Dynamics of $\mathcal{B}$-free sets: a view through the window

Stanisław Kasjan*†1, Gerhard Keller†2, and Mariusz Lemańczyk*†1

[1]Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Toruń, Poland
[2]Department of Mathematics, University of Erlangen-Nürnberg, Germany

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Abstract

Let $\mathcal{B}$ be an infinite subset of $\{1, 2, \dots\}$. We characterize arithmetic and dynamical properties of the $\mathcal{B}$-free set $\mathcal{F}_\mathcal{B}$ through group theoretical, topological and measure theoretic properties of a set W (called the *window*) associated with $\mathcal{B}$. This point of view stems from the interpretation of the set $\mathcal{F}_\mathcal{B}$ as a weak model set. Our main results are: $\mathcal{B}$ is taut if and only if the window is Haar regular; the dynamical system associated to $\mathcal{F}_\mathcal{B}$ is a Toeplitz system if and only if the window is topologically regular; the dynamical system associated to $\mathcal{F}_\mathcal{B}$ is proximal if and only if the window has empty interior; and the dynamical system associated to $\mathcal{F}_\mathcal{B}$ has the "naïvely expected" maximal equicontinuous factor if and only if the interior of the window is aperiodic.

1 Introduction and main results

For any given set $\mathcal{B} \subseteq \mathbb{N} = \{1, 2, \dots\}$ one can define its *set of multiples*

$$\mathcal{M}_\mathcal{B} := \bigcup_{b\in\mathcal{B}} b\mathbb{Z}$$

and the set of *$\mathcal{B}$-free numbers*

$$\mathcal{F}_\mathcal{B} := \mathbb{Z} \setminus \mathcal{M}_\mathcal{B} .$$

The investigation of structural properties of $\mathcal{M}_\mathcal{B}$ or, equivalently, of $\mathcal{F}_\mathcal{B}$ has a long history (see the monograph [10] and the recent paper [4] for references), and dynamical systems theory provides some useful tools for this. Namely, denote by $\eta \in \{0, 1\}^\mathbb{Z}$ the characteristic function of $\mathcal{F}_\mathcal{B}$, i.e. $\eta(n) = 1$ if and only if $n \in \mathcal{F}_\mathcal{B}$, and consider the orbit closure X_η of η in the shift dynamical system $(\{0, 1\}^\mathbb{Z}, \sigma)$, where σ stands for the left shift. Then topological dynamics and ergodic theory provide a wealth of concepts to describe various aspects of the structure of η, see [16] which originated this point of view by studying the set of square-free numbers, and also [1], [4] which continued this line of research.

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In this paper we continue to provide a dictionary that characterizes arithmetic properties of $\mathcal{B}$ in terms of dynamical properties of X_η, and, as an intermediate step, also in terms of topological and measure theoretic properties of a pair (H, W) associated with the passage from $\mathcal{B}$ to X_η, where H is a compact abelian group and W a compact subset of H. This latter point of view is borrowed from the theory of weak model sets, which applies here, because $\mathcal{F}_\mathcal{B}$ is a particular example of such a set, see e.g. [3, 13]. Finally the Chinese Remainder Theorem allows us to interpret our dynamical results combinatorially.

In order to formulate our main results, we need to recall some notions from the theory of sets of multiples [10] and also to introduce some further notation. Let $\mathcal{B}$ be a non-empty subset of $\mathbb{N}$.

- $\mathcal{B}$ is *primitive*, if there are no $b, b' \in \mathcal{B}$ with $b \mid b'$. From any set $\mathcal{B} \subseteq \mathbb{N}$ one can remove all multiples of other numbers in $\mathcal{B}$, which results in the set

$$\mathcal{B}^{prim} := \mathcal{B} \setminus \bigcup_{b\in\mathcal{B}} b \cdot (\mathbb{N} \setminus \{1\}) . \tag{1}$$

  $\mathcal{B}^{prim}$ is primitive by construction, and $\mathcal{M}_\mathcal{B} = \mathcal{M}_{\mathcal{B}^{prim}}$.

- $\mathcal{B}$ is *taut*, if $\boldsymbol{\delta}(\mathcal{M}_{\mathcal{B}\setminus\{b\}}) < \boldsymbol{\delta}(\mathcal{M}_\mathcal{B})$ for each $b \in \mathcal{B}$, where $\boldsymbol{\delta}(\mathcal{M}_\mathcal{B}) := \lim_{n\to\infty} \frac{1}{\log n} \sum_{k\leqslant n, k\in\mathcal{M}_\mathcal{B}} k^{-1}$ denotes the logarithmic density of this set, which is known to exist by the Theorem of Davenport and Erdös [6, 7]. So a set is taut, if removing any single point from it changes its set of multiples drastically and not only by “a few points”.

- $\tilde{H} := \prod_{b\in\mathcal{B}} \mathbb{Z}/b\mathbb{Z}$ and $\Delta : \mathbb{Z} \to \tilde{H}$, $\Delta(n) = (n, n, \dots)$ – the canonical diagonal embedding.

- $H := \overline{\Delta(\mathbb{Z})}$ is a compact abelian group, and we denote by m_H its normalised Haar measure.

- $R_{\Delta(1)} : H \to H$ denotes the rotation by $\Delta(1)$, i.e. $(R_{\Delta(1)}h)_b = (h_b + 1) \bmod b$ for all $b \in \mathcal{B}$.

- The *window* is defined as

$$W := \{h \in H : h_b \neq 0 \ (\forall b \in \mathcal{B})\}. \tag{2}$$

- $\varphi : H \to \{0, 1\}^\mathbb{Z}$ is the coding function: $\varphi(h)(n) = 1$, if and only if $R^n_{\Delta(1)}h \in W$, equivalently, if and only if $h_b + n \neq 0 \bmod b$ for all $b \in \mathcal{B}$.

- By $S, S' \subset \mathcal{B}$ we always mean *finite* subsets.

- The topology on H is generated by the (open and closed) cylinder sets

$$U_S(h) := \{h' \in H : \forall b \in S : h_b = h'_b\}, \text{ defined for finite } S \subset \mathcal{B} \text{ and } h \in H .$$

A recurring theme of the main results in this paper is to characterize arithmetic and dynamical properties of a $\mathcal{B}$-free set $\mathcal{F}_\mathcal{B}$ through group theoretical, topological and measure theoretic properties of the window W defined above.

Remark 1.1. With the notation introduced above, we can write

$$X_\eta = \overline{\varphi(\Delta(\mathbb{Z}))} .$$

This is certainly a subset of $X_\varphi := \overline{\varphi(H)}$, the set studied in [13] under the name $\mathcal{M}^G_W$. In Proposition 2.2 we show that $X_\eta = X_\varphi$ when $\mathcal{B}$ has *light tails* (see Subsection 2.5 for a definition), but we do not know whether also tautness of $\mathcal{B}$ suffices (see also Subsection 2.5).

1.1 Tautness as a measure theoretic property

Theorem A. [1] *Suppose that the set $\mathcal{B}$ is primitive. Then the following are equivalent:*

(i) *$\mathcal{B}$ is taut.*

(ii) *The window W is* Haar regular, *i.e.* $\mathrm{supp}(m_H|_W) = W$.

Moreover, these properties imply

(iii) $\overline{\Delta(\mathbb{Z}) \cap W} = W$.

The proof of the theorem is provided in Section 2. The concept of a *Haar regular window* was introduced in [14] in the context of general weak model sets.

Given a set $\mathcal{B} \subset \mathbb{N}$, one says that $h := (h_b)_{b\in\mathcal{B}} \in \mathbb{Z}^{\mathcal{B}}$ satisfies the CRT (Chinese Remainder Theorem) if for each finite $S \subset \mathcal{B}$ there exists $n \in \mathbb{Z}$ such that

$$h_b = n \bmod b \quad \text{for each } b \in S. \tag{3}$$

Clearly,

$$h \text{ satisfies the CRT iff } h \in H.$$

We are looking for solutions of (3) with $n \in \mathcal{F}_{\mathcal{B}}$. If for h as above we can solve (3) with $n = n_S \in \mathcal{F}_{\mathcal{B}}$ for all finite $S \subset \mathcal{B}$, then we say that h satisfies the $\mathcal{B}$-free CRT. A necessary condition for $h = (h_b)_{b\in\mathcal{B}}$ to satisfy the $\mathcal{B}$-free CRT is, of course, that $h_b \neq 0 \bmod b$ for each $b \in \mathcal{B}$, and a moment's reflection shows that

$$h \text{ satisfies the } \mathcal{B}\text{-free CRT iff } h \in \overline{\Delta(\mathbb{Z}) \cap W}\,. \tag{4}$$

Therefore the implication $(i) \Rightarrow (iii)$ of Theorem A is an immediate consequence of the following proposition.

Proposition 1.1. *Assume that $\mathcal{B}$ is taut. Let $h \in W$ and $S \subset \mathcal{B}$ finite. Then the set of $\mathcal{B}$*-free *integers n that solve $n = h_b \bmod b$ for $b \in S$ has asymptotic density $m_H(U_S(h) \cap W) > 0$.*

In Subsection 2.4 we provide a sequence $\mathcal{B}$, which is not taut, but for which $\overline{\Delta(\mathbb{Z}) \cap W} = W$ (Example 2.2). Hence (iii) of Theorem A is not equivalent to (i) and (ii). Here we provide two simpler examples which throw some light on property (iii). Denote by $\mathcal{P} \subseteq \mathbb{N}$ the set of all *prime numbers*.

Example 1.1. If $\mathcal{B} = \mathcal{P}$ then $H = \prod_{p\in\mathcal{P}} \mathbb{Z}/p\mathbb{Z}$, W is uncountable (although of Haar measure zero) and $\overline{\Delta(\mathbb{Z}) \cap W} \neq W$, since for each n we find $p \in \mathcal{P}$ such that $p \mid n$, so $n = 0 \bmod p$.

Example 1.2. If $\mathcal{B} \subset \mathcal{P}$ is *thin*, i.e. if $\sum_{p\in\mathcal{B}} 1/p < +\infty$, then $\overline{\Delta(\mathbb{Z}) \cap W} = W$ in view of (4), because each $h \in H$ satisfies the $\mathcal{B}$-free CRT. Indeed, if $S \subset \mathcal{B}$ is finite and $n = h_b \bmod b$ for $b \in S$, then $n + \mathrm{lcm}(S)\mathbb{Z}$ is the set of all solutions to this system of congruences. Moreover, if $h \in W$, then $\gcd(n, b) = 1$ for all $b \in S$. We only need to find $r \in \mathbb{Z}$ so that $n + r\,\mathrm{lcm}(S)$ is a prime number which is not in $\mathcal{B}$. The latter follows from Dirichlet's theorem: The set of prime numbers contained in $n + \mathrm{lcm}(S)\mathbb{Z}$ is not thin. Of course this is a special case of Theorem A.

Remark 1.2. Denote by $\nu_\eta := m_H \circ \varphi^{-1}$ the *Mirsky measure* on X_η. There are two independent proofs of the fact that the two equivalent conditions from Theorem A imply that the measure preserving dynamical system $(X_\eta, \sigma, \nu_\eta)$ is isomorphic to the group rotation $(H, R_{\Delta(1)}, m_H)$: In [4, Theorem F] it is proved that this is implied by (i). That it is also a direct consequence of (ii) follows - in the more

[1] The authors are indebted to J. Kułaga-Przymus for pointing out the relevance of [10, Lemma 1.17] for the proof of this theorem.

general context of model sets - from [14]. The proof uses our observation that W is aperiodic (see Proposition 5.1). To see this, denote by $H_W := \{h \in H : W + h = W\}$ the period group of W and by $H_W^{Haar} := \{h \in H : m_H((W+h)\triangle W) = 0\}$ its group of *Haar periods*. It is easily seen that $H_W = H_W^{Haar}$ for Haar regular W, in particular whenever the sequence $\mathcal{B}$ is taut. Hence, if W is aperiodic, it is also Haar aperiodic, and this is what is needed to apply the general theorem from [14] to the present context.

A word of caution is in order at this point: Althoug, in the $\mathcal{B}$-free context, the window W is always aperiodic (Proposition 5.1), this is not necessarily true for its Haar regularization $W_{reg} := \operatorname{supp}(m_H|_W)$, because that window is not of the same arithmetic type as W. On the other hand, as proved in [4, Theorem C], each non-taut set $\mathcal{B}$ can be modified into a taut set $\mathcal{B}'$ whose corresponding Mirsky measure $\nu_{\eta'}$ coincides with ν_η (as a measure on $\{0,1\}^{\mathbb{Z}}$). The (arithmetic!) window $W' \subseteq H'$ defined by $\mathcal{B}'$ is then aperiodic and Haar regular, and we suspect that it to be closely related to $W_{reg} \subseteq H$.

1.2 The proximal and the Toeplitz case

From [4, Theorem A] we know that X_η has a unique minimal subset M. In Lemma 3.10, we prove that $M = \overline{\varphi(C_\varphi)}$, where C_φ denotes the set of continuity points of $\varphi : H \to \{0,1\}^{\mathbb{Z}}$, see also [13, Lemma 6.3]. M is degenerate to a singleton, namely to $M = \{(\dots,0,0,0,\dots)\}$, if and only if $\operatorname{int}(W) = \emptyset$ [13], and we collect a number of equivalent characterizations of this extreme case in Theorem C below. Assuming primitivity of $\mathcal{B}$ and property (*iii*) of Theorem A, we prove the following equivalent characterizations of minimality of (X_η, σ), i.e. of $M = X_\eta$, in Subsection 3.2. For $S \subset \mathcal{B}$ let

$$\mathcal{A}_S := \{\gcd(b, \operatorname{lcm}(S)) : b \in \mathcal{B}\}, \tag{5}$$

and note that $\mathcal{F}_{\mathcal{A}_S} \subseteq \mathcal{F}_{\mathcal{B}}$, because $b \mid m$ for some $b \in \mathcal{B}$ implies $\gcd(b, \operatorname{lcm}(S)) \mid m$ for any $S \subset \mathcal{B}$. Let

$$\mathcal{A}_\infty := \{n \in \mathbb{N} : \forall_{S \subset \mathcal{B}}\ \exists_{S' : S \subseteq S'} : n \in \mathcal{A}_{S'} \setminus S'\}. \tag{6}$$

In Lemma 3.2 we prove: If $(S_k)_k$ is a filtration of $\mathcal{B}$ with finite sets, then

$$\limsup_{k\to\infty} (\mathcal{A}_{S_k} \setminus S_k) = \mathcal{A}_\infty\ . \tag{7}$$

Theorem B. *Suppose that $\mathcal{B}$ is primitive. Consider the following list of properties:*

(B1) The window W is topologically regular, i.e. $\overline{\operatorname{int}(W)} = W$.
(B2) $\mathcal{F}_{\mathcal{B}} = \bigcup_{S \subset \mathcal{B}\ finite} \mathcal{F}_{\mathcal{A}_S}$.
(B3) $\mathcal{A}_\infty = \emptyset$.
(B4) There are no $d \in \mathbb{N}$ and no infinite pairwise coprime set $\mathcal{A} \subseteq \mathbb{N} \setminus \{1\}$ such that $d\,\mathcal{A} \subseteq \mathcal{B}$.
(B5) $\eta = \varphi(0)$ is a Toeplitz sequence (see [8], [12] for the definition) different from $(\dots,0,0,0,\dots)$.
(B6) $0 \in C_\varphi$ and $\varphi(0) \neq (\dots,0,0,0,\dots)$.
(B7) $\eta \in M$ and $\eta \neq (\dots,0,0,0,\dots)$.
(B8) X_η is minimal., i.e. $X_\eta = M$, and $\operatorname{card}(X_\eta) > 1$.
(B9) The dynamics on X_η is a minimal almost 1-1 extension of $(H, R_{\Delta(1)})$, the rotation by $\Delta(1)$ on H.

a) (B1) - (B6) are all equivalent, and each of these conditions implies that $\mathcal{B}$ is taut.
b) (B7) and (B8) are equivalent.
c) Each of (B1) - (B6) implies (B9).
d) (B9) implies (B7) and (B8).
e) If $\overline{\Delta(\mathbb{Z}) \cap W} = W$ (in particular if $\mathcal{B}$ is taut), then (B1) - (B9) are all equivalent.

Remark 1.3. One ingredient of the proof of Theorem B is the observation that the set $\mathcal{B}$ is taut whenever η is a Toeplitz sequence. This was pointed out to us by A. Bartnicka who also gave a proof of it, which we recall in Lemma 3.7 below.

Moreover, we can interpret the result purely arithmetically as follows: If $\mathcal{B}$ is primitive and satisfies (B4) then the set of elements for which the $\mathcal{B}$-free CRT holds is topologically regular, i.e. it contains a dense subset of points for which all sufficiently close points satisfying the CRT satisfy also the $\mathcal{B}$-free CRT.

The following characterization of regular Toeplitz sequences is included in Proposition 4.1 in Subsection 4.2, where also the precise definition of regularity of a Toeplitz sequence is recalled.

Proposition 1.2. *Assume that $\overline{\mathrm{int}(W)} = W$. Then the Toeplitz sequence η is regular, if and only if $m_H(\partial W) = 0$.*

In Subsection 4.2 we also provide examples of sets $\mathcal{B}$ that give rise to regular Toeplitz sequences and others giving rise to irregular Toeplitz sequences. Note also that $m_H(\partial W) = 0$ if and only if $\inf_{S \subset \mathcal{B}} \overline{d}(\mathcal{M}_{\mathcal{A}_S} \setminus \mathcal{M}_{\mathcal{B}}) = 0$, see Lemma 4.3, and observe that $m_H(\partial W) = 0$ implies unique ergodicity of the dynamics on X_η [13, Theorem 2c].

The next theorem is complementary to Theorem B. Most of its equivalences follow from results in [4] and [13] and are proved in Subsection 3.3. They do not rely on the more advanced arithmetic concept of tautness.

Theorem C. *The following are equivalent:*
(C1) $\mathrm{int}(W) = \emptyset$
(C2) $\bigcup_{S \subset \mathcal{B}\ \text{finite}} \mathcal{F}_{\mathcal{A}_S} = \emptyset$*, i.e.* $\mathcal{F}_{\mathcal{A}_S} = \emptyset$ *for all finite* $S \subset \mathcal{B}$.
(C3) $\forall S \subset \mathcal{B} : 1 \in \mathcal{A}_S$.
(C4) $\mathcal{B}$ *contains an infinite pairwise coprime subset.*
(C5) If $C \subseteq \mathbb{N}$ *is finite and if* $\mathcal{B} \subseteq \mathcal{M}_C$*, then* $1 \in C$.
(C6) $M = \{(\dots, 0, 0, 0, \dots)\}$.
(C7) The dynamics on X_η *are proximal.*

Remark 1.4. Under the conditions of Theorem C no element of W is stable, that is, for each h staisfying the $\mathcal{B}$-free CRT there is an element $h' \in \mathbb{Z}^{\mathcal{B}}$ arbitrarily close to h which satisfies the CRT but not the $\mathcal{B}$-free CRT.

