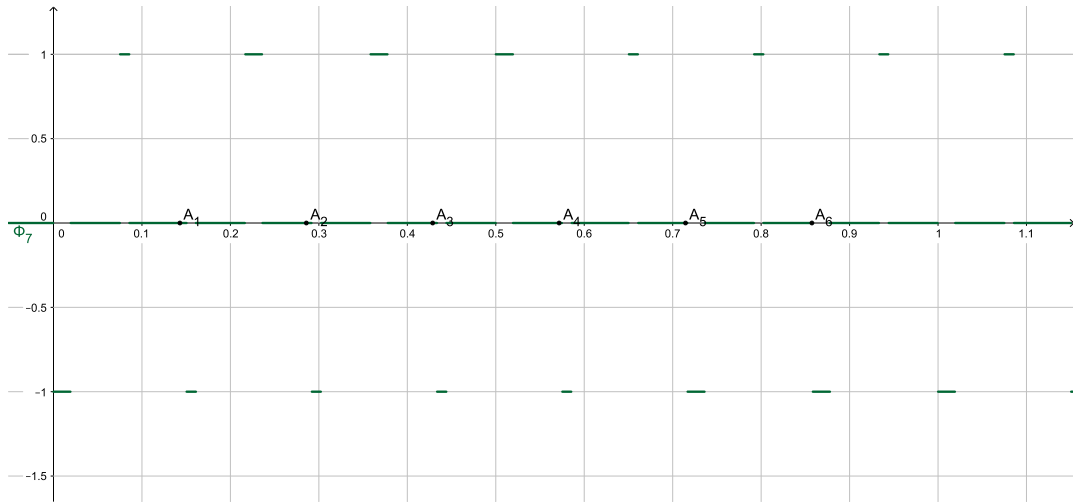


Fig.2  $\Phi_7 = \Phi_\beta^{(7)}$ ,  $A_k = \frac{k}{7}$ , graph of  $\Phi_\beta^{(7)}$

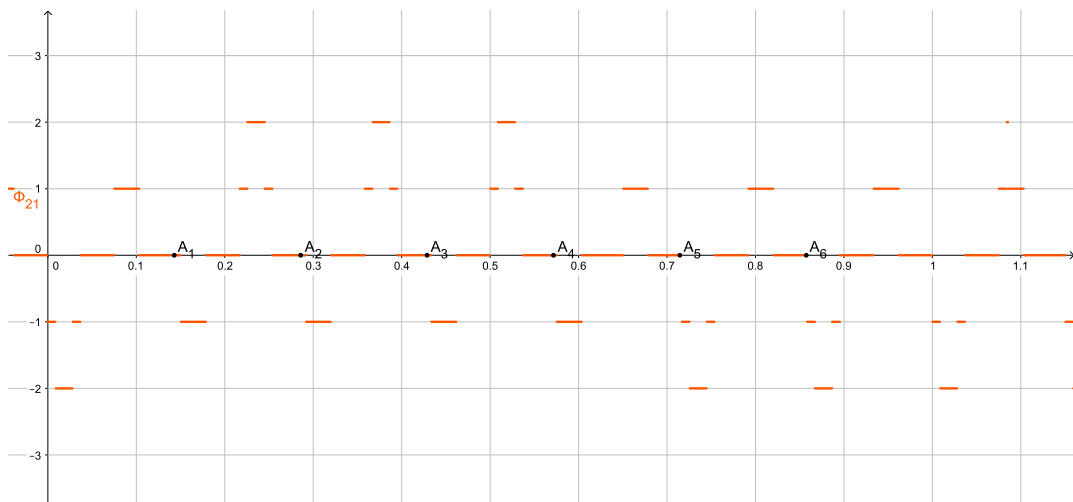


Fig.3  $\Phi_{21} = \Phi_\beta^{(21)} = \Phi_\beta^{(7)} + \Phi_\beta^{(7)}(\cdot + 7\alpha) + \Phi_\beta^{(7)}(\cdot + 14\alpha)$ ,  $A_k = \frac{k}{7}$ , graph of  $\Phi_\beta^{(21)}$

JEAN-PIERRE CONZE, IRMAR, UMR CNRS 6625, UNIVERSITÉ DE RENNES I

CAMPUS DE BEAULIEU, 35042 RENNES CEDEX, FRANCE

*E-mail address:* `conze@univ-rennes1.fr`

MARIUSZ LEMAŃCZYK, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, NICOLAUS COPERNICUS UNIVERSITY,

CHOPIN STREET 12/18, 87-100 TORUŃ, POLAND

*E-mail address:* `mlem@mat.umk.pl`