1.3 The maximal equicontinuous factor

We finish with a result that identifies the maximal equicontinuous factor of the dynamics on X_η and answers Question 3.14 in [4]. Given a subset $A \subseteq H$, denote by

$$H_A := \{h \in H : A + h = A\}$$

the *period group* of A. The set $A \subseteq H$ is *topologically aperiodic*, if $H_A = \{0\}$. Observe also that $H_{\mathrm{int}(A)}$ is a closed subgroup of H, whenever A is closed [14, Lemma 6.1].

In Proposition 5.1 we prove that $H_W = \{0\}$ whenever $\mathcal{B}$ is primitive. If $\mathrm{int}(W) = \emptyset$, then of course $H_{\mathrm{int}(W)} = H$. If $\mathrm{int}(W) \neq \emptyset$, the situation is more complicated: $H_{\mathrm{int}(W)}$ is obviously always a strict subgroup of H, and very often $H_{\mathrm{int}(W)} = \{0\}$, but there are examples where $H_{\mathrm{int}(W)}$ is a non-trivial strict subgroup of H, see Subsection 5.3. In any case, however, $H_{\mathrm{int}(W)}$ determines the maximal equicontinuous factor. The following is proved in [14, Theorem A2]:

Theorem *The translation by* $\Delta(1) + H_{\mathrm{int}(W)}$ *on* $H/H_{\mathrm{int}(W)}$ *is the maximal equicontinuous factor of the dynamics on* X_η. (8)

Let $S_1 \subset S_2 \subset \dots$ be any filtration of $\mathcal{B}$ by finite sets. In Subsection 5.1 we define divisors d_k of $\mathrm{lcm}(S_k)$:

$$d_k := \lim_{j\to\infty} \gcd(s_k, c_{k+j}), \quad \text{where} \quad s_k := \mathrm{lcm}(S_k) \quad \text{and} \quad c_l := \text{minimal period of } \mathcal{M}_{\mathcal{A}_{S_l}}. \tag{9}$$

By Remark 5.1 we have $\frac{s_k}{d_k} \mid \frac{s_{k+1}}{d_{k+1}}$ for any k. The sequences (s_k), (d_k) and (c_k) determine $H_{\mathrm{int}(W)}$ in the following way:

Proposition 1.3. *a)*

$$0 \to H_{\mathrm{int}(W)} \to H \cong \varprojlim \mathbb{Z}/s_k\mathbb{Z} \to \varprojlim \mathbb{Z}/d_k\mathbb{Z} \to 0$$

is an exact sequence.[2]

b) $H_{\mathrm{int}(W)} \cong \varprojlim \mathbb{Z}/\frac{s_k}{d_k}\mathbb{Z}$.

c) $H/H_{\mathrm{int}(W)} \cong \varprojlim \mathbb{Z}/d_k\mathbb{Z}$.

d) $H_{\mathrm{int}(W)} = \{0\}$ *if and only if* $s_k = d_k$ *for each* $k \in \mathbb{N}$*, equivalently if for each* $b \in \mathcal{B}$ *there is* $n > 0$ *such that* b *divides* c_n.

Theorem D. *a) The translation by* $(1, 1, \dots)$ *on* $H/H_{\mathrm{int}(W)} \cong \varprojlim \mathbb{Z}/d_k\mathbb{Z}$ *is the maximal equicontinuous factor of the dynamics on* X_η.

b) In case d) of Proposition 1.3, the translation by $\Delta(1)$ *on* $H \cong \varprojlim \mathbb{Z}/s_k\mathbb{Z}$ *is the maximal equicontinuous factor of the dynamics on* X_η.

In Subsection 5.3 we provide a number of examples illustrating this theorem.

Remark 1.5. In [4], the following set Y is defined: [3]

$$Y := \left\{x \in \{0,1\}^{\mathbb{Z}} : \mathrm{card}(\mathrm{supp}(x) \bmod b) = b - 1 \ \forall b \in \mathcal{B}\right\}.$$

Observe that $\mathrm{card}(\mathrm{supp}(x) \bmod b) \leqslant b - 1$ for all $x \in X_\eta$ and $b \in \mathcal{B}$. [4] Proposition 3.27 of [4] asserts that $(H, R_{\Delta(1)})$ is the maximal equicontinuous factor of (X_η, S), whenever $X_\eta \subseteq Y$. Hence, in that case, $H_{\overline{\mathrm{int}\, W}} = H_{\mathrm{int}\, W} = \{0\} = H_W$ by Theorem D and Proposition 5.1. This is the second one of the following two implications:

$$W = \overline{\mathrm{int}\, W} \quad \Rightarrow \quad X_\eta \subseteq Y \quad \Rightarrow \quad H_W = H_{\overline{\mathrm{int}\, W}}. \tag{10}$$

The first one is proved in Proposition 3.3.

[2] A sequence of abelian groups and homomorphisms $\dots \longrightarrow M_{k-1} \xrightarrow{f_{k-1}} M_k \xrightarrow{f_k} M_{k+1} \to \dots$ is called *exact* if the kernel of f_k is equal to the image of f_{k-1} for any k. In particular, a sequence

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

is exact, when f is injective, the kernel of g equals the image of f and g is surjective. We say that it is a "short exact sequence". In particular, the homomorphism g induces an isomorphism $M'' \cong M/f(M')$ in this case.

[3] Versions of this set occur also in [15] and [2].

[4] Indeed, if $\mathrm{card}(\mathrm{supp}(x) \bmod b) = b$ for some $x \in X_\eta$ and $b \in \mathcal{B}$, then this happens on some integer interval $[-M, M]$, and hence $\mathrm{card}(\mathrm{supp}(\eta) \bmod b) = b$, which contradicts the fact that $\mathrm{supp}(\eta) \subseteq \mathcal{F}_{\mathcal{B}}$.

2 Tautness of $\mathcal{B}$ and Haar regularity of W

2.1 Arithmetic of $\mathcal{B}$ and topology of W, part I

Definition 2.1. *Let $\mathcal{M} \subseteq \mathbb{N}$.*
a) The upper *resp.* lower density *of $\mathcal{M}$ is*

$$\overline{d}(\mathcal{M}) = \limsup_{N\to\infty} \frac{1}{N} \operatorname{card}(\mathcal{M} \cap \{1,\dots,N\}) \ \textit{resp.}\ \underline{d}(\mathcal{M}) = \liminf_{N\to\infty} \frac{1}{N} \operatorname{card}(\mathcal{M} \cap \{1,\dots,N\})$$

If the limit exists, we write $d(\mathcal{M})$.
b) The logarithmic density *of $\mathcal{M}$ is*

$$\delta(\mathcal{M}) = \lim_{N\to\infty} \frac{1}{\log N} \sum_{n\in\mathcal{M}\cap\{1,\dots,N\}} \frac{1}{n}$$

whenever the limit exists.

The theorem of Davenport and Erdös [6, 7] asserts that $\delta(\mathcal{M}_\mathcal{B}) = \underline{d}(\mathcal{M}_\mathcal{B})$ exists for any subset $\mathcal{B} \subseteq \mathbb{N}$.

Definition 2.2. *$\mathcal{B} \subseteq \mathbb{N} \setminus \{1\}$ is a* Behrend sequence, *if $\delta(\mathcal{M}_\mathcal{B}) = 1$.*

Recall that $\mathcal{B}$ is taut, if $\delta(\mathcal{M}_{\mathcal{B}\setminus\{b\}}) < \delta(\mathcal{M}_\mathcal{B})$ for each $b \in \mathcal{B}$. The following is a corollary to a theorem of Behrend [5]:

Proposition 2.1. *A set $\mathcal{B} \subseteq \mathbb{N}$ is taut, if and only if it is primitive and there are no $q \in \mathbb{N}$ and no Behrend set $\mathcal{A} \subseteq \mathbb{N} \setminus \{1\}$ such that $q\mathcal{A} \subseteq \mathcal{B}$ [10, Corollary 0.19].*

This motivates the next definition:

Definition 2.3. *A set $\mathcal{B} \subseteq \mathbb{N}$ is* pre-taut, *if there are no $q \in \mathbb{N}$ and Behrend set $\mathcal{A} \subseteq \mathbb{N} \setminus \{1\}$ such that $q\mathcal{A} \subseteq \mathcal{B}$.*

Lemma 2.1. *Let $\mathcal{B} \subseteq \mathbb{N}$ and $c \in \mathbb{N}$.*
a) If $c\mathcal{B}$ is pre-taut, then also $\mathcal{B}$ is pre-taut. Moreover, $\mathcal{B}$ is taut if and only if $c\mathcal{B}$ is taut.
b) Each subset of a (pre-)taut set is (pre-)taut.
c) A finite union of pre-taut sets is pre-taut.
d) If $\mathcal{B}$ is taut, then $\mathcal{B} = \{1\}$ or $d(\mathcal{M}_\mathcal{B}) \neq 1$ (possibly non-existing). Equivalently, if $d(\mathcal{M}_\mathcal{B}) = 1$, then $\mathcal{B} = \{1\}$ or $\mathcal{B}$ is not taut.
e) If $\mathcal{B}$ is pre-taut, then $1 \in \mathcal{B}$ or $d(\mathcal{M}_\mathcal{B}) \neq 1$ (possibly non-existing). Equivalently, if $d(\mathcal{M}_\mathcal{B}) = 1$, then $1 \in \mathcal{B}$ or $\mathcal{B}$ is not pre-taut.

Proof. a) The first implication is obvious. It is also clear that $\mathcal{B}$ is primitive if and only if $c\mathcal{B}$ is primitive. Moreover,

$$\begin{aligned}\mathcal{B} \text{ is taut} &\Leftrightarrow \forall b \in \mathcal{B} : \underline{d}(\mathcal{M}_\mathcal{B}) > \underline{d}(\mathcal{M}_{\mathcal{B}\setminus\{b\}}) \Leftrightarrow \forall b \in \mathcal{B} : c^{-1}\underline{d}(\mathcal{M}_\mathcal{B}) > c^{-1}\underline{d}(\mathcal{M}_{\mathcal{B}\setminus\{b\}})\\ &\Leftrightarrow \forall b \in \mathcal{B} : \underline{d}(\mathcal{M}_{c\mathcal{B}}) > \underline{d}(\mathcal{M}_{c(\mathcal{B}\setminus\{b\})}) = \underline{d}(\mathcal{M}_{c\mathcal{B}\setminus\{cb\}})\\ &\Leftrightarrow \forall b' \in c\mathcal{B} : \underline{d}(\mathcal{M}_{c\mathcal{B}}) > \underline{d}(\mathcal{M}_{c\mathcal{B}\setminus\{b'\}})\\ &\Leftrightarrow c\mathcal{B} \text{ is taut.}\end{aligned}$$

b) is obvious (see Remark 2.1).
c) follows from [10, Corollary 0.14], see also [4, Proposition 2.33].
d) Suppose that $\mathcal{B}$ is taut. Then $\mathcal{B}$ is primitive, and $d(\mathcal{M}_{\mathcal{B}}) \neq 1$ unless $1 \in \mathcal{B}$ by [10, Corollary 0.19]. Hence $d(\mathcal{M}_{\mathcal{B}}) \neq 1$ or $\mathcal{B} = \{1\}$.
e) follows directly from Definition 2.3 [5]. □

Remark 2.1. $\mathcal{B}$ is taut if and only if it is pre-taut and primitive. If $\mathcal{B}$ is pre-taut, then $\mathcal{B}^{prim}$ is taut in view of Lemma 2.1b.

For $q \in \mathbb{N}$ and $\mathcal{B} \subseteq \mathbb{Z}$ let

$$\mathcal{B}'(q) = \left\{ \frac{b}{\gcd(b,q)} : b \in \mathcal{B} \right\},$$

and note that $1 \in \mathcal{B}'(q)$ if and only if $q \in \mathcal{M}_{\mathcal{B}}$.

Lemma 2.2. *Let $q \in \mathbb{N}$, $\mathcal{B}, C \subseteq \mathbb{Z}$, and $qC \subseteq \mathcal{M}_{\mathcal{B}}$. Then $\mathcal{M}_C \subseteq \mathcal{M}_{\mathcal{B}'(q)}$.*

Proof. Let $c \in C$. There are $\ell \in \mathbb{Z}$ and $b \in \mathcal{B}$ such that $qc = \ell b$. Since $q \mid \ell b$, it follows that $q \mid \ell \gcd(b,q)$, thus $k = \frac{\ell \gcd(b,q)}{q}$ is an integer. We have

$$c = \frac{\ell b}{q} = k \cdot \frac{b}{\gcd(b,q)} \in \mathcal{M}_{\mathcal{B}'(q)}\ .$$

This shows that $C \subseteq \mathcal{M}_{\mathcal{B}'(q)}$ and hence also $\mathcal{M}_C \subseteq \mathcal{M}_{\mathcal{B}'(q)}$. □

Lemma 2.3. *Let $\mathcal{B} \subseteq \mathbb{N}$ and $q \in \mathbb{N}$.*
a) If $\mathcal{B}$ is pre-taut, then $\mathcal{B}'(q)$ is pre-taut.
b) If $\mathcal{B}$ is taut, then $\mathcal{B}'(q)$ is a finite disjoint union of taut sets $\mathcal{B}_i'$ defined below in the proof of a).
c) If $d(\mathcal{M}_{\mathcal{B}}) = 1$, then $d(\mathcal{M}_{\mathcal{B}'(q)}) = 1$.
d) If $\mathcal{B} = \bigcup_{i=1}^N C_i$ and if $d(\mathcal{M}_{\mathcal{B}}) = 1$, then $d(\mathcal{M}_{C_i}) = 1$ for at least one $i \in \{1, \dots, N\}$.

Proof. Let $I := \left\{ \frac{q}{\gcd(b,q)} : b \in \mathcal{B} \right\}$. For $i \in I$ denote $\mathcal{B}_i := \left\{ b \in \mathcal{B} : \frac{q}{\gcd(b,q)} = i \right\}$ and $\mathcal{B}_i' := \left\{ \frac{b}{\gcd(b,q)} : b \in \mathcal{B}_i \right\}$. Then I is finite, $\mathcal{B} = \bigcup_{i \in I} \mathcal{B}_i$, and $\mathcal{B}'(q) = \bigcup_{i \in I} \mathcal{B}_i'$. Moreover, $\mathcal{B}_i = \left\{ \frac{q}{i} b' : b' \in \mathcal{B}_i' \right\} = \frac{q}{i} \mathcal{B}_i'$.

a) If $\mathcal{B}$ is pre-taut, then all $\mathcal{B}_i$ are pre-taut (Lemma 2.1b), then all $\mathcal{B}_i'$ are pre-taut (Lemma 2.1a), and then $\mathcal{B}'(q)$ is pre-taut (Lemma 2.1c).
b) If $\mathcal{B}$ is taut, then all $\mathcal{B}_i$ are taut (Lemma 2.1b), and then all $\mathcal{B}_i'$ are taut (Lemma 2.1a).
c) As $\mathcal{B} \subseteq \mathcal{M}_{\mathcal{B}'(q)}$, we have also $\mathcal{M}_{\mathcal{B}} \subseteq \mathcal{M}_{\mathcal{B}'(q)}$.
d) If $\mathcal{B}$ is Behrend, then at least one of the sets C_i is Behrend [10, Corollary 0.14], and so $d(\mathcal{M}_{C_i}) = 1$. Otherwise $1 \in \mathcal{B}$, so that $1 \in C_i$ for some i, whence $\mathcal{M}_{C_i} = \mathbb{Z}$. □

Lemma 2.4. *(compare [4, Proposition 4.25]) Assume that $\mathcal{B} \subseteq \mathbb{N}$ is taut and $d(\mathcal{M}_C) = 1$ for some $C \subseteq \mathbb{Z}$. If $qC \subseteq \mathcal{M}_{\mathcal{B}}$ for some $q \geqslant 1$, then $b \mid q$ for some $b \in \mathcal{B}$.*

Proof. By Lemma 2.2, $\mathcal{M}_C \subseteq \mathcal{M}_{\mathcal{B}'(q)}$, so that $d(\mathcal{M}_{\mathcal{B}'(q)}) = 1$. Then $\mathcal{B}'(q) = \{1\}$ or $\mathcal{B}'(q)$ is not taut (Lemma 2.1d). If $\mathcal{B}'(q) = \{1\}$, then $\gcd(b,q) = b$ for all $b \in \mathcal{B}$, i.e $b \mid q$ for all $b \in \mathcal{B}$, which is impossible because $\mathcal{B}$ is infinite. Hence $\mathcal{B}'(q)$ is not taut. On the other hand, as $\mathcal{B}$ is taut by assumption, $\mathcal{B}'(q)$ is a finite union of taut sets $\mathcal{B}_i'$ (Lemma 2.3b). As $d(\mathcal{M}_{\mathcal{B}'(q)}) = 1$, also $d(\mathcal{M}_{\mathcal{B}_i'}) = 1$ for at least one of the sets $\mathcal{B}_i'$ (Lemma 2.3d), so that $\mathcal{B}_i' = \{1\}$ for this set (Lemma 2.1d). This implies $q \in \mathcal{M}_{\mathcal{B}}$. □

[5] Note that d) follows from e) and Remark 2.2.

Recall that the topology on H is generated by the (open and closed) cylinder sets

$$U_S(h) := \{h' \in H : \forall b \in S : h_b = h'_b\}, \text{ defined for finite } S \subset \mathcal{B} \text{ and } h \in H\,,$$

and recall also the definition of $\mathcal{A}_S := \{\gcd(b, \mathrm{lcm}(S)) : b \in \mathcal{B}\}$. Note that $\mathcal{A}_S$ is finite and $S \subseteq \mathcal{A}_S$.

Lemma 2.5. *Let $U = U_S(\Delta(n))$ for some $S \subset \mathcal{B}$ and $n \in \mathbb{Z}$.*
a) If $n \in \mathcal{M}_S$, then $U \cap W = \emptyset$.
b) If $U \cap W = \emptyset$, then $n + \mathrm{lcm}(S) \cdot \mathbb{Z} \subseteq \mathcal{M}_{\mathcal{B}\cap\mathcal{A}_S}$.
c) There is a filtration of $\mathcal{B}$ by finite sets S for which $\mathcal{B} \cap \mathcal{A}_S = S$.
d) If $\mathcal{B} \cap \mathcal{A}_S = S$, then $n \in \mathcal{M}_S$ iff $U \cap W = \emptyset$ iff $n + \mathrm{lcm}(S) \cdot \mathbb{Z} \subseteq \mathcal{M}_S$.

Proof. a) This follows immediately from the definitions of $U_S(\Delta(n))$ and W.
b) For each $h \in U$ there is $b \in \mathcal{B}$ such that $h_b = 0$. As U is compact, the Heine-Borel argument produces a finite set $S' \subset \mathcal{B}$ such that for each $h \in U$ there is $b \in S'$ such that $h_b = 0$. Let $s = \mathrm{lcm}(S)$. This observation applies in particular to all $h \in \Delta(n + s\mathbb{Z}) \subseteq U_S(\Delta(n)) = U$. That means, for each $k \in \mathbb{Z}$ there is $b_k \in S'$ such that $b_k \mid n + sk$. In other words: $n + s\mathbb{Z} \subset \mathcal{M}_{S'}$. The set S' need not be primitive automatically, but we can replace it w.l.o.g. by a primitive subset without changing its set of multiples. Then, as S' is finite, it is taut. Denote $q = \gcd(n, s)$ and $C = \frac{n}{q} + \frac{s}{q} \cdot \mathbb{Z}$. Then $qC = n + s\mathbb{Z} \subseteq \mathcal{M}_{S'}$, and as $\gcd(n/q, s/q) = 1$, $d(\mathcal{M}_C) = 1$ (Dirichlet, see [4, Corollary 4.24]). Now Lemma 2.4 shows that $b \mid q = \gcd(n, s)$ for some $b \in S'$. In particular, $n \in b\mathbb{Z}$ and $b \mid s = \mathrm{lcm}(S)$ for that $b \in S'$, so that $b = \gcd(b, s) \in \mathcal{B} \cap \mathcal{A}_S$ and $n + \mathrm{lcm}(S) \cdot \mathbb{Z} \subseteq b\mathbb{Z} \subseteq \mathcal{M}_{\mathcal{B}\cap\mathcal{A}_S}$.
c) It suffices to prove that for any finite $S \subset \mathcal{B}$ there exists a finite $S' \subset \mathcal{B}$ with $\mathcal{B} \cap \mathcal{A}_{S'} = S'$. So let $S \subset \mathcal{B}$ and $S' := \mathcal{B} \cap \mathcal{A}_S$. S' is finite, because $\mathcal{A}_S$ is finite, and obviously $S \subseteq S' \subseteq \mathcal{B} \cap \mathcal{A}_{S'}$. As each $b' \in S' \subseteq \mathcal{A}_S$ divides $\mathrm{lcm}(S)$, also $\mathrm{lcm}(S')$ divides $\mathrm{lcm}(S)$. Therefore $\mathrm{lcm}(S') = \mathrm{lcm}(S)$, so that $\mathcal{A}_{S'} = \mathcal{A}_S$. Hence $S' = \mathcal{B} \cap \mathcal{A}_{S'}$.
d) This follows from a) and b). □

2.2 Proof of Theorem A

Let $\mathcal{B} = \{b_1, b_2, \dots\}$ be primitive, and denote $S_1 \subset S_2 \subset \dots \subset \mathcal{B}$ a filtration of $\mathcal{B}$ by finite sets S_k. Let $s_k = \mathrm{lcm}(S_k)$. We can assume without loss of generality that $b \mid s_k \Rightarrow b \in S_k$ holds for all $b \in \mathcal{B}$ and all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, the collection of all cylinder sets $U_{S_k}(h)$, $h \in H$, can be written explicitly as

$$\mathcal{Z}_k := \{U_{S_k}(\Delta(n)) : n = 1, \dots, s_k\}\,.$$

Suppose first that $\mathcal{B}$ is not taut. Then it contains a scaled copy $c\mathcal{A}$ of a Behrend set $\mathcal{A} \subseteq \{2, 3, \dots\}$. Enlarging $\mathcal{A}$, if necessary, we can assume that $c\mathcal{A} = \mathcal{B} \cap c\mathbb{Z}$. (As $\mathcal{B}$ is primitive, also the enlarged $\mathcal{A}$ does not contain the number 1.) Let $a_0 > 1$ be the smallest element of $\mathcal{A}$ and denote $b_0 = ca_0$. Let $H_0 = \{h \in H : h_{b_0} \in c\mathbb{Z}\}$. Then H_0 is open and closed, and we will show that $H_0 \cap W \neq \emptyset$ but $m_H(H_0 \cap W) = 0$, so that W is not Haar regular.

First observe that $(\Delta(c))_{b_0} = c \in c\mathbb{Z}$, so that $\Delta(c) \in H_0$. Suppose for a contradiction that $H_0 \cap W = \emptyset$. Then $\Delta(c) \notin W$, i.e. there is $b \in \mathcal{B}$ such that $c \in b\mathbb{Z}$. Hence $c\mathcal{A} \subseteq b\mathbb{Z}$, so that $c\mathcal{A} = \{b\}$, because $b \in \mathcal{B}$ and $\mathcal{B}$ is primitive. Hence $b = ca_0 = b_0$, so that $\mathcal{A} = \{a_0\}$, a contradiction, as $\mathcal{A}$ is Behrend.

We turn to the proof of $m_H(H_0 \cap W) = 0$. Let $\mathcal{H}_W^\ell = \{n \in \{0, \dots, s_\ell - 1\} : U_{S_\ell}(\Delta(n)) \cap H_0 \cap W \neq \emptyset\}$. It suffices to show that $\sum_{n\in\mathcal{H}_W^\ell} m_H(U_{S_\ell}(\Delta(n)) \to 0$ as $\ell \to \infty$. As all cylinder sets $U_{S_\ell}(\Delta(n))$ have identical Haar measure s_ℓ^{-1}, this is equivalent to $\#\mathcal{H}_W^\ell / s_\ell \to 0$ as $\ell \to \infty$. So let ℓ be so large that $b_0 \in S_\ell$. Denote $\mathcal{A}^\ell = \{a \in \mathcal{A} : ca \mid s_\ell\}$. As $c\mathcal{A} \subseteq \mathcal{B}$, the sequence $(\mathcal{A}^\ell)_\ell$ is increasing and exhausts the set $\mathcal{A}$.

If $n \in \mathcal{H}_W^\ell$, then $n \in c\mathbb{Z}$ and, by Lemma 2.5a, $n \in \mathcal{F}_{S_\ell}$. Hence $n = cn' \in \mathcal{F}_{S_\ell}$ for some $n' \in \mathbb{Z}$. Suppose for a contradiction that $n' \in \mathcal{M}_{\mathcal{A}^\ell}$, i.e. there are $k \in \mathbb{Z}$ and $a \in \mathcal{A}_\ell$ such that $n' = ka$. Then $n = kca$, where $ca \in \mathcal{B}$ and $ca \mid s_\ell$, so that $ca \in S_\ell$, which contradicts $n \in \mathcal{F}_{S_\ell}$. Hence $n' \in \mathcal{F}_{\mathcal{A}^\ell}$ so that $n \in c\mathcal{F}_{\mathcal{A}^\ell} = c(\mathbb{Z} \setminus \mathcal{M}_{\mathcal{A}^\ell})$. As $\mathcal{A}$ is Behrend, $\overline{d}(\mathbb{Z} \setminus \mathcal{M}_{\mathcal{A}^\ell}) \to 0$ as $\ell \to \infty$. Hence

$$\#\mathcal{H}_W^\ell / s_\ell \leqslant \#\left(c(\mathbb{Z} \setminus \mathcal{M}_{\mathcal{A}^\ell}) \cap [0, s_l)\right)/s_\ell \leq \#\left((\mathbb{Z} \setminus \mathcal{M}_{\mathcal{A}^\ell}) \cap [0, s_l)\right)/s_\ell = d(\mathbb{Z} \setminus \mathcal{M}_{\mathcal{A}^\ell}) \to 0 \,.$$

Suppose now that $\mathcal{B}$ is taut. We must show that for any $k \in \mathbb{N}$ and $U \in \mathcal{Z}_k$

$$U \cap W = \emptyset \quad \text{or} \quad m_H(U \cap W) > 0 \,.$$

So fix some $U = U_{S_k}(\Delta(n))$ such that $m_H(U \cap W) = 0$. We have to show that $U \cap W = \emptyset$. Observe first that $U_{S_k}(\Delta(m)) = U$ if and only if $m \in s_k\mathbb{Z} + n$. For $\ell > k$ let

$$\mathcal{G}_\ell := (s_k\mathbb{Z} + n) \cap \{m \in \mathbb{Z} : U_{S_\ell}(\Delta(m)) \cap W = \emptyset\} = (s_k\mathbb{Z} + n) \cap \mathcal{M}_{S_\ell},$$

where we used Lemma 2.5c for the last equality. Observe that

$$\mathcal{G}_\ell = \mathcal{G}_\ell + s_\ell\mathbb{Z} = (\mathcal{G}_\ell \cap [0, s_\ell)) + s_\ell\mathbb{Z} \,.$$

Hence, for each $\ell > k$,

$$\begin{aligned} \underline{d}\left((s_k\mathbb{Z} + n) \cap \mathcal{M}_\mathcal{B}\right) &= \liminf_{t\to\infty} \frac{\#\left((s_k\mathbb{Z} + n) \cap \mathcal{M}_\mathcal{B} \cap [0, t)\right)}{t} \geqslant \liminf_{t\to\infty} \frac{\#\left((s_k\mathbb{Z} + n) \cap \mathcal{M}_{S_\ell} \cap [0, t)\right)}{t} \\ &= \liminf_{t\to\infty} \frac{\#\left(\mathcal{G}_\ell \cap [0, t)\right)}{t} = \frac{\#\left(\mathcal{G}_\ell \cap [0, s_\ell)\right)}{s_\ell}. \end{aligned}$$

As all $U' \in \mathcal{Z}_\ell$ have identical Haar measure $m_H(U') = s_\ell^{-1}$ and as $m_H(U \setminus W) = m_H(U)$ by assumption, it follows that

$$\begin{aligned} \underline{d}\left((s_k\mathbb{Z} + n) \cap \mathcal{M}_\mathcal{B}\right) &\geqslant \limsup_{\ell\to\infty} \frac{\#\left(\mathcal{G}_\ell \cap [0, s_\ell)\right)}{s_\ell} = \limsup_{\ell\to\infty} m_H\left(\bigcup_{U' \in \mathcal{Z}_\ell, U' \subseteq U \setminus W} U'\right) \\ &= m_H(U \setminus W) = m_H(U) = s_k^{-1} = d(s_k\mathbb{Z} + n) \,, \end{aligned}$$

so that

$$d\left((s_k\mathbb{Z} + n) \cap \mathcal{M}_\mathcal{B}\right) = d(s_k\mathbb{Z} + n) \,.$$

Let $q = \gcd(s_k, n)$, $a' = s_k/q$ and $r' = n/q$. Then $\gcd(a', r') = 1$ and $q\mathbb{Z} \cap \mathcal{M}_\mathcal{B} = q\mathbb{Z} \cap \mathcal{M}_{q\cdot\mathcal{B}'(q)}$, in particular $(s_k\mathbb{Z} + n) \cap \mathcal{M}_\mathcal{B} = (s_k\mathbb{Z} + n) \cap \mathcal{M}_{q\cdot\mathcal{B}'(q)}$. Hence

$$\begin{aligned} d\left((a'\mathbb{Z} + r') \cap \mathcal{M}_{\mathcal{B}'(q)}\right) &= q \cdot d\left(q \cdot \left((a'\mathbb{Z} + r') \cap \mathcal{M}_{\mathcal{B}'(q)}\right)\right) = q \cdot d\left((s_k\mathbb{Z} + n) \cap \mathcal{M}_{q\cdot\mathcal{B}'(q)}\right) \\ &= q \cdot d\left((s_k\mathbb{Z} + n) \cap \mathcal{M}_\mathcal{B}\right) = q \cdot d(s_k\mathbb{Z} + n) = q \cdot d\left(q(a'\mathbb{Z} + r')\right) \\ &= d(a'\mathbb{Z} + r') = 1/a' \,. \end{aligned}$$

In view of Lemma 1.17 in [10], this suffices to conclude that $\mathcal{B}'(q)$ is Behrend.

On the other hand, as $\mathcal{B}$ is taut, $\mathcal{B}'(q)$ is pre-taut (Lemma 2.3), so that $1 \in \mathcal{B}'(q)$ or $\mathcal{B}'(q)$ is not Behrend (Lemma 2.1e). Hence $1 \in \mathcal{B}'(q)$. This implies $q \in \mathcal{M}_\mathcal{B}$, which in turn implies $U \cap W = U_{S_k}(\Delta(n)) \cap W = \emptyset$ (the property to be proved): Indeed, if $q \in \mathcal{M}_\mathcal{B}$, then there is some $b \in \mathcal{B}$ with $b \mid q$, and as $q \mid s_k$, this implies $b \mid s_k$, so that $b \in S_k$. From $b \mid q \mid n$ we then conclude that $n \in \mathcal{M}_{S_k}$, and Lemma 2.5a implies $U_{S_k}(\Delta(n)) \cap W = \emptyset$.

It remains to show that the implication $(i) \Rightarrow (iii)$ follows from Proposition 1.1, which will be proved in the next subsection. So let $h \in W$. By the proposition there exists $n \in \mathcal{F}_\mathcal{B}$ such that $\Delta(n) \in U_S(h)$, hence $\Delta(n) \in U_S(h) \cap (\Delta(\mathbb{Z}) \cap W)$. As this holds for all finite $S \subset \mathcal{B}$, this proves the claim.

2.3 Tautification of the set $\mathcal{B}$ and regularization of the window W

In [4, Section 4.2] the authors provide a construction that associates to each (non-taut) set $\mathcal{B}$ a taut set $\mathcal{B}'$ such that $\mathcal{F}_{\mathcal{B}'} \subseteq \mathcal{F}_{\mathcal{B}}$ but $\overline{d}(\mathcal{F}_{\mathcal{B}} \setminus \mathcal{F}_{\mathcal{B}'}) = 0$, and such that the two Mirsky measures ν_η and $\nu_{\eta'}$ determined by $\mathcal{B}$ and $\mathcal{B}'$ coincide. $\mathcal{B}$ and $\mathcal{B}'$ determine groups H resp. H' with windows W resp. W', and while the window W is not Haar regular (if $\mathcal{B}$ is non-taut), the window W' is Haar regular because of Theorem A.

On the abstract level one can also pass from the window $W \subseteq H$ to its *Haar regularization* $W_{reg} := \mathrm{supp}(m_H|_W)$ (introduced in [14]), which also determines the same Mirsky measure on $\{0,1\}^{\mathbb{Z}}$. However, W_{reg} will not be a window of the particular arithmetic type defined in (2), in particular it need not be aperiodic. The construction of $\mathcal{B}'$ given $\mathcal{B}$ in [4] suggests an obvious factor map $f : H \to H'$, and we expect that also $f(W_{reg}) = W'$, so that in this sense the regularization of W and the tautification of $\mathcal{B}$ are two sides of the same medal.

The following example illustrates this discussion.

Example 2.1. Let $\mathcal{P} = \{p_1, p_2, \dots\}$ denote the set of primes. Let $\mathcal{B} := \bigcup_{i\geq 1} p_i^2(\mathcal{P} \setminus \{p_i\})$. Note that $\mathcal{B}$ is primitive. It is not taut, because it contains rescalings of Behrend sets. The corresponding taut set is $\mathcal{B}' = \{p_i^2 : i \geq 1\}$, which generates the square-free system. [6]

2.4 The property $\overline{\Delta(\mathbb{Z}) \cap W} = W$

Proof of Proposition 1.1. Given $h \in W$, we need to show that for each finite $S \subset \mathcal{B}$ the set

$$\mathcal{L}_S(h) := \{n \in \mathcal{F}_{\mathcal{B}} : h_b = n \bmod b \text{ for each } b \in S\}$$

has asymptotic density $m_H(U_S(h) \cap W) > 0$.

By Theorem A, the tautness assumption on $\mathcal{B}$ implies that W is Haar regular, so that indeed

$$m_H(U_S(h) \cap W) > 0 \ .$$

Let $\mathcal{B} = \{b_1, b_2, \dots\}$ and, for $K \geq 1$, $W_K := \{g \in H : g_i \neq 0 \text{ for } i = 1, \dots, K\}$. Then W_K is clopen and $W \subseteq W_K$. Moreover, $W_K \supseteq W_{K+1}$ and $\bigcap_K W_K = W$. Fix $\varepsilon > 0$. We now choose $K \geq 1$ so that

$$m_H(W_K \setminus W) < \varepsilon \ . \tag{11}$$

Since $U_S(h) \cap W_K$ is clopen (and T is strictly ergodic)

$$\left| \frac{1}{N} \sum_{n \leq N} \mathbb{1}_{U_S(h)\cap W_K}(T^n 0) - m_H\left(U_S(h) \cap W_K\right) \right| < \varepsilon \tag{12}$$

for all $N \geq N_0$. Moreover, we can choose N_1 so that for $N \geq N_1$, we also have

$$\left| \frac{1}{N} \sum_{n \leq N} \mathbb{1}_{U_S(h)\cap W_K}(T^n 0) - \frac{1}{N} \sum_{n \leq N} \mathbb{1}_{U_S(h)\cap W}(T^n 0) \right| < \varepsilon \ . \tag{13}$$

Indeed, if

$$T^n 0 = \Delta(n) \in (U_S(h) \cap W_K) \setminus (U_S(h) \cap W) \subset W_K \setminus W \ ,$$

[6] Note that $\eta(n) = 1$ at all square-free numbers and also at p_i^k for $i \geq 1$ and $k \geq 2$.

then (by setting $\mathcal{B}_K = \{b_1, \ldots, b_K\}$), we have

$$n \in \mathcal{F}_{\mathcal{B}_K} \cap \mathcal{M}_{\mathcal{B}} = \mathcal{M}_{\mathcal{B}} \setminus \mathcal{M}_{\mathcal{B}_K} .$$

Therefore, by the Davenport-Erdös theorem [10, Eq. (0.67)], we can choose first $K \geq 1$ sufficiently large so that $\overline{d}(\mathcal{M}_{\mathcal{B}} \setminus \mathcal{M}_{\mathcal{B}_K}) < \varepsilon$ and then N_1 so that

$$\frac{1}{N} \sum_{n \leq N} \mathbb{1}_{W_K \setminus W}(T^n 0) = \frac{1}{N} \sum_{n \leq N} \mathbb{1}_{\mathcal{M}_{\mathcal{B}} \setminus \mathcal{M}_{\mathcal{B}_K}}(n) < \varepsilon$$

for all $N \geq N_1$, so in particular (13) holds. In view of (11), (12) and (13), it follows that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n \leq N} \mathbb{1}_{U_S(h) \cap W}(T^n 0) = m_H(U_S(h) \cap W) .$$

As $T^n 0 = \Delta(n) \in U_S(h) \cap W$ if and only if $n \in \mathcal{L}_S(h)$, this finishes the proof. □

Example 2.2. ($\overline{\Delta(\mathbb{Z}) \cap W} = W$ does not imply tautness)
Suppose that (m_k, r_k), $k \in \mathbb{N}$, is an enumeration of all coprime pairs of natural numbers. For any k choose a prime $p_k \in r_k + m_k\mathbb{Z}$ such that $p_k > 2^{k+1}$. Let $\mathcal{B} = \mathcal{P} \setminus \{p_k : k \in \mathbb{N}\}$. Clearly $\mathcal{B}$ is primitive, and $\mathcal{M}_{\{p_k : k \in \mathbb{N}\}}$ has upper density less than or equal to $\sum_{k=1}^{\infty} 1/2^{k+1} = 1/2$. Thus $d(\mathcal{M}_{\mathcal{B}}) = 1$ and $\mathcal{B}$ is not taut [10, Corollary 0.14]. But $\overline{\Delta(\mathbb{Z}) \cap W} = W$. Indeed, let $h = (h_b)_{b \in \mathcal{B}} \in W$ and take any finite set $S \subset \mathcal{B}$. We are going to show that $U_S(h) \cap W \cap \Delta(\mathbb{Z}) \neq \emptyset$. Let $n \in \mathbb{Z}$ be such that $n = h_b$ mod b for $b \in S$. Since $h \in W$, b does not divide n for any $b \in S$, i.e. $\mathrm{lcm}(S)$ and n are coprime. Then $(\mathrm{lcm}(S), n) = (m_k, r_k)$ for some k, and the prime number p_k belongs to arithmetic progression $r_k + m_k\mathbb{Z} = n + \mathrm{lcm}(S)\mathbb{Z}$, in other words $\Delta(p_k) \in U_S(\Delta(n)) = U_S(h)$. Finally, $\Delta(p_k) \in W$, because the prime number p_k does not belong to $\mathcal{B}$ and hence also not to $\mathcal{M}_{\mathcal{B}}$.

2.5 X_η and X_φ

The set $\mathcal{B} \subseteq \mathbb{N}$ has *light tails*, if

$$\lim_{K \to \infty} \overline{d}\,(\mathcal{M}_{\{b \in \mathcal{B} : b > K\}}) = 0 . \tag{14}$$

If $\mathcal{B}$ has light tails, then $\mathcal{B}$ is taut, but the converse doses not hold [4, Section 4.3]. Here we prove:

Proposition 2.2. *If $\mathcal{B}$ has light tails, then $X_\eta = X_\varphi$.*

Proof. Let $H = (h_k) \in H$ and $n \in \mathbb{N}$. We are going to show that $\varphi(h)[-n, n] = \eta[l + 1, l + 2n + 1]$ for some $l \in \mathbb{Z}$. We know that $\varphi(h)(i) = 1$ if and only if $h_j + i$ is not a multiple of b_j for any $j \in \mathbb{N}$. For any $i \in [-n, n]$ such that $\varphi(h)(i) = 0$ let k_i be such that $b_{k_i} | h_{k_i} + i$.

Let $K \in \mathbb{N}$ be such that the set $\mathscr{A} := \{b_1, \ldots, b_K\}$ contains b_{k_i}, for $i \in [-n, n]$ and any b_k with $k > K$ has a prime factor $p > 2n + 1$. Since $h \in H$, there exists $m \in Z$ such that

$$m = h_k \bmod b_k \tag{15}$$

for all $k \leq K$. It follows that

$$(\operatorname{supp} \varphi(h) \cap [-n, n]) + m = [-n + m, n + m] \cap \mathcal{F}_{\mathcal{A}}$$

Indeed, if $i \in \operatorname{supp} \varphi(h) \cap [-n, n]$, then $h_k + i$ is not a multiple of b_k for any $k \in \mathbb{N}$. By (15) we get that $m + i$ is not a multiple of b_k for any $k \leq K$, that is, $m + i \in \mathcal{F}_{\mathcal{A}}$. On the other hand, if

$i \notin \text{supp}\varphi(h) \cap [-n,n]$, then $b_{k_i}|h_{k_i} + i$. Since $k_i \le K$, again by (15), we obtain $b_{k_i}|m + i$, that is $m + i \notin \mathcal{F}_{\mathcal{A}}$.

By [4, Proposition 5.11] [7] there exists $l \in \mathbb{Z}$ such that

$$([-n+m, n+m] \cap \mathcal{F}_{\mathcal{A}}) + l + n + 1 - m = [l+1, l+2n+1] \cap \mathcal{F}_{\mathcal{B}}$$

It follows that $\varphi(h)[-n,n] = \eta[l+1, l+2n+1]$. □

We now present a Behrend set (hence a non-taut set), for which X_η is a strict subeset of X_φ.

Example 2.3. Let $\mathcal{B} = \{p_2, p_3, \ldots\} = \{3, 5, 7, 11, \ldots\}$ - the set of all odd prime numbers. Since we are in the coprime case,

$$H = \prod_{k=2}^{\infty} \mathbb{Z}/p_k\mathbb{Z}.$$

Now, $\eta = \varphi(\Delta(0))$ is the characteristic function of the $\mathcal{B}$-free set $\{\pm 2^m : m \ge 0\}$. We compute an initial block of $\varphi(h)$ for

$$h = (0, 1, 0, 0, \ldots) \in H.$$

We have $\varphi(h)(0) = 0$, $\varphi(h)(1) = 1$, $\varphi(h)(2) = 1$, $\varphi(h)(3) = 0$, $\varphi(h)(4) = 0$[8], $\varphi(h)(5) = 1$, $\varphi(h)(6) = 0$, $\varphi(h)(7) = 0$ and $\varphi(h)(8) = 1$. It follows that the block 11001001 appears on $\varphi(h)$. But there is no block $\underline{a}$ of length 8 appearing on η and such that $11001001 \le \underline{a}$. Indeed, the two neighboring 1's at the beginning of $\underline{a}$ could only appear at the positions 1,2 or -2,-1 in η. In the both cases this would force $\eta(5) = 1$, which is not true. This shows that $\varphi(h) \notin X_\eta$, although it belongs to X_φ.[9]

Question 2.1. If $\mathcal{B}$ is taut, is then $X_\eta = X_\varphi$? [10]

3 Minimality/proximality of X_η and topological properties of W

Throughout this section we assume that $\mathcal{B}$ is primitive.

3.1 Arithmetic of $\mathcal{B}$ and topology of W, part II

Recall from (5) that $\mathcal{A}_S := \{\gcd(b, \text{lcm}(S)) : b \in \mathcal{B}\}$ and $\mathcal{F}_{\mathcal{A}_S} \subseteq \mathcal{F}_{\mathcal{B}}$ for $S \subset \mathcal{B}$. If $S \subseteq S' \subset \mathcal{B}$, then the following inclusions and implications are obvious:

$$S \subseteq S' \subseteq \mathcal{A}_{S'} \subseteq \mathcal{M}_{\mathcal{A}_S} \Rightarrow \mathcal{M}_S \subseteq \mathcal{M}_{S'} \subseteq \mathcal{M}_{\mathcal{A}_{S'}} \subseteq \mathcal{M}_{\mathcal{A}_S} \Rightarrow \mathcal{F}_{\mathcal{A}_S} \subseteq \mathcal{F}_{\mathcal{A}_{S'}} \subseteq \mathcal{F}_{S'} \subseteq \mathcal{F}_S \, . \tag{17}$$

Let $\mathcal{E} := \bigcup_{S \subset \mathcal{B}} \mathcal{F}_{\mathcal{A}_S}$ and observe that $\mathcal{E} \subseteq \mathcal{F}_{\mathcal{B}}$.

Lemma 3.1. *a) For all $S \subset \mathcal{B}$ and $n \in \mathbb{Z}$ we have: $U_S(\Delta(n)) \subseteq W \Leftrightarrow n \in \mathcal{F}_{\mathcal{A}_S}$.*

---

[7] Assume that $\mathcal{B} \subset \mathbb{N}$ has light tails and $\mathcal{B}^{(n)} \subset \mathcal{A} \subset \mathcal{B}$. Suppose that

$$\{k+1, \ldots, k+n\} \cap \mathcal{M}_{\mathcal{A}} = \{k+i_0, k+i_1, \ldots, k+i_r\} \tag{16}$$

for some $1 \le i_0, \ldots, i_r \le n$, $r < n$. Then the density of $k' \in \mathbb{N}$ such that

$$\{k'+1, \ldots, k'+n\} \cap \mathcal{M}_{\mathcal{B}} = \{k'+i_0, k'+i_1, \ldots, k'+i_r\}$$

is positive. (Here $\mathcal{B}^{(n)} := \{b \in \mathcal{B} : p \le n \text{ for any } p \in \text{Spec}(b)\}$. If $\mathcal{B}$ is primitive, then $\mathcal{B}^{(n)}$ is finite.)

[8] If we add 4 to each coordinate of h, we obtain the sequence $(1, 0, 4, 4, \ldots)$, whence $\varphi(h)(4) = 0$.

[9] Indeed, $\varphi(h)$ does not even belong to $\widetilde{X}_\eta$, the hereditary closure of X_η, see [4].

[10] We recall that in case of $\mathcal{B}$ taut, the Mirsky measure is supported on X_η.

b) *If* $(S_k)_k$ *is a filtration of* $\mathcal{B}$ *with finite sets and* $\lim_k \Delta(n_{S_k}) = h$ *(see Remark 5.2), then* $h \in \operatorname{int}(W)$ *if and only if* $n_{S_k} \in \mathcal{F}_{\mathcal{A}_{S_k}}$ *for some k.*
c) *For all* $n \in \mathbb{Z}$ *we have:* $\Delta(n) \in \operatorname{int}(W) \Leftrightarrow n \in \mathcal{E}$.
d) $\operatorname{int}(W) = \emptyset \Leftrightarrow \mathcal{E} = \emptyset \Leftrightarrow \forall S \subset \mathcal{B} : \mathcal{F}_{\mathcal{A}_S} = \emptyset \Leftrightarrow \forall S \subset \mathcal{B} : 1 \in \mathcal{A}_S$

Proof. a) As $U_S(\Delta(n))$ is clopen,

$$\begin{aligned} U_S(\Delta(n)) \not\subseteq W &\Leftrightarrow \exists m \in n + \operatorname{lcm}(S) \cdot \mathbb{Z}\ \exists c \in \mathcal{B} : c \mid m \\ &\Leftrightarrow \exists c \in \mathcal{B}\ \exists k \in \mathbb{Z} : c \mid n + k \cdot \operatorname{lcm}(S) \\ &\Rightarrow \exists c \in \mathcal{B} :\ \gcd(c, \operatorname{lcm}(S)) \mid n \\ &\Leftrightarrow \exists k \in \mathcal{A}_S : k \mid n \\ &\Leftrightarrow n \notin \mathcal{F}_{\mathcal{A}_S} . \end{aligned}$$

That the only implication is also an equivalence is a consequence of the CRT. Indeed, if $\gcd(c, \operatorname{lcm}(S)) \mid n$, then there exist $k, l \in \mathbb{Z}$ such that $l \cdot c - k \cdot \operatorname{lcm}(S) = n$, thus $c \mid n + k \cdot \operatorname{lcm}(S)$.
b) Assume that $h \in \operatorname{int}(W)$, that is $U_S(h) \subseteq W$ for some S. Then, for k such that $S \subseteq S_k$, we have $U_{S_k}(\Delta(n_{S_k})) = U_{S_k}(h) \subseteq W$, which is equivalent to $n_{S_k} \in \mathcal{F}_{\mathcal{A}_{S_k}}$ by a). Conversely, if $n_{S_k} \in \mathcal{F}_{\mathcal{A}_{S_k}}$ then, again by a), $U_{S_k}(\Delta(n_{S_k})) = U_{S_k}(h) \subseteq W$ and $h \in \operatorname{int}(W)$.
c) Follows from a).
d) Follows from c).

□

Recall from (6) that $\mathcal{A}_\infty := \{n \in \mathbb{N} : \forall_{S \subset \mathcal{B}}\ \exists_{S' : S \subseteq S'} : n \in \mathcal{A}_{S'} \setminus S'\}$.

Lemma 3.2. *a) If* $(S_k)_k$ *is a filtration of* $\mathcal{B}$ *with finite sets, then*

$$\limsup_{k\to\infty} \mathcal{A}_{S_k} \setminus S_k = \mathcal{A}_\infty .$$

b) For each $n \in \mathcal{A}_\infty$ *there is a filtration* $(S_k)_k$ *of* $\mathcal{B}$ *with finite sets such that*

$$n \in \bigcap_{k\in\mathbb{N}} \mathcal{A}_{S_k} \setminus S_k .$$

Proof. a) Assume that $n \in \mathcal{A}_{S_k} \setminus S_k$ for infinitely many k, and let $S \subset \mathcal{B}$. Then there is k such that $S \subseteq S_k$ and $n \in \mathcal{A}_{S_k} \setminus S_k$. Hence $n \in \mathcal{A}_\infty$. Conversely, let $n \in C_\infty$. There is a finite set S_1 such that $n \in \mathcal{A}_{S_1} \setminus S_1$. Assume that we have constructed sets $S_1 \subset S_2 \subset \ldots \subset S_k$ with the property that $n \in \mathcal{A}_{S_i} \setminus S_i$ for $i = 1, \ldots, k$ and $\{1, \ldots, k\} \cap \mathcal{B} \subset S_k$. Then there is a set S_{k+1} containing $S_k \cup \{k+1\}$ and such that $n \in \mathcal{A}_{S_{k+1}} \setminus S_{k+1}$. In this inductive way we construct a filtration $(S_k)_k$ as required.
b) follows from a). □

Lemma 3.3. *The sets* $\mathcal{E}$ *and* $\mathcal{A}_\infty$ *are related by the identity*

$$\mathcal{E} = \mathcal{F}_{\mathcal{B}\cup\mathcal{A}_\infty} = \mathcal{F}_{\mathcal{B}} \cap \mathcal{F}_{\mathcal{A}_\infty}.$$

Proof. Let $n \in \mathcal{E}$ and chose S such that $n \in \mathcal{F}_{\mathcal{A}_S}$. Take arbitrary $b \in \mathcal{B}$ and $c \in \mathcal{A}_\infty$. There exists a finite set S' such that $S \cup \{b\} \subseteq S'$ and $c \in \mathcal{A}_{S'} \setminus S'$. Since $\mathcal{F}_{\mathcal{A}_S} \subseteq \mathcal{F}_{\mathcal{A}_{S'}}$, $n \in \mathcal{F}_{\mathcal{A}_{S'}}$, hence neither b nor c divides n. We have proved that $\mathcal{E} \subseteq \mathcal{F}_{\mathcal{B}\cup\mathcal{A}_\infty}$. In order to prove the other inclusion assume that $n \in \mathbb{N}$ and that for any S there exists $c_S \in \mathcal{A}_S$ dividing n. As n has only finitely many divisors, it has a divisor c such that there exists a filtration $(S_k)_k$ of $\mathcal{B}$ such that $c \in \mathcal{A}_{S_k}$ for any $k \in \mathbb{N}$. If $c \notin \mathcal{B}$, then $c \in \mathcal{A}_{S_k} \setminus S_k$ for any $k \in \mathbb{N}$. This proves $n \notin \mathcal{F}_{\mathcal{B}\cup\mathcal{A}_\infty}$. □

Lemma 3.4. $\mathcal{A}_\infty = \emptyset$ *if and only if* $\mathcal{E} = \mathcal{F}_\mathcal{B}$.

Proof. If $\mathcal{A}_\infty = \emptyset$, then $\mathcal{E} = \mathcal{F}_\mathcal{B}$ by Lemma 3.3. Conversely, assume that $\mathcal{E} = \mathcal{F}_\mathcal{B}$. Then $\mathcal{F}_\mathcal{B} \subseteq \mathcal{F}_{\mathcal{A}_\infty}$ by Lemma 3.3, so that $\mathcal{A}_\infty \subseteq \mathcal{M}_{\mathcal{A}_\infty} \subseteq \mathcal{M}_\mathcal{B}$. Suppose for a contradiction that there exists some $n \in \mathcal{A}_\infty$. Then there is $b \in \mathcal{B}$ such that $n \in b\mathbb{Z}$, i.e. $b \mid n$, and there is a finite set $S = S_k \subset \mathcal{B}$ such that $n \in \mathcal{A}_S \setminus S$, see Lemma 3.2b. Hence there exists $b' \in \mathcal{B}$ such that $n = \gcd(\mathrm{lcm}(S), b')$. It follows that $b \mid n \mid b'$, which is impossible, because $\mathcal{B}$ is assumed to be primitive. □

Proposition 3.1. *The following conditions are equivalent:*

(i) $W \neq \overline{\mathrm{int}(W)}$

(ii) For any filtration $S_0 \subset S_1 \subset \ldots \subset \mathcal{B}$ *of* $\mathcal{B}$ *with finite subsets* S_k*, there exists a number* d *such that* $d \in \mathcal{A}_{S_k} \setminus S_k$*, for infinitely many* $k \in \mathbb{N}$*.*

(iii) There exists a filtration $S_0 \subset S_1 \subset \ldots \subset \mathcal{B}$ *of* $\mathcal{B}$ *with finite subsets* S_k *and there exists a number* d *such that* $d \in \mathcal{A}_{S_k} \setminus S_k$*, for every* $k \in \mathbb{N}$*.*

(iv) There are $d \in \mathbb{N}$ *and an infinite pairwise coprime set* $\mathcal{A} \subseteq \mathbb{N} \setminus \{1\}$ *such that* $d\mathcal{A} \subseteq \mathcal{B}$*.*

Proof. $(i) \Rightarrow (ii)$: Let $h = (h_b) \in W \setminus \overline{\mathrm{int}(W)}$. There exists S such that $U_S(h) \cap \mathrm{int}(W) = \emptyset$. We can assume that any $b \in \mathcal{B}$ such that $b | \mathrm{lcm}(S)$, belongs to S.[11] Let n be a number such that

$$n = h_b \bmod b \tag{18}$$

for $b \in S$.

Then $\Delta(n + k\,\mathrm{lcm}(S)) \in U_S(h)$, hence $\Delta(n + k\,\mathrm{lcm}(S)) \notin \mathrm{int}(W)$ for any $k \in \mathbb{Z}$. This means (see Lemma 3.1) that for any finite set T, in particular for any $T = S_k$, the arithmetic progression $n + \mathrm{lcm}(S)\mathbb{Z}$ is contained in $\mathcal{M}_{\mathcal{A}_T}$. Since the set $\mathcal{A}_T$ is finite, it follows that $\mathcal{A}_T$ contains a divisor of $\gcd(n, \mathrm{lcm}(S))$[12]. There is only finitely many divisors of $\gcd(n, \mathrm{lcm}(S))$, hence one of them, denote it by d, appears in $\mathcal{A}_{S_k}$ for infinitely many k. To finish the proof it is enough to observe that $d \notin \mathcal{B}$ (consequently, $d \notin S_k$, for any k). Indeed, otherwise $d \in S$, by our assumption on S. Moreover, $d|n$ and then, by (18), $d|h_b$, where $b = d$, which leads to a contradiction with the assumption $h \in W$.

$(ii) \Rightarrow (iii)$: obvious

$(iii) \Rightarrow (i)$: Assume that $d \in \mathcal{A}_{S_k} \setminus S_k$ for any k. Then $d \notin \mathcal{M}_\mathcal{B}$[13], hence $\Delta(d) \in W$. We prove that $\Delta(d) \notin \overline{\mathrm{int}(W)}$. It is enough to show that $U_{S_0}(\Delta(d)) \cap \mathrm{int}(W) = \emptyset$.

Assume that $h = (h_b) \in U_{S_0}(\Delta(d)) \cap \mathrm{int}(W)$. It means that

$$d = h_b \bmod b \text{ for any } b \in S_0\,, \tag{19}$$

and there exists a finite set $T \subset \mathcal{B}$ such that $U_T(h) \subset W$. We can assume that $T = S_k$ for some k.

Let $m \in \mathbb{Z}$ be such that

$$m = h_b \bmod b \text{ for any } b \in S_k\,. \tag{20}$$

Let $c \in \mathcal{B}$ be such that $\gcd(c, \mathrm{lcm}(S_k)) = d$. Clearly, $c \notin S_k$, since $d \notin \mathcal{B}$. Since $U_{S_k}(h) \subset W$, it follows that there exists $b \in S_k$ such that

$$\gcd(c, b) \text{ does not divide } h_b\,, \tag{21}$$

[11] Otherwise we can incorporate all such b's into S, there are finitely many of them.

[12] Apply Dirichlet theorem on primes in arithmetic progressions.

[13] Otherwise d is divisible by some $b \in \mathcal{B}$. On the other hand, d divides some $b' \in \mathcal{B}$ as a member of $\mathcal{A}_{S_k}$, which in view of the fact that $\mathcal{B}$ is primitive, leads to the conclusion that $d = b = b' \in \mathcal{B}$. But it is not true, since $d \notin S_k$ for any k.

Indeed, otherwise there would exist $l \in \mathbb{Z}$ such that $l \equiv h_b \bmod b$ for $b \in S_k$ and $l = 0 \bmod c$, hence $\Delta(l) \in U_{S_k}(h)$, but $\Delta(l) \notin W$, a contradiction. Thus, in view of (20),

$$\gcd(c, b) \text{ does not divide } m\,. \tag{22}$$

On the other hand,

$$\gcd(c, b) | \gcd(c, \operatorname{lcm}(S_k)) = d\,. \tag{23}$$

Since $d \in \mathcal{A}_{S_0} \setminus S_0$ we get

$$d | \operatorname{lcm}(S_0)\,. \tag{24}$$

By (19) and (20),

$$\operatorname{lcm}(S_0) | m - d\,. \tag{25}$$

Now, (23), (24) and (25) imply $\gcd(c, b)|m$, a contradiction with (22).

(*iii*) $\Rightarrow$ (*iv*): Assume that $d \in \mathcal{A}_{S_k} \setminus S_k$ for any k. Then

$$\forall k \in \mathbb{N}\ \exists b_k \in \mathcal{B} \setminus S_k :\ d = \gcd(b_k, \operatorname{lcm}(S_k))\,. \tag{26}$$

As $d \notin S_k$, we have $b_k \neq d$ for all k. We choose a subsequence $b_{k_1}, b_{k_2}, \ldots$ of $(b_k)_k$ in the following way: Let $k_1 = 1$, and given $k_1, \ldots, k_j$, let

$$k_{j+1} := \min\left\{k \in \mathbb{N} : b_{k_1}, \ldots, b_{k_j} \in S_{k_{j+1}}\right\}.$$

Let $a_j = b_{k_j}/d$ for all $j \in \mathbb{N}$ and denote $\mathcal{A} = \{a_j : j \in \mathbb{N}\}$. Then $\mathcal{A} \subseteq \mathbb{N}$ and $d\mathcal{A} \subseteq \mathcal{B}$ by construction. Suppose that $1 \in \mathcal{A}$. Then $d \in \mathcal{B}$, a contradiction to (26), as $\mathcal{B}$ is primitive. Hence $\mathcal{A} \subseteq \mathbb{N} \setminus \{1\}$.

It remains to prove that $\mathcal{A}$ is pairwise coprime. Suppose for a contradiction that there is a prime number p dividing some a_i and a_j, $i < j$. Then $pd \mid b_{k_i}$ and $pd \mid b_{k_j}$. As $b_{k_i} \in S_{k_j}$, it follows that $pd \mid \operatorname{lcm}(S_{k_j})$, so that $pd \mid \gcd(b_{k_j}, \operatorname{lcm}(S_{k_j})) = d$ (see (26)), which is impossible.

(*iv*) $\Rightarrow$ (*iii*): Let $d \in \mathbb{N}$ and $\mathcal{A} = \{a_1 < a_2 < \ldots\}$ be as in (*iv*). Then $d \notin \mathcal{B}$, because $\mathcal{B}$ is primitive. For $k \in \mathbb{N}$ let $S_k = \mathcal{B} \cap \{1, \ldots, k\} \cup \{da_k\}$. As all a_j are pairwise coprime, there are $j_1 < j_2 < \cdots \in \mathbb{N}$ such that a_{j_k} is coprime to $\operatorname{lcm}(S_k)$. On the other hand, $d \mid \operatorname{lcm}(S_k)$. Hence $d = \gcd(da_{j_k}, \operatorname{lcm}(S_k)) \in \mathcal{A}_{S_k}$. As $d \notin \mathcal{B}$, we see that $d \in \mathcal{A}_{S_k} \setminus S_k$ for all $k \in \mathbb{N}$. □

Proposition 3.2. *The following conditions are equivalent:*

(i) *W is topologically regular, i.e. $W = \overline{\operatorname{int}(W)}$.*
(ii) *There are no $d \in \mathbb{N}$ and no infinite pairwise coprime set $\mathcal{A} \subseteq \mathbb{N} \setminus \{1\}$ such that $d\mathcal{A} \subseteq \mathcal{B}$.*
(iii) *$\mathcal{A}_\infty = \emptyset$.*
(iv) *$\mathcal{E} = \mathcal{F}_{\mathcal{B}}$.*

Proof. The equivalence of (*i*) and (*ii*) follows from Proposition 3.1, that of (*iii*) and (*iv*) from Lemma 3.4.. In view of Lemma 3.2, Proposition 3.1 finally implies the equivalence of (*i*) and (*iii*), too. □

Lemma 3.5. $\Delta(\mathbb{Z}) \cap \left(\overline{\operatorname{int}(W)} \setminus \operatorname{int}(W)\right) = \emptyset$*.*

Proof. Assume $\Delta(m) \in \overline{\operatorname{int}(W)} \setminus \operatorname{int}(W)$. Then for any $S \subset \mathcal{B}$ there exists $n_S \in \mathbb{Z}$ such that $\Delta(n_S) \in U_S(\Delta(m)) \cap \operatorname{int}(W)$. It means that for any S there exist: a finite set $T_S \subset \mathcal{B}$, (we can assume that $S \subset T_S$), $b_S \in \mathcal{B}$ and $n_S \in \mathbb{Z}$ such that (see Lemma 3.1 c)):

- $\operatorname{lcm}(S)|m - n_s$ (that is, $\Delta(n_s) \in U_S(\Delta(m))$)
- $\gcd(b_S, \operatorname{lcm}(T_S))$ does not divide n_S ($\Delta(n_S)$ is chosen to be an element of $U_S(\Delta(m)) \cap \operatorname{int}(W)$)

- $\gcd(b_S, \mathrm{lcm}(T_S))|m$ (since $\Delta(m) \notin \mathrm{int}(W)$)

Then $\gcd(b_S, \mathrm{lcm}(T_S))$ does not divide $\mathrm{lcm}(S)$. Let us iterate: S_0 is arbitrary and $S_{k+1} := T_{S_k}$, $c_k := \mathrm{lcm}(S_{k+1})$, $d_k := \gcd(b_{S_k}, \mathrm{lcm}(S_{k+1}))$. We have:

- $c_k | c_{k+1}$
- $d_k | m$
- $d_k | c_k$
- d_k does not divide c_{k-1}

Since $d_k | m$ for every k, the sequence $(\mathrm{lcm}(d_1, \ldots d_k))_k$ stabilizes on $\mathrm{lcm}(d_1, \ldots d_{k_0})$ for some k_0, which means d_l divides $\mathrm{lcm}(d_1, \ldots d_{k_0})$, and consequently d_l divides $\mathrm{lcm}(c_1, \ldots, c_{k_0}) = c_{k_0}$, for any l, a contradiction. □

For $x \in \{0,1\}^{\mathbb{Z}}$ denote $\mathrm{supp}\, x := \{n \in \mathbb{Z} : x(n) = 1\}$. Following [4] we consider the set

$$\begin{aligned} Y &:= \left\{x \in \{0,1\}^{\mathbb{Z}} : |\,\mathrm{supp}\, x \bmod b| = b-1 \text{ for all } b \in \mathcal{B}\right\} \\ &= \left\{x \in \{0,1\}^{\mathbb{Z}} : \text{for all } b \in \mathcal{B} \text{ there is exactly one } r \in \{0,\ldots,b-1\} \text{ with } \mathrm{supp}\, x \cap (b\mathbb{Z}+r) = \emptyset\right\}. \end{aligned}$$

As $\mathrm{supp}\, \eta = \mathcal{F}_{\mathcal{B}}$ is disjoint from $b\mathbb{Z}$ for all $b \in \mathcal{B}$, we have

$$\eta \in Y \Leftrightarrow \forall b \in \mathcal{B}\, \forall r \in \{1,\ldots,b-1\} : \mathcal{F}_{\mathcal{B}} \cap (b\mathbb{Z}+r) \neq \emptyset\,. \tag{27}$$

Lemma 3.6. *If $\overline{\Delta(\mathbb{Z}) \cap W} = W$, then $\eta \in Y$.*

Proof. For $b \in \mathcal{B}$ and $r \in \{0,\ldots,b-1\}$ let $V_b(r) := \{h \in H : h_b = r\}$ and observe that these sets are open and closed in H. Hence $\overline{\Delta(\mathbb{Z}) \cap V_b(r) \cap W} = V_b(r) \cap W$, because $\overline{\Delta(\mathbb{Z}) \cap W} = W$.

Suppose for a contradiction that $\eta \notin Y$. Then (27) implies that there are $b \in \mathcal{B}$ and $r \in \{1,\ldots,b-1\}$ such that

$$\Delta(\mathbb{Z}) \cap V_b(r) \cap W = \emptyset\,,$$

which implies that also $V_b(r) \cap W = \emptyset$. Hence

$$V_b(r) \subseteq W^c = \bigcup_{b' \in \mathcal{B}} V_{b'}(0)\,,$$

and as $V_b(r)$ is compact and the $V_{b'}(0)$ are open, there is a finite $S \subset \mathcal{B}$ such that

$$V_b(r) \subseteq \bigcup_{b' \in S} V_{b'}(0)\,,$$

In other words, whenever $h_b = r$ for some $h \in H$, then $h_{b'} = 0$ for some $b' \in S$. Applied to any $h = \Delta(n)$ this yields:

$$n \in b\mathbb{Z} + r \;\Rightarrow\; n \in \bigcup_{b' \in S} b'\mathbb{Z}\,.$$

Since r is not divisible by b, we can assume that $b \notin S$. Let $q = \gcd(b, r)$, $\tilde{b} = b/q$, $\tilde{r} = r/q$. Then $q(\tilde{b}\mathbb{Z} + \tilde{r}) = b\mathbb{Z} + r \subseteq \mathcal{M}_S$, so that $\mathcal{M}_{\tilde{b}\mathbb{Z}+\tilde{r}} \subseteq \mathcal{M}_{S'(q)}$ by Lemma 2.2. But $d(\mathcal{M}_{\tilde{b}\mathbb{Z}+\tilde{r}}) = 1$ by Dirichlet's theorem, whereas $d(\mathcal{M}_{S'(q)}) < 1$, because $S'(q) \subseteq \{1,\ldots,\max S\}$ is finite and $1 \notin S'(q)$ [14]. This is a contradiction. □

Remark 3.1. Together with Theorem A this shows that $\eta \in Y$ whenever $\mathcal{B}$ is taut. This implication was proved previously in [4, Corollary 4.27].

[14] As $q|b$ and $\mathcal{B}$ is primitive, $q \notin S$, thus $1 \notin S'(q)$.

Lemma 3.6 provides the implication

$$\overline{\Delta(\mathbb{Z}) \cap W} = W \Rightarrow \eta \in Y.$$

The reverse implication does not hold, as is shown by the next example.

Example 3.1. Observe that for every $k \in \mathbb{Z}$ there exists a prime divisor p_k of $5 + 12k$ such that

$$p_k \neq 1 \bmod 12 \ \text{ and } \ p_k \neq -1 \bmod 12 \tag{28}$$

Let

$$\mathcal{B} = \{4, 6\} \cup \{p_k : k \in \mathbb{Z}\}$$

Let us enumerate the elements of $\mathcal{B}$ as $b_0, b_1, b_2, \ldots$ and $b_0 = 4, b_1 = 6$. Observe that

$$5 + 12\mathbb{Z} \subset \mathcal{M}_{\mathcal{B}} \tag{29}$$

Since niether 2 nor 3 divides an element of the progression $5 + 12\mathbb{Z}$, in view of (28) we see that $1, 2, 3, 11, 22 \in \mathcal{F}_{\mathcal{B}}$. It follows that

$$|\operatorname{supp} \mathcal{F}_{\mathcal{B}} \bmod 4| = 3 \ \text{ and } \ |\operatorname{supp} \mathcal{F}_{\mathcal{B}} \bmod 6| = 5 \tag{30}$$

We claim that

$$|\operatorname{supp} \mathcal{F}_{\mathcal{B}} \bmod b_k| = b_k - 1 \text{ for any } k \geq 2 \tag{31}$$

It is clear that $\gcd(12, b_k) = 1$ for any $k \geq 2$. Let $k \geq 2$ and take arbitrary $r \in \{1, \ldots, b_k - 1\}$. There exists $r' \in \mathbb{Z}$ such that

$$\begin{cases} r' \equiv r \bmod b_k \\ r' \equiv 1 \bmod 12 \end{cases} \tag{32}$$

Then $\gcd(12b_k, r') = 1$ and, by Dirichlet Theorem, there exists a prime number q of the form $q = 12b_k l + r'$ for some $l \in \mathbb{Z}$. Since, by (32), $q \equiv 1 \bmod 12$, $q \in \mathcal{F}_{\mathcal{B}}$ by (28). Moreover, $q \equiv r \bmod b_k$ by (32).

Thus the claim (31) follows. Clearly, (31) and (30) yield $\eta \in Y$.

We shall construct $h \in W$ such that $h \notin \overline{\Delta(\mathbb{Z}) \cap W}$. We denote $S_k = \{b_0, b_1, \ldots b_k\}$. Inductively we construct a sequence (n_{S_k}) of integers satisfying:

a) $n_{S_1} = 5$

b) $\operatorname{lcm}(S_k) | n_{S_{k+1}} - n_{S_k}$ for $k = 1, 2, \ldots$

c) $n_{S_k} \in \mathcal{F}_{S_k}$ for $k = 1, 2, \ldots$

Assume that $n_{S_1}, \ldots, n_{S_k}$ have been constructed. If b_{k+1} does not divide n_{S_k}, we set $n_{S_{k+1}} = n_{S_k}$. Otherwise we set $n_{S_{k+1}} = n_{S_k} + \operatorname{lcm}(S_k)$. The conditions a), b), c) follow easily by induction. Let

$$h = \lim_k \Delta(n_{S_k})$$

Thanks to c), $h \in W$.

But

$$U_{S_1}(h) \cap \Delta(\mathbb{Z}) \cap W = U_{S_1}(\Delta(5)) \cap \Delta(\mathbb{Z}) \cap W = \Delta(5 + \mathbb{1}2\mathbb{Z}) \cap W = \emptyset$$

the last equality by (29). (Clearly, $d(\mathcal{M}_{\mathcal{B}}) = 1$ and $\mathcal{B}$ is not taut.)

3.2 Proof of Theorem B

Lemma 3.7. [15] *If $\mathcal{B}$ is primitive and η is a Toeplitz sequence, then $\mathcal{B}$ is taut.*

Proof. Suppose that $\mathcal{B}$ is not taut. Then there are $c \in \mathbb{N}$ and a Behrend set $\mathcal{A}$ such that $c\mathcal{A} \subseteq \mathcal{B}$. Hence

$$d(\mathcal{M}_{\mathcal{B}} \cap c\mathbb{Z}) = c^{-1}, \tag{33}$$

because $\mathcal{M}_{\mathcal{A}}$ has density one. As $\mathcal{B}$ is primitive, c must be $\mathcal{B}$-free. So $\eta(c) = 1$, and (since η is Toeplitz) there exists $m \in \mathbb{N}$ such that $c + m\mathbb{Z} \subseteq \mathcal{F}_{\mathcal{B}}$. But then

$$\underline{d}(\mathcal{F}_{\mathcal{B}} \cap c\mathbb{Z}) \geqslant \underline{d}((c + m\mathbb{Z}) \cap c\mathbb{Z}) = d(\mathrm{lcm}(c, m)\mathbb{Z}) = \mathrm{lcm}(c, m)^{-1} > 0,$$

which contradics (33). □

Lemma 3.8. *Assume that $\eta \in Y$. If $\eta = \mathbb{1}_{\mathcal{F}_{\mathcal{B}}}$ is almost periodic (i.e. if the orbit closure of η is minimal), then $X_\eta \subseteq Y$.*

Proof. Fix $k \geq 1$. Since $\eta \in Y$, the support of η taken mod b_k misses exactly one residue class mod b_k (that is, it misses zero). Let B be a block on η such that its support mod b_k misses exactly one residue class mod b_k. Since η is almost periodic, the block B appears on η with bounded gaps. It follows that if C is any sufficiently long block that appears on η, its support misses exactly one residue class. Clearly this property passes to limits in the product topology, so each $y = \lim S^{m_i}\eta$ is also in Y. □

In general, we can define a map $\theta : Y \to \prod_{k\geq 1} \mathbb{Z}/b_k\mathbb{Z}$ by setting

$$\theta(y) = g = (g_k)_{k\geq 1} \text{ iff } \operatorname{supp} y \cap (b_k\mathbb{Z} - g_k) = \emptyset \text{ for all } k \geq 1.$$

Remark 2.51 in [4] tells us that

$$\theta(Y \cap X_\eta) \subset H,$$

while Remark 2.52 says that θ is continuous.

Corollary 3.1. *By the definitions of φ and θ, we have $\theta \circ \varphi(h) = h$ provided $\varphi(h) \in Y$. In particular, $\theta(\eta) = 0$ and θ is continuous at η. Moreover, θ is equivariant.*

For any map $\psi : X \to Y$ denote by $C_\psi \subseteq X$ the set of continuity points of this map.

Lemma 3.9. *Let (X, S) and (Y, T) be compact dynamical systems and assume that (X, S) is minimal. Let $\psi : X \to Y$ be a map satisfying $\psi \circ S = T \circ \psi$. Then $\overline{\psi(C_\psi)}$ is a minimal subset of Y.*

Proof. Denote by $Z := \overline{\{(x, \psi(x)) : x \in X\}}$ the closure of the graph of ψ and note that a fibre $Z_x = \{(x, y) : y \in Z\}$ is a singleton, if and only if $x \in C_\psi$. Let $Z_0 := \overline{\{(x, \psi(x)) : x \in C_\psi\}}$. We claim that $Z_0 \subseteq A$ whenever A is a non-empty closed $S \times T$-invariant subset of Z. Indeed, $\pi_X(A)$ is a non-empty closed S-invariant subset of X, so $\pi_X(A) = X$ by minimality of (X, S). In particular, $C_\psi \subseteq \pi_X(A)$. As all $A_x \subseteq Z_x$ with $x \in C_\psi$ are singletons, $\{(x, \psi(x)) : x \in C_\psi\} \subseteq A$. Hence also $Z_0 \subseteq A$.

This shows that Z_0 is a minimal subset of $X \times Y$ (and, by the way, that it is the only minimal subset of Z). It follows that $\pi_Y(Z_0)$ is a minimal subset of Y, and so it remains to show that $\psi(C_\psi) \subseteq \pi_Y(Z_0)$. But, for $x \in C_\psi$, $(x, \psi(x)) \in Z_0$, and so $\psi(x) \in \pi_Y(Z_0)$. □

Denote by C_φ the set of all points in H at which $\varphi : H \to \{0, 1\}^{\mathbb{Z}}$ is continuous.

[15] The authors are indebted to A. Bartnicka for pointing out and proving this lemma.

Lemma 3.10. *a)* $C_\varphi = \{h \in H : (h + \Delta(\mathbb{Z})) \cap \partial W = \emptyset\}$.
b) $C_\varphi + \Delta(1) = C_\varphi$.
c) $\overline{\varphi(C_\varphi)}$ *is the unique minimal subset* M.

Proof. a) This is proved by direct inspection, see e.g. [13, Lemma 6.1].
b) This is obvious.
c) This follows from Lemma 3.9. □

Proof of Theorem B. We start with a list of implications, which, when suitably combined, prove the assertions a) - e) of Theorem B. Most of these implications can be proved without assuming that $\mathcal{B}$ is primitive and that $\overline{\Delta(\mathbb{Z}) \cap W} = W$. Therefore we indicate explicitly, for which implications we use these extra assumptions.

Proof of the equivalence of B1 – B4: These equivalences follow from Proposition 3.2.

Proof of B1 ⇒ B6: Observe first that $0 \in H$ belongs to C_φ if and only if $\Delta(\mathbb{Z}) \cap \partial W = \emptyset$, see Lemma 3.10. But $\Delta(\mathbb{Z}) \cap \partial W = \Delta(\mathbb{Z}) \cap \left(\overline{\operatorname{int}(W)} \setminus \operatorname{int}(W)\right)$ in view of B1, and this intersection is empty by Lemma 3.5. As $\operatorname{int}(W) \neq \emptyset$ and as $H = \overline{\Delta(\mathbb{Z})}$, $\Delta(\mathbb{Z}) \cap W \neq \emptyset$ and hence $\varphi(0) \neq (\dots, 0, 0, 0, \dots)$.

Proof of B6 ⇒ B5: Let $\mathcal{B} = \{b_1, b_2, \dots\}$ and assume (B6) that $0 \in C_\varphi$, i.e. $\Delta(\mathbb{Z}) \cap \partial W = \emptyset$, and $\eta \neq (\dots, 0, 0, 0, \dots)$. Now, take $n \in \mathbb{Z}$. Either $n \in \mathcal{M}_\mathcal{B}$ - then $\eta(n) = 0$, so $b_s \mid n$ for some $s \geq 1$ and $\eta(n + jb_s) = 0$ for each $j \in \mathbb{Z}$. Or $n \in \mathcal{F}_\mathcal{B}$, i.e. $\Delta(n) \in W$. As $\Delta(\mathbb{Z}) \cap \partial W = \emptyset$ by assumption, this implies $\Delta(n) \in \operatorname{int}(W)$, so that $n \in \mathcal{E} = \bigcup_{S \subset \mathcal{B}} \mathcal{F}_{\mathcal{A}_S}$ by Lemma 3.1. Hence there is a finite subset $S \subset \mathcal{B}$ such that $n \in \mathcal{F}_{\mathcal{A}_S}$. As $\operatorname{lcm}(\mathcal{A}_S) = \operatorname{lcm}(S)$, this implies

$$n + \operatorname{lcm}(S)\,\mathbb{Z} \subseteq \mathcal{F}_{\mathcal{A}_S} \subseteq \mathcal{E} \subseteq \mathcal{F}_\mathcal{B}\,.$$

Hence $\eta(n + j\operatorname{lcm}(S)) = 1$ for each $j \in \mathbb{Z}$. This proves that η is a Toeplitz sequence different from $(\dots, 0, 0, 0, \dots)$.

Proof of B5 ⇒ B1: Assume that η is a Toeplitz sequence. Then $\mathcal{B}$ is taut by Lemma 3.7, hence $\overline{\Delta(\mathbb{Z}) \cap W} = W$ by Theorem A. Now B1 follows from the chain of the next three implications.

Proof of B5 ⇒ B8: Each Toeplitz sequence is almost periodic [8], [12, Theorem 4], i.e. its orbit closure is minimal.

Proof of B8 ⇒ B7: If $X_\eta = M$, then $\eta \in M$, and $\eta \neq (\dots, 0, 0, 0, \dots)$, because otherwise the minimality of X_η implies $X_\eta = \{(\dots, 0, 0, 0, \dots)\}$, contradicting $\operatorname{card}(X_\eta) > 1$.

Proof of B7 ⇒ B1 (assuming that $\overline{\Delta(\mathbb{Z}) \cap W} = W$): Assume that $(\dots, 0, 0, 0, \dots) \neq \eta \in M = \overline{\varphi(C_\varphi)}$. Then $M = X_\eta \subseteq Y$ by Lemma 3.8, and there is a sequence $h_1, h_2, \dots \in C_\varphi$ such that $\eta = \lim_{i \to \infty} \varphi(h_i)$. Consider $n \in \mathbb{Z}$ with $\Delta(n) \in W$, i.e. such that $\eta(n) = 1$. In particular $\eta = \varphi(0) \neq (\dots, 0, 0, 0, \dots)$. Corollary 3.1 implies $\lim_{i \to \infty} h_i = \lim_{i \to \infty} \theta(\varphi(h_i)) = \theta(\eta) = 0$. Then $1 = \eta(n) = \lim_{i \to \infty} \varphi(h_i)(n)$, i.e. $h_i + \Delta(n) \in W$ for all sufficiently large i. As $h_i \in C_\varphi$, we have $h_i + \Delta(\mathbb{Z}) \cap \partial W = \emptyset$ (Lemma 3.10). Hence $h_i + \Delta(n) \in \operatorname{int}(W)$ for all sufficiently large i, what implies that $\Delta(n) = \lim_{i \to \infty} h_i + \Delta(n) \in \overline{\operatorname{int}(W)}$. This proves that $\Delta(\mathbb{Z}) \cap W \subseteq \overline{\operatorname{int}(W)}$. Hence $W = \overline{\Delta(\mathbb{Z}) \cap W} \subseteq \overline{\operatorname{int}(W)}$, i.e. W is topologically regular.

Proof of B7 ⇒ B8: As $\eta \in M$, also $X_\eta \subseteq M$, and hence $X_\eta = M$. As $\eta \neq (\dots, 0, 0, 0, \dots)$, X_η contains no fixed point. Hence $\operatorname{card}(X_\eta) > 1$.

Proof of B1 ⇒ B9 (assuming that $\mathcal{B}$ is primitive): The window W is aperiodic because of Proposition 5.1, and it is topologically regular by B1. As B1 ⇒ B8, X_η is minimal. Therefore Corollary 1a) of [13], together with Lemmas 4.5 and 4.6 of the same reference, implies B9.

Proof of B9 ⇒ B8: This is trivial. □

Proposition 3.3. *Assume that the window W is topologically regular. Then $X_\eta \subseteq Y$.*

Proof. We start proving that $\eta \in Y$. Assume the contrary, that is, there are $b_0 \in \mathcal{B}$ and $r \in \{1, \dots, b_0-1\}$ such that

$$r + b_0\mathbb{Z} \subset \mathcal{M}_\mathcal{B}. \tag{34}$$

Let $a = \gcd(r, b_0)$ and $r' = r/a$, $b_0' = b_0/a$. (34) yields that for any $k \in \mathbb{N}$ there exists $b_k \in \mathcal{B}$ such that

$$b_k \mid a(r' + kb_0').$$

Let $J = \{k \in \mathbb{N} : r' + kb_0' \text{ is prime}\}$. By Dirichlet Theorem the set J is infinite. As $\mathcal{B}$ is primitive, b_k does not divide $a = \gcd(r, b_0)$. Hence

$$b_k = \gcd(a, b_k)\,(r' + kb_0')$$

for any $k \in J$. Since a has only finitely many divisors, there exists a divisor a' such that

$$b_k = a'(r' + kb_0')$$

for infinitely many $k \in J$. Thus we obtain a contradiction with the condition (B4) of Theorem B, which is equivalent to (B1) $W = \overline{\operatorname{int} W}$. Thus $\eta \in Y$.

Assume now that $x \in X_\eta$ and let $b \in \mathcal{B}$. As $\eta \in Y$, there is $N_b \in \mathbb{N}$ such that $\operatorname{card}(\operatorname{supp}(\eta|_{[0:N_b]}) \bmod b) = b-1$. As X_η is minimal by (B8) of Theorem B, there is $n \in \mathbb{N}$ such that $\operatorname{supp}(x|_{[n:n+N_b]}) = \operatorname{supp}(\eta|_{[0:N_b]})$. Hence

$$\operatorname{card}(\operatorname{supp}(x) \bmod b) \geqslant \operatorname{card}(\operatorname{supp}(x|_{[n:n+N_b]}) \bmod b) \geqslant \operatorname{card}(\operatorname{supp}(\eta|_{[0:N_b]}) \bmod b) = b - 1\,,$$

so that $x \in Y$, because $\operatorname{card}(\operatorname{supp}(x) \bmod b) \leqslant b - 1$ for all $x \in X_\eta$, see Footnote 4 to Remark 1.5. □

3.3 Proof of Theorem C

The equivalence of C1, C2 and C3 follows from Lemma 3.1. If C1 holds, i.e. if $\operatorname{int}(W) = \emptyset$, then $\varphi(C_\varphi) = \{(\dots, 0, 0, 0, \dots)\}$ is a shift invariant set [13, Proposition 3.3d with Remark 3.2b], so that $M = \overline{\varphi(C_\varphi)} = \{(\dots, 0, 0, 0, \dots)\}$. This is C6, and Theorem 3.8 in [4] shows that C4, C5, C6 and C7 are all equivalent.

We finish by proving C5 ⇒ C3: Consider any finite $S \subset \mathcal{B}$. As $\mathcal{B} \subseteq \mathcal{M}_{\mathcal{A}_S}$ by definition of the set $\mathcal{A}_S$, C5 implies that $1 \in \mathcal{A}_S$.

4 The sequence $\mathcal{B}$ and Haar measure

4.1 Measure and density

Lemma 4.1. $m_H(W) = 1 - \underline{d}(\mathcal{M}_\mathcal{B}) = \overline{d}(\mathcal{F}_\mathcal{B})$.

Proof. For $S \subset \mathcal{B}$ denote by $\mathcal{U}_S$ the family of all sets $U_S(\Delta(n))$ that are contained in W^c and by $\cup\mathcal{U}_S$ the union of these sets. Then

$$m_H(W^c) = \sup_S m_H(\cup\mathcal{U}_S) = \sup_S \frac{\#\mathcal{U}_S}{\operatorname{lcm}(S)} \geqslant \sup_S \frac{\#(\mathcal{M}_S \cap \{1, \dots, \operatorname{lcm}(S)\})}{\operatorname{lcm}(S)} = \sup_S d(\mathcal{M}_S) = \underline{d}(\mathcal{M}_\mathcal{B})$$

by Lemma 2.5a, and similarly

$$m_H(W^c) \leqslant \sup_S \frac{\#(\mathcal{M}_{\mathcal{B}\cap\mathcal{A}_S} \cap \{1,\dots,\mathrm{lcm}(S)\})}{\mathrm{lcm}(S)} = \sup_S d(\mathcal{M}_{\mathcal{B}\cap\mathcal{A}_S}) \leqslant \underline{d}(\mathcal{M}_{\mathcal{B}})$$

by Lemma 2.5b. □

Corollary 4.1. ***[4, Theorem 4.1]*** *$\mathcal{B}$ is a Besicovich sequence if and only if $\mathcal{F}_{\mathcal{B}}$ is generic for the Mirsky measure. (As $n \in \mathcal{F}_{\mathcal{B}}$ iff $\Delta(n) \in W$, it would be more precise to say that the sequence $(\Delta(n))_n$ is generic for the Mirsky measure.)*

Proof. If $\mathcal{B}$ is Besicovich, then $d(\mathcal{F}_{\mathcal{B}}) = m_H(W)$, so that $\mathcal{F}_{\mathcal{B}}$ has maximal density. Hence it is generic for the Mirsky measure, see [13, Theorem 5b]. On the other hand, if $\mathcal{F}_{\mathcal{B}}$ is generic for (any) measure, then its frequency of ones converges in particular, which means that its asymptotic density exists. □

Lemma 4.2. $m_H(\mathrm{int}(W)) = \sup_S d(\mathcal{F}_{\mathcal{A}_S}) \leqslant \underline{d}(\mathcal{E})$

Proof. For $S \subset \mathcal{B}$ denote by $\mathcal{U}_S^o$ the family of all sets $U_S(\Delta(n))$ that are contained in $\mathrm{int}(W)$ and by $\cup\mathcal{U}_S^o$ the union of these sets. Recall from Lemma 3.1a that $\#\mathcal{U}_S^o = \#(\mathcal{F}_{\mathcal{A}_S} \cap \{1,\dots,\mathrm{lcm}(S)\})$. Then

$$m_H(\mathrm{int}(W)) = \sup_S m_H(\cup\mathcal{U}_S^o) = \sup_S \frac{\#\mathcal{U}_S^o}{\mathrm{lcm}(S)} = \sup_S \frac{\#(\mathcal{F}_{\mathcal{A}_S} \cap \{1,\dots,\mathrm{lcm}(S)\})}{\mathrm{lcm}(S)} = \sup_S d(\mathcal{F}_{\mathcal{A}_S})$$

□

Lemma 4.3. $m_H(\partial W) = \inf_S \bar{d}(\mathcal{M}_{\mathcal{A}_S} \setminus \mathcal{M}_{\mathcal{B}}) \leqslant \inf_S d(\mathcal{M}_{\mathcal{A}_S\setminus\mathcal{B}})$.

Proof.

$$\begin{aligned} m_H(\partial W) &= m_H(W) - m_H(\mathrm{int}(W)) = \bar{d}(\mathcal{F}_{\mathcal{B}}) - \sup_S d(\mathcal{F}_{\mathcal{A}_S}) = \inf_S \left(\bar{d}(\mathcal{F}_{\mathcal{B}}) - d(\mathcal{F}_{\mathcal{A}_S})\right) \\ &= \inf_S \bar{d}(\mathcal{M}_{\mathcal{A}_S} \setminus \mathcal{M}_{\mathcal{B}}) \end{aligned}$$

□

4.2 Regular Toeplitz sequences

Let $\mathcal{B} = \{b_1, b_2, \dots\}$. For each $k \geq 1$, consider the sequence

$$b_1, \dots, b_k, c_{k+1}^{(k)}, c_{k+2}^{(k)}, \dots,$$

where

$$c_{k+i}^{(k)} := \gcd(\mathrm{lcm}(b_1,\dots,b_k), b_{k+i}),\ \ i \geq 1.$$

Then:

$$c_i^{(k)} | \mathrm{lcm}(b_1,\dots,b_k), \text{whence } \{c_{k+i}^{(k)} :\ i \geq 1\} \text{ is finite.}$$

$$\mathscr{A}_{\{b_1,\dots,b_k\}} = \{b_1,\dots,b_k\} \cup \{c_{k+i}^{(k)} :\ i \geq 1\}.$$

$$c_{k+1}^{(k)} | b_{k+1}.$$

$$c_{k+1+i}^{(k)} | c_{k+1+i}^{(k+1)},\ \text{ for each } i \geq 1.$$

Moreover, following Lemma 2.5c, there is an increasing sequence (k_n) such that

$$\mathscr{B} \cap \mathscr{A}_{\{b_1,\dots,b_{k_n}\}} = \{b_1, \dots, b_{k_n}\}. \tag{35}$$

We assume that $W \subset H$ is topologically regular, so by Remark 1.3, $\eta = \mathbb{1}_{\mathcal{F}_\mathscr{B}}$ is a Toeplitz sequence. We set $s_k := \mathrm{lcm}(b_1, \dots, b_k)$ and would like now to examine the sequence (s_k) as a periodic structure of η. More precisely, we would like to see for how many $n \in [1, s_k]$, we have $\eta(n) = \eta(n + js_k)$ for each $j \in \mathbb{Z}$. We call any such n to be "good". Now, if $n \in \mathcal{F}_{\mathscr{A}_{\{b_1,\dots,b_k\}}}$, then $n + s_k\mathbb{Z} \subset \mathcal{F}_{\mathscr{A}_{\{b_1,\dots,b_k\}}}$, so n is good. Otherwise, $n \in \mathcal{M}_{\mathscr{A}_{\{b_1,\dots,b_k\}}}$. Then either $n \in \mathcal{M}_{b_1,\dots,b_k}$ and then clearly $\eta(n + js_k) = 0$ for each $j \in \mathbb{Z}$, so again n is good, or

$$n \in \mathcal{M}_{\{c^{(k)}_{K+i}:\, i\geq 1\}} \setminus \mathcal{M}_{\{b_1,\dots,b_k\}}.$$

Only for such n, we are not sure that n is good. Moreover, note that in view of (4.2), we have

$$\mathcal{M}_{\{c^{(k+1)}_{k+1+i}:\, i\geq 1\}} \subset \mathcal{M}_{\{c^{(k)}_{k+i}:\, i\geq 1\}},$$

so the sequence $(d(\mathcal{M}_{\{c^{(k)}_{k+i}:\, i\geq 1\}}))_k$ is decreasing, and so is the sequence $(\overline{d}(\mathcal{M}_{\{c^{(k)}_{k+i}:\, i\geq 1\}} \setminus \mathcal{M}_\mathscr{B}))$. Therefore, by taking into account (35), the infimum of this sequence is equal to the liminf, in fact to the limit and we have

$$\inf_{S\subset\mathscr{B}} \overline{d}(\mathcal{M}_{\mathscr{A}_S}) \setminus \mathcal{M}_\mathscr{B}) = \liminf_{k\to\infty} \overline{d}(\mathcal{M}_{\{c^{(k)}_{k+i}:\, i\geq 1\}\setminus\mathcal{M}_{\{b_1,\dots,b_k\}}}) \,. \tag{36}$$

Definition 4.1. *Let $\eta = \mathbb{1}_{\mathcal{F}_\mathscr{B}}$ be a Toeplitz sequence. It is a* regular Toeplitz sequence *for the periodic structure (s_k), $s_k = \mathrm{lcm}(b_1, \dots, b_k)$, if the* lim inf *in* (36) *is zero.*

Now, using Lemma 4.3, the identity in (36) shows the following result.

Proposition 4.1. *If W is topologically regular, then $\eta = \mathbb{1}_{\mathcal{F}_\mathscr{B}}$ is a regular Toeplitz sequence for the periodic structure (s_k), $s_k = \mathrm{lcm}(b_1, \dots, b_k)$, if and only if $m_H(\partial W) = 0$.*

Example 4.1. Assume that $\{b'_k : k \geq 1\}$ is a coprime set of odd numbers and let $b_k = 2^k b'_k$. Then $c^{(k)}_{k+i} = 2^k$ for each $i \geq 1$. Hence, we have even $d(\mathcal{M}_{\{c^{(k)}_{k+i}:\, i\geq 1\}}) \to 0$, in particular η is a regular Toeplitz sequence for the periodic structure (s_k) with $s_k = 2^k b'_1 \cdots b'_k$. This example comes from [4].

We will now show that we can obtain Toeplitz sequences also in case $m_H(\partial W) > 0$.

Example 4.2. We will construct $\mathscr{B} = \{b_1, b_2, \dots\}$ such that this set is thin (hence taut) and that $\lim_{k\to\infty} \inf(\{c^{(k)}_{k+i} : i \geq 1\} \setminus \{b_1, \dots, b_k\}) = \infty$, which, by Proposition 3.2, implies that W is topologically regular (and hence η is Toeplitz by Remark 1.3). Let $\delta_k > 0$ and $\sum_{k\geq 1} \delta_k < 1/16$.

We start with $b_1 = 2^3$ and set $c^{(1)}_{1+i} = 2$ for each $i \geq 1$. Suppose that a sequence

$$b_1, \dots, b_k, c^{(k)}_{k+1}, c^{(k)}_{k+2}, \dots$$

has been defined. We require that this sequence satisfies:

$$c^{(k)}_{k+i} | \mathrm{lcm}(b_1, \dots, b_k),\ i \geq 1,$$

$$c^{(k)}_{k+i} \notin \{b_1, \dots, b_k\},\ i \geq 1,$$

$$\text{for each } i \geq 1, |\{j \geq 1 : c^{(k)}_{k+j} = c^{(k)}_{k+i}\}| = +\infty.$$

We will now show how to define $c^{(k+1)}_{k+2}, c^{(k+1)}_{k+3}, \dots$ and then b_{k+1}. Recall an elementary lemma.

Lemma 4.4. *Let F_1, F_2 be finite sets of natural numbers such that* $\gcd(f_1, f_2) = 1$ *for each $f_i \in F_i$, $i = 1, 2$. Then $d(\mathcal{M}_{F_1 \cdot F_2}) = d(\mathcal{M}_{F_1} \cap \mathcal{M}_{F_2}) = d(\mathcal{M}_{F_1})d(\mathcal{M}_{F_2})$.*

Choose $P \subset \mathscr{P} \setminus \operatorname{spec}\{b_1, \dots, b_k\}$, so that (by Lemma 4.4)

$$d(\mathcal{M}_{P \cdot \{c^{(k)}_{k+i} :\, i \geq 1\}}) \geq d(\mathcal{M}_{\{c^{(k)}_{k+i} :\, i \geq 1\}}) - \delta_k.$$

In view of (4.2),

$$\{i \geq 2 : \ c^{(k)}_{k+i} = c^{(k)}_{k+1}\} = \{r_1, r_2, \dots\}$$

Let $P = \{q_1, \dots, q_t\}$. Then set

$$c^{(k+1)}_{k+r_1+tj} := c^{(k)}_{k+1} q_1, c^{(k+1)}_{k+r_2+tj} := c^{(k)}_{k+1} q_2, \dots, c^{(k+1)}_{k+r_t+tj} := c^{(k)}_{k+1} q_t$$

for each $j = 0, 1, \dots$ If $2 \notin \{i \geq 2 : \ c^{(k)}_{k+i} = c^{(k)}_{k+1}\}$ then repeat the same construction with the set $\{i \geq 2 : \ c^{(k)}_{k+i} = c^{(k)}_{k+2}\}$. Since (by (4.2)) the set $\{c^{(k)}_{k+i} : \ i \geq 1\}$ is finite, our construction of the sequence $(c^{(k+1)}_{k+1+i})_i$ is done in finitely many steps. Finally, we set $b_{k+1} := c^{(k)}_{k+1} \prod_{q \in P} q$ (or, if needed, $b_{k+1} := c^{(k)}_{k+1} \prod_{q \in P} q^{\alpha_{k+1}}$ for any $\alpha_{k+1} \in \mathbb{N}$). Note that

$$\gcd(\operatorname{lcm}(b_1, \dots, b_k), b_{k+1}) = c^{(k)}_{k+1}$$

since $P \cap \{b_1, \dots, b_k\} = \emptyset$. More than that, by the construction, we also have

$$\gcd(\operatorname{lcm}(b_1, \dots, b_k), b_{k+i}) = c^{(k)}_{k+i} \text{ for each } i \geq 1.$$

Moreover, it is not hard to see that the new sequence

$$b_1, \dots, b_k, b_{k+1}, c^{(k+1)}_{k+2}, c^{(k+1)}_{k+3}, \dots$$

satisfies (4.2)-(4.2). Furthermore, $\mathcal{B} = \{b_1, b_2, \dots\}$ satisfies the other requirements mentioned at the beginning of the construction so that η is a Toeplitz sequence and W is topologically regular. Note that in our construction $d(\mathcal{M}_{\{c^{(1)}_{1+i} :\, i \geq 1\}}) = 1/2$. Moreover, by (4.2)

$$d(\mathcal{M}_{\{c^{(k)}_{k+i} :\, i \geq 1\}}) \geq d(\mathcal{M}_{\{c^{(1)}_{1+i} :\, i \geq 1\}}) - \sum_{j=1}^{k} \delta_k \geq \frac{1}{4}$$

for each $k \geq 1$. Finally notice that $d(\mathcal{M}_{\mathcal{B}}) \leq \sum_{k \geq 1} 1/b_k$, which (by construction) can be made smaller than 1/8. It follows that $\lim_{k \to \infty} \overline{d}(\mathcal{M}_{\mathscr{A}_{\{b_1, \dots, b_k\}}} \setminus \mathcal{M}_{\mathcal{B}}) > 0$, whence $m_H(\partial W) > 0$.

5 The maximal equicontinuous factor of X_η

5.1 The period groups of W and of $\operatorname{int}(W)$

Given a subset $A \subseteq H$, denote by

$$H_A := \{h \in H : W + h = W\}$$

the *period group* of A. The set $A \subseteq H$ is *topologically aperiodic*, if $H_A = \{0\}$. The following simple observations are proved in [14, Lemma 6.1]:

- $H_A \subseteq H_{\bar{A}} \cap H_{\mathrm{int}(A)}$.
- If A is closed, then $H_{\mathrm{int}(A)} = H_{\overline{\mathrm{int}(A)}}$ is closed.

Proposition 5.1. *Assume that $\mathcal{B}$ is primitive. Then the window W is topologically aperiodic.*

Proof. Suppose that $h = (h_b)_{b\in\mathcal{B}} \neq 0$ and

$$W + h = W\,. \tag{37}$$

Since $h \neq 0$, there is $b \in \mathcal{B}$ such that b does not divide h_b. Let $n := \gcd(b, h_b)$. Then $n \in \mathcal{F}_{\mathcal{B}}$, as otherwise there exists $b' \in \mathcal{B}$ such that $b' \mid n$; but then $b' \mid b$ a contradiction ($\mathcal{B}$ is assumed to be primitive). Hence $\Delta(n) \in W$. There are $x, y \in \mathbb{Z}$ such that $n = xh_b + yb$, whence $b \mid n - xh_b$. It follows that $\Delta(n) - xh \notin W$, a contradiction with (37). □

If W is topologically regular, then clearly $\mathrm{int}(W)$ is topologically aperiodic, as well. Otherwise $H_{\mathrm{int}(W)}$ may be non-trivial, as we will see in the course of this section.

Recall from (1) that for any set $A \subseteq \mathbb{N}$,

$$\mathcal{M}_A = \mathcal{M}_{A^{prim}}$$

If A^{prim} is finite, then $\mathcal{M}_A$ is a union of finitely many arithmetic progressions. Let c_A denote the period of $\mathcal{M}_A$, that is, the least natural number such that $c_A + \mathcal{M}_A = \mathcal{M}_A$.

Lemma 5.1. *Assume that $A, B \subset \mathbb{N}$ are finite*
a) If $c + \mathcal{M}_A = \mathcal{M}_A$ for some $c \in \mathbb{N}$, then $\mathrm{lcm}(A^{prim}) \mid c$.
b) $c_A = \mathrm{lcm}(A^{prim})$
c) if $A \subset \mathcal{M}_B$ then $c_B \mid c_A$

Proof. a) For any $a \in A^{prim}$ we have $a + c\mathbb{Z} \subseteq \mathcal{M}_{A^{prim}}$, from which it follows that there exists $a' \in A^{prim}$ such that $a' \mid \gcd(a, c)$. But, as A^{prim} is primitive, that means that $a' = a$ and $a \mid c$. We conclude that $\mathrm{lcm}(A^{prim}) \mid c$.
b) Clearly $\mathcal{M}_A + \mathrm{lcm}(A^{prim}) = \mathcal{M}_A$, thus $c_A \mid \mathrm{lcm}(A^{prim})$ and the assertion follows from a).
c) If $A \subset \mathcal{M}_B$ then $A^{prim} \subset \mathcal{M}_{B^{prim}}$, hence $\mathrm{lcm}(B^{prim}) \mid \mathrm{lcm}(A^{prim})$, and we finish by b).

□

Lemma 5.2. *Assume that $S \subseteq S'$ are finite subsets of $\mathcal{B}$, then $\mathcal{A}_S = \{\gcd(a, \mathrm{lcm}(S)) : a \in \mathcal{A}_{S'}\}$.*

Proof. Since $\mathrm{lcm}(S) \mid \mathrm{lcm}(S')$, $\gcd(b, \mathrm{lcm}(S)) = \gcd(\gcd(b, \mathrm{lcm}(S')), \mathrm{lcm}(S))$ for any $b \in \mathcal{B}$ and the assertion follows. □

Let $S_1 \subset S_2 \subset \ldots \subset S_k \subset \ldots$ be a filtration of $\mathcal{B}$ with finite sets and denote

$$s_k := \mathrm{lcm}(S_k),\ c_k := c_{\mathcal{A}_{S_k}}$$

By Lemma 5.1 c) we have $c_l \mid c_{l+1}$ for any l. It follows that, for any k, the sequence $(\gcd(s_k, c_l))_{l\geq 1}$ stabilizes on a divisor d_k of s_k. Clearly, since $c_k \mid s_k$,

$$c_k \mid d_k \mid s_k\,. \tag{38}$$

Observe that

$$d_k = \gcd(s_k, d_{k+1})\,. \tag{39}$$

Indeed, there is $l_0 \in \mathbb{N}$ such that $d_{k+1} = \gcd(s_{k+1}, c_l)$ for all $l > l_0$. Since $s_k \mid s_{k+1}$, we get

$$\gcd(s_k, c_l) = \gcd(s_k, \gcd(s_{k+1}, c_l)) .$$

It follows that $\gcd(s_k, c_l) = \gcd(s_k, d_{k+1})$ for $l > l_0$, and (39) follows.

By applying (39) we prove by induction that

$$d_k = \gcd(s_k, d_{k+j}) . \tag{40}$$

for $j \geq 0$.

Lemma 5.3. *Let $(n_k)_{k\in\mathbb{N}}$ be a sequence of integers. The following are equivalent:*

$$\forall k \in \mathbb{N}: \; c_k \mid n_k \;\textit{ and }\; s_k \mid n_{k+1} - n_k , \tag{41}$$

and

$$\forall k \in \mathbb{N}: \; d_k \mid n_k \;\textit{ and }\; s_k \mid n_{k+1} - n_k . \tag{42}$$

Proof. As $c_k \mid d_k$, (42) implies (41). Conversely, assume that (41) holds. We show inductively that for all $j \geqslant 0$

$$\forall k \in \mathbb{N}: \; \gcd(s_k, c_{k+j}) \mid n_k , \tag{43}$$

and this implies (42) immediately.

For $j = 0$, (43) follows from (41), because $c_k \mid s_k$. So suppose that (43) holds for some $j \geqslant 0$. Then

$$\begin{aligned} n_{k+1} &= 0 \mod \gcd(s_{k+1}, c_{k+1+j}) \text{ and} \\ n_{k+1} &= n_k \mod s_k . \end{aligned}$$

Hence $n_k = 0 \mod \gcd(s_k, s_{k+1}, c_{k+1+j}) = \gcd(s_k, c_{k+j+1})$ for all $k \in \mathbb{N}$, i.e. (43) for $j + 1$. □

Recall that $H_{\mathrm{int}(W)} = \{h \in H : \mathrm{int}(W) + h = \mathrm{int}(W)\}$ denotes the period group of $\mathrm{int}(W)$.

Proposition 5.2. *a) $h \in H_{\mathrm{int}(W)}$ if and only if $h = \lim_k \Delta(n_k)$ for some sequence $(n_k)_k$ satisfying*

$$\forall k \in \mathbb{N}: \; d_k \mid n_k \;\textit{ and }\; s_k \mid n_{k+1} - n_k . \tag{44}$$

Moreover, sequences $(n_k)_k$ can be defined inductively: For n_1 there are s_1/d_1 choices and, given $n_1, \ldots, n_k$, there are precisely $s_{k+1}/\operatorname{lcm}(s_k, d_{k+1})$ many choices for n_{k+1}.

b) $H_{\mathrm{int}(W)} = \{0\}$ if and only if $s_k = d_k$ for all $k \in \mathbb{N}$.

Remark 5.1. Observe that, in view of (39),

$$\frac{s_k}{d_k} \cdot \frac{s_{k+1}}{\operatorname{lcm}(s_k, d_{k+1})} = \frac{s_k\, s_{k+1} \gcd(s_k, d_{k+1})}{d_k\, s_k\, d_{k+1}} = \frac{s_k\, s_{k+1}\, d_k}{d_k\, s_k\, d_{k+1}} = \frac{s_{k+1}}{d_{k+1}},$$

so that

$$\frac{s_k}{d_k} \mid \frac{s_{k+1}}{d_{k+1}}$$

$$\frac{s_k}{d_k} = \frac{s_1}{d_1} \cdot \prod_{j=1}^{k-1} \frac{s_{j+1}}{\operatorname{lcm}(s_j, d_{j+1})}.$$

Proof of Proposition 5.2. a) For each S_k denote by $W_k := \bigcup_{n\in\mathcal{F}_{\mathcal{A}_{S_k}}} U_{S_k}(\Delta(n))$. Then $\mathrm{int}(W)$ is the increasing union of the sets W_k, see Lemma 3.1, and $U_{S_k}(\Delta(n)) \subseteq W_k$ if and only if $U_{S_k}(\Delta(n)) \subseteq \mathrm{int}(W)$. Let $h = \lim_k \Delta(n_k)$, where n_k stands for n_{S_k}, which was defined in Lemma 3.1b. Then

$$\forall k \in \mathbb{N}:\ s_k \mid n_{k+1} - n_k\ , \tag{45}$$

and $h \in H_{\mathrm{int}(W)}$, if and only if

$$\forall k \in \mathbb{N}:\ \mathcal{F}_{\mathcal{A}_{S_k}} + n_k = \mathcal{F}_{\mathcal{A}_{S_k}}\ . \tag{46}$$

Indeed, let $k \in \mathbb{N}$, $m \in \mathcal{F}_{\mathcal{A}_{S_k}}$, and let $g = (g_b)_{b\in\mathcal{B}}$ be any element from $U_{S_k}(\Delta(m)) \subseteq \mathrm{int}(W)$. Then $g_b = m \bmod b$ for all $b \in S_k$. Assume now that $h \in H_{\mathrm{int}(W)}$. Then $g + h \in \mathrm{int}(W)$ and $(g + h)_b = m + n_k \bmod b$ for all $b \in S_k$, so that $g+h \in U_{S_k}(\Delta(m+n_k))$. Hence $U_{S_k}(\Delta(m))+h \subseteq U_{S_k}(\Delta(m+n_k)) = U_{S_k}(\Delta(m)) + \Delta(n_k)$. In particular, $U_{S_k}(\Delta(m))$ and $U_{S_k}(\Delta(m + n_k))$ have identical Haar measure, and so do $U_{S_k}(\Delta(m)) + h$ and $U_{S_k}(\Delta(m + n_k))$. As both are open sets and one is contained in the other, they must coincide. Hence $U_{S_k}(\Delta(m + n_k)) = U_{S_k}(\Delta(m)) + h \subseteq \mathrm{int}(W) + h = \mathrm{int}(W)$, so that $m + n_k \in \mathcal{F}_{\mathcal{A}_{S_k}}$. This proves that $\mathcal{F}_{\mathcal{A}_{S_k}} + n_k \subseteq \mathcal{F}_{\mathcal{A}_{S_k}}$. As $\mathcal{A}_{S_k}$ is a finite set, this implies $\mathcal{F}_{\mathcal{A}_{S_k}} + n_k = \mathcal{F}_{\mathcal{A}_{S_k}}$.

Conversely, assume that (46) holds, and let $U_{S_k}(\Delta(m)) \subseteq \mathrm{int}(W)$. Recall that this implies $U_{S_k}(\Delta(m)) \subseteq W_k$, i.e. $m \in \mathcal{F}_{\mathcal{A}_{S_k}}$. Hence, by assumption, also $m + n_k \in \mathcal{F}_{\mathcal{A}_{S_k}}$, so that $U_{S_k}(\Delta(m + n_k)) \subseteq W_k \subseteq \mathrm{int}(W)$. Let $g \in U_{S_k}(\Delta(m))$. Then $g_b = m \bmod b$ for all $b \in S_k$, so that $(g + h)_b = m + n_k \bmod b$ for all $b \in S_k$, i.e. $g + h \in U_{S_k}(\Delta(m + n_k))$. Hence $U_{S_k}(\Delta(m)) + h \subseteq U_{S_k}(\Delta(m + n_k)) \subseteq \mathrm{int}(W)$. As this argument applies to all k and all $U_{S_k}(\Delta(m)) \subseteq \mathrm{int}(W)$, it proves that $\mathrm{int}(W)+h \subseteq \mathrm{int}(W)$. The same Haar measure argument as before, applied to the open set $\mathrm{int}(W)$, shows that $\mathrm{int}(W) + h = \mathrm{int}(W)$, i.e. $h \in H_{\mathrm{int}(W)}$.

Condition (46) is equivalent to

$$\forall k \in \mathbb{N}:\ c_k = \mathrm{lcm}(\mathcal{A}_{S_k}^{prim}) \mid n_k\ . \tag{47}$$

Invoking Lemma 5.3, we conclude

$$h \in H_{\mathrm{int}(W)} \quad\Leftrightarrow\quad \forall k \in \mathbb{N}:\ d_k \mid n_k \ \text{ and }\ s_k \mid n_{k+1} - n_k\ . \tag{48}$$

This proves the claimed equivalence.

Now we describe all sequences $(n_k)_{k\in\mathbb{N}}$ which satisfy(44) and $n_k \in \{0, \dots, s_k - 1\}$ for all k. Denote $q_k := s_k/d_k$.

n_1: Let $n_1 = m_1 d_1$ for any $m_1 \in \{0, \dots, q_1 - 1\}$.

n_2: n_2 must be chosen such that $n_2 = 0 \bmod d_2$ and $n_2 = n_1 \bmod s_1$. As $\gcd(s_1, d_2) = d_1 \mid n_1$ in view of (39), the CRT guarantees the existence of at least one solution n_2, and if n_2 is one particular solution, then the set of all solutions is precisely $n_2 + \mathrm{lcm}(s_1, d_2)\cdot\mathbb{Z}$. As n_2 is to be chosen in $\{0, \dots, s_2 - 1\}$, there are exactly $s_2/\,\mathrm{lcm}(s_1, d_2)$ possible choices for n_2.

⋮

n_{k+1}: n_{k+1} must be chosen such that $n_{k+1} = 0 \bmod d_{k+1}$ and $n_{k+1} = n_k \bmod s_k$. As $\gcd(s_k, d_{k+1}) = d_k \mid n_k$ in view of (39), the CRT guarantees the existence of at least one solution n_{k+1}, and if n_{k+1} is one particular solution, then the set of all solutions is precisely $n_{k+1} + \mathrm{lcm}(s_k, d_{k+1})\cdot\mathbb{Z}$. As n_{k+1} is to be chosen in $\{0, \dots, s_{k+1} - 1\}$, there are exactly $s_{k+1}/\,\mathrm{lcm}(s_k, d_{k+1})$ possible choices for n_{k+1}.

b) $H_{\mathrm{int}(W)} = \{0\} \Leftrightarrow$ there is unique choice of the numbers n_k described in a) $\Leftrightarrow s_1/d_1 = 1$ and $s_{k+1}/\,\mathrm{lcm}(s_k, d_{k+1}) = 1$ for any $k \Leftrightarrow d_k = s_k$ for any k, the last equivalence by Remark 5.1. □

5.2 Proof of Theorem D

Remark 5.2. If $(S_k)_k$ is a filtration of $\mathcal{B}$ by finite sets and if $h = (h_b)_{b \in \mathcal{B}} \in H$, then we write $\lim_k \Delta(n_{S_k}) = h$, whenever $n_{S_k} \in \mathbb{Z}$ are numbers such that for every $k \in \mathbb{N}$:

$$h_b = n_{S_k} \bmod b \text{ for all } b \in S_k$$

Let us denote $s_k = \mathrm{lcm}(S_k)$. There is an inverse system of groups

$$\ldots \mathbb{Z}/s_{k+1}\mathbb{Z} \to \mathbb{Z}/s_k\mathbb{Z} \to \ldots \to \mathbb{Z}/s_1\mathbb{Z}$$

The homomorphisms are the canonical projections. Observe that $s_k | n_{S_{k+1}} - n_{S_k}$ for any k and the sequence $(n_{S_k} + s_k\mathbb{Z})_k$ is an element of the inverse limit $\varprojlim \mathbb{Z}/s_k\mathbb{Z}$. In this way we obtain an isomorphism of topological groups

$$\sigma : \varprojlim \mathbb{Z}/s_k\mathbb{Z} \cong H \tag{49}$$

given by $(n_{S_k} + s_k\mathbb{Z})_k \mapsto \lim_k \Delta(n_{S_k})$. Compare Remark 2.32 [4]. In particular, the inverse limit does not depend on the filtration $(S_k)_k$.[16]

Proof of Proposition 1.3. Let $\beta_k : \mathbb{Z}/s_k\mathbb{Z} \to \mathbb{Z}/d_k\mathbb{Z}$ be the map given by $n + s_k\mathbb{Z} \mapsto n + d_k\mathbb{Z}$, let M_k be the kernel of β_k and let $\alpha_k : M_k \to \mathbb{Z}/s_k\mathbb{Z}$ be the canonical embedding. There is a commutative diagram of abelian groups

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
\ldots & \xrightarrow{f'_{k+1}} & M_k & \xrightarrow{f'_k} & M_{k-1} & \xrightarrow{f'_{k-1}} & \ldots & \xrightarrow{f'_2} & M_1 \\
 & & \downarrow{\scriptstyle \alpha_k} & & \downarrow{\scriptstyle \alpha_{k-1}} & & & & \downarrow{\scriptstyle \alpha_1} \\
\ldots & \xrightarrow{f_{k+1}} & \mathbb{Z}/s_k\mathbb{Z} & \xrightarrow{f_k} & \mathbb{Z}/s_{k-1}\mathbb{Z} & \xrightarrow{f'_{k-1}} & \ldots & \xrightarrow{f'_2} & \mathbb{Z}/s_1\mathbb{Z} \\
 & & \downarrow{\scriptstyle \beta_k} & & \downarrow{\scriptstyle \beta_{k-1}} & & & & \downarrow{\scriptstyle \beta_1} \\
\ldots & \xrightarrow{f''_{k+1}} & \mathbb{Z}/d_k\mathbb{Z} & \xrightarrow{f''_k} & \mathbb{Z}/d_{k-1}\mathbb{Z} & \xrightarrow{f''_{k-1}} & \ldots & \xrightarrow{f''_2} & \mathbb{Z}/d_1\mathbb{Z} \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
 & & 0 & & 0 & & & & 0
\end{array}$$

where $f_k(n + s_k\mathbb{Z}) = n + s_{k-1}\mathbb{Z}$, f'_k is the restriction of f_k to M_k and $f''_k(n + d_k\mathbb{Z}) = n + d_{k-1}\mathbb{Z}$.

The columns of the diagram are exact sequences of groups, in other words, the diagram can be interpreted as an exact sequence of inverse systems of abelian groups.

Since inverse limit is a left exact functor, see [9, Chapter II, Theorem 12.3], we obtain an exact sequence

$$0 \to \varprojlim M_k \xrightarrow{\alpha} \varprojlim \mathbb{Z}/s_k\mathbb{Z} \xrightarrow{\beta} \varprojlim \mathbb{Z}/d_k\mathbb{Z} \tag{50}$$

The condition (39) yields that the homomorphism γ in (50) is surjective, thus we have an exact sequence

$$0 \to \varprojlim M_k \xrightarrow{\alpha} \varprojlim \mathbb{Z}/s_k\mathbb{Z} \xrightarrow{\beta} \varprojlim \mathbb{Z}/d_k\mathbb{Z} \to 0 \tag{51}$$

[16] The last statement follows from a general property of inverse limits: the inverse limits of cofinal inverse systems are isomorphic, [9, Chapter II, Section 12].

Indeed, let $(n_k + d_k\mathbb{Z})_k \in \varprojlim \mathbb{Z}/d_k\mathbb{Z}$. By induction we construct the numbers $m_1, m_2, \ldots$ such that $d_k | m_k - n_k$ and $s_k | m_{k+1} - m_k$, for any k. Then $\beta((m_k + s_k\mathbb{Z})_k) = (n_k + d_k\mathbb{Z})_k$. We set $m_1 = n_1$. Assume that $m_1, \ldots m_k$ have been defined. Since $d_k \mid n_{k+1} - n_k$, $d_k \mid m_k - n_k$ and $\gcd(d_{k+1}, s_k) = d_k$, there exists integers x, y such that $xd_{k+1} + ys_k = m_k - n_{k+1}$. We set $m_{k+1} = m_k - ys_k$.

There are group isomorphisms $g_k : \mathbb{Z}/\frac{s_k}{d_k}\mathbb{Z} \to M_k$ given by $g_k(n + \frac{s_k}{d_k}\mathbb{Z}) = d_k n + s_k\mathbb{Z}$ and making the following diagram commutative

$$\begin{array}{ccccccccc} \ldots & \to & \mathbb{Z}/\frac{s_{k+1}}{d_{k+1}}\mathbb{Z} & \to & \mathbb{Z}/\frac{s_k}{d_k}\mathbb{Z} & \to & \ldots & \to & \mathbb{Z}/\frac{s_1}{d_1}\mathbb{Z} \\ & & \downarrow g_{k+1} & & \downarrow g_k & & & & \downarrow g_1 \\ \ldots & \longrightarrow & M_{k+1} & \xrightarrow{f'_{k+1}} & M_k & \xrightarrow{f'_k} & \ldots & \xrightarrow{f'_2} & M_1 \end{array}$$

(the arrows in the upper row represent the canonical projections). It follows that there is an isomorphism

$$\varprojlim M_k \cong \varprojlim \mathbb{Z}/\frac{s_k}{d_k}\mathbb{Z} \tag{52}$$

By Proposition 5.2 a) it follows that $\varprojlim M_k$ is isomorphic to $H_{\mathrm{int}(W)}$. There is an isomorphism given by $\sigma\alpha$, where σ is the isomorphism defined in Remark 5.2.

Now a), b) and c) follow from (51), (52) and Remark 5.2. In order to prove d) it is enough to note that $s_k = d_k$ if and only if $s_k \mid c_{k+j}$ for some $j \geq 0$. □

Proof of Theorem D. This is an immediate corollary to Proposition 1.3. □

5.3 Examples

Remark 5.3. Given a prime number p and $m \in \mathbb{Z}$ we denote by $v_p(m)$ be the p-valuation of m, that is, if $m \neq 0$ then $v_p(m)$ is the maximal integer such that $p^{v_p(m)} \mid m$ and $v_p(0) = +\infty$. Assume that $t = (t_k)$ is a sequence of natural numbers such that $t_k \mid t_{k+1}$ for any k. Set $v_p(t) = \sup_k v_p(t_k)$. The sequence t yields an inverse system of abelian groups

$$\ldots \to Z/t_{k+1}\mathbb{Z} \to Z/t_k\mathbb{Z} \to \ldots \to Z/t_1\mathbb{Z}$$

where the arrows represent the canonical projections $n + t_{k+1}\mathbb{Z} \mapsto n + t_k\mathbb{Z}$. The inverse limit $\varprojlim \mathbb{Z}/t_k\mathbb{Z}$ of this system is isomorphic to the group

$$\prod_{p \in \mathcal{P}} G_p$$

where $G_p = \mathbb{Z}/p^{v_p(t)}\mathbb{Z}$ if $v_p(t) < +\infty$ and $G_p = \widehat{\mathbb{Z}}_p$ (the group of p-adic numbers) otherwise, i.e. when $\lim_k v_p(t_k) = +\infty$.

Recall from (6) that

$$\mathcal{A}_\infty = \{c \in \mathbb{N} : \forall_{S \subset \mathcal{B}} \exists_{S' : S \subseteq S'} : c \in \mathcal{A}_{S'} \setminus S'\}. \tag{53}$$

Our first exaxmple has a finite, non-trivial maximal equicontinuous factor and a finite set $\mathcal{A}_\infty$.

Example 5.1. $\mathcal{B} = \{36\} \cup \{2p_1, 2p_2, \ldots\} \cup \{3q_1, 3q_2, \ldots\}$, where $p_1, q_1, p_2, q_2, \ldots$ are pairwise different primes. Let $S_k = \{36, 2p_1, \ldots, 2p_k, 3q_1, \ldots, 3q_k\}$. Then

$$s_k = 36 p_1 \cdots p_k q_1 \cdots q_k,\ \mathcal{A}_{S_k} = \{2, 3\},\ c_k = d_k = 6\ ,$$

so that

$$\frac{s_k}{d_k} = 6 p_1 \cdots p_k q_1 \cdots q_k \, .$$

In particular, the maximal equicontinuous factor of X_η is the translation by 1 on $\mathbb{Z}/6\mathbb{Z}$. Moreover, $\mathcal{A}_\infty = \{2, 3\}$, so that $\emptyset \neq \overline{\mathrm{int}(W)} \neq W$ by Theorems B and C.

Our next example has an infinite maximal equicontinuous factor different from H and an infinite set $\mathcal{A}_\infty$.

Example 5.2. Let $p_1, q_1, p_2, q_2, \ldots$ be pairwise different primes. Let

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \ldots$$

where

$$\begin{aligned}
\mathcal{B}_1 &= \{p_1 q_1\} \\
\mathcal{B}_2 &= \{p_1 p_2^2, p_1 q_2^2, q_1 q_2^2\} \\
\mathcal{B}_3 &= \{p_1 p_2 p_3^2, p_1 p_2 q_3^2, p_1 q_2 q_3^2, q_1 q_3^2\} \\
\mathcal{B}_4 &= \{p_1 p_2 p_3 p_4^2, p_1 p_2 p_3 q_4^2, p_1 p_2 q_3 q_4^2, p_1 q_2 q_4^2, q_1 q_4^2\} \\
\mathcal{B}_5 &= \{p_1 p_2 p_3 p_4 p_5^2, p_1 p_2 p_3 p_4 q_5^2, p_1 p_2 p_3 q_4 q_5^2, p_1 p_2 q_3 q_5^2, p_1 q_2 q_5^2, q_1 q_5^2\} \\
&\ldots
\end{aligned}$$

That is,

$$\mathcal{B}_{k+1} = \{p_1 \ldots p_k p_{k+1}^2,\ p_1 \ldots p_k q_{k+1}^2,\ p_1 \ldots p_{k-1} q_k q_{k+1}^2\} \cup \{\frac{b q_{k+1}^2}{q_k^2} : b \in \mathcal{B}_k \setminus \{p_1 \ldots p_{k-1} p_k^2, p_1 \ldots p_{k-1} q_k^2\}\}$$

for $k \geq 2$.

Let $S_k = \mathcal{B}_1 \cup \ldots \cup \mathcal{B}_k$. Then $s_k = \mathrm{lcm}(S_k) = p_1 p_2^2 \ldots p_k^2 q_1 q_2^2 \ldots q_k^2$ and

$$\mathcal{A}_{S_k} = S_k \cup \{p_1 \ldots p_k,\ p_1 \ldots p_{k-1} q_k,\ p_1 \ldots p_{k-2} q_{k-1},\ \ldots, p_1 q_2, q_1\} \, ,$$

so that

$$\mathcal{A}_{S_k}^{prim} = \{p_1 \ldots p_k,\ p_1 \ldots p_{k-1} q_k,\ p_1 \ldots p_{k-2} q_{k-1},\ \ldots, p_1 q_2, q_1\} \, .$$

Hence

$$c_k = p_1 \cdots p_k q_1 \cdots q_k \quad \text{and} \quad d_k = \gcd(s_k, c_{k+j}) = c_k \, ,$$

so that

$$\frac{s_k}{d_k} = p_2 \cdots p_k q_2 \cdots q_k \, .$$

Hence $H_{\mathrm{int}(W)} \cong \prod_{i=2}^{+\infty} \mathbb{Z}/p_i q_i \mathbb{Z}$ and $H/H_{\mathrm{int}(W)} \cong \prod_{i=1}^{+\infty} \mathbb{Z}/p_i q_i \mathbb{Z}$ are infinite compact groups. Moreover,

$$\mathcal{A}_\infty = \limsup_{k \to \infty} \mathcal{A}_{S_k} \setminus S_k = \{q_1, p_1 q_2, p_1 p_2 q_3, p_1 p_2 p_3 q_4, \ldots\}$$

is infinite and does not contain the number 1, thus $\emptyset \neq \overline{\mathrm{int}(W)} \neq W$ by Theorems B and C.

We end with a non-trivial example where the maximal equicontinuous factor equals H and $\mathcal{A}_\infty$ is an infinite set.

Example 5.3. Let $q, p_1, p_2, \ldots$ be pairwise different odd primes. Let

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \ldots$$

where

$$\begin{aligned}
\mathcal{B}_1 &= \{p_1 q\} \\
\mathcal{B}_2 &= \{p_2 q, p_1 p_2\} \\
\mathcal{B}_3 &= \{p_3 q, p_1 p_3, p_2 p_3\} \\
\mathcal{B}_4 &= \{p_4 q, p_1 p_4, p_2 p_4, p_3 p_4\} \\
&\ldots
\end{aligned}$$

That is,

$$\mathcal{B}_k = \{p_k q, p_1 p_k, \ldots, p_{k-1} p_k\}$$

for $k \geq 1$. Let $S_k = \mathcal{B}_1 \cup \ldots \cup \mathcal{B}_k$. Then $s_k = \mathrm{lcm}(S_k) = q p_1 \ldots p_k$ and

$$\mathcal{A}_{S_k} = S_k \cup \{p_1, \ldots, p_k\} \cup \{q\}$$

hence $c_{\mathcal{A}_{S_k}} = q p_1 \ldots p_k = \mathrm{lcm}(S_k)$, so that $s_k = c_k = d_k$ for all k. In particular $\mathrm{int}(W)$ is aperiodic by Proposition 5.2. Moreover,

$$\mathcal{A}_\infty = \limsup_{k\to\infty} (\mathcal{A}_{S_k} \setminus S_k) = \{q, p_1, p_2, \ldots\}$$

is infinite and does not contain the number 1, thus $\emptyset \neq \overline{\mathrm{int}(W)} \neq W$ by Theorems B and C.

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