Recurrence and ergodicity of cocycles over IETs

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Cocycles over interval exchange transformations and multivalued Hamiltonian flows

Multivalued Hamiltonian flows

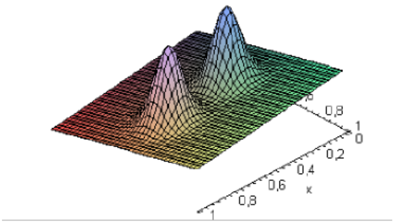
Let (M, ω) be a compact symplectic smooth surface and β be a closed 1-form on M. Denote by $X : M \to TM$ the **multivalued Hamiltonian vector field** determined by

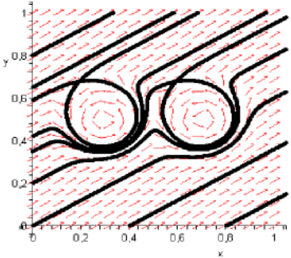
$$\beta = i_X\omega = \omega(X, \cdot).$$

Let $(\phi_t)_{t\in\mathbb{R}}$ stand for the **multivalued Hamiltonian flow** on M associated to the vector field X. Since $d\beta = di_X\omega = 0$, the flow $(\phi_t)_{t\in\mathbb{R}}$ preserves the symplectic form ω, and hence it preserves the smooth measure $\nu = \nu_\omega$ determined by ω.
Denote by $\pi : \widehat{M} \to M$ the universal cover of M and by $\widehat{\beta}$ the pullback of β by $\pi : \widehat{M} \to M$. Since $\widehat{M}$ is simply connected and $\widehat{\beta}$ is also a closed form, there exists a smooth function $H : \widehat{M} \to \mathbb{R}$, called a **multivalued Hamiltonian**, such that $dH = \widehat{\beta}$.
By Darboux's theorem, in local coordinates $\omega = dx \wedge dy$, and then

$$X(x, y) = \left(\frac{\partial}{\partial y}H(x, y), -\frac{\partial}{\partial x}H(x, y)\right).$$

Assume that H is a Morse function. Then the flow $(\phi_t)_{t\in\mathbb{R}}$ has finitely many fixed points (equal to zeros of β and equal to images of critical points of H by the map π). The set of fixed points $\mathcal{F}(\beta)$ consists of centers or non-degenerated saddles. Assume that any two different saddles are not connected by a separatrix of the flow (called a saddle connection). Nevertheless, the flow can have saddle connections which are loops. Each such saddle connection gives a decomposition of M into two nontrivial invariant subsets.



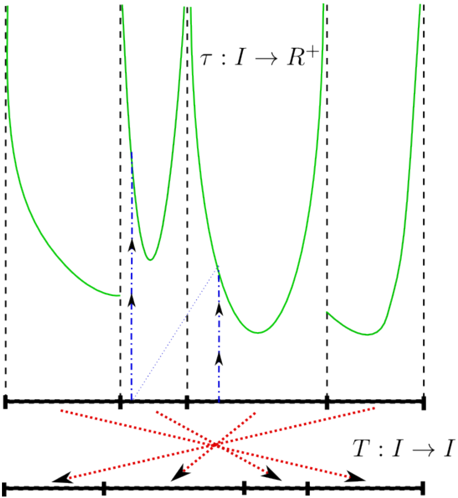


The surface M can be represented as the finite union of disjoint $(\phi_t)_{t\in\mathbb{R}}$–invariant sets as follows

$$M = \mathcal{P} \cup \mathcal{S} \cup \bigcup_{\mathcal{T}\in\mathfrak{T}} \mathcal{T},$$

where $\mathcal{P}$ is an open set consisting of periodic orbits, $\mathcal{S}$ is a finite union of fixed points or saddle connections, and for each $\mathcal{T} \in \mathfrak{T}$ its closure $\overline{\mathcal{T}}$ is a transitive component of $(\phi_t)_{t\in\mathbb{R}}$.
We will consider the multivalued Hamiltonian flow $(\phi_t)_{t\in\mathbb{R}}$ only on such transitive component $\mathcal{T}$. Each such flow has a special representation over a minimal IET $T : I \to I$ and under a roof function $\tau : I \to \mathbb{R}^+$ which is piecewise C^∞ and it has singularities of logarithmic type at discontinuities of T.

Special representation



Extensions of multivalued Hamiltonian flows

Let us consider a system of differential equations on $M \times \mathbb{R}^\ell$ of the form

$$\begin{cases} \frac{dx}{dt} &= X(x), \\ \frac{dy}{dt} &= f(x), \end{cases}$$

for $(x, y) \in M \times \mathbb{R}^\ell$, where $f : M \to \mathbb{R}^\ell$ is a smooth function. Then the associated flow $(\Phi_t^f)_{t\in\mathbb{R}}$ on $M \times \mathbb{R}^\ell$ is given by

$$\Phi_t^f(x, y) = \left(\phi_t x, y + \int_0^t f(\phi_s x)\, ds\right).$$

It follows that $(\Phi_t^f)_{t\in\mathbb{R}}$ is a skew product flow with the base flow $(\phi_t)_{t\in\mathbb{R}}$ on M and the cocycle $F : \mathbb{R} \times M \to \mathbb{R}^\ell$ given by

$$F(t, x) = \int_0^t f(\phi_s x)\, ds.$$

The deviation of the cocycle F was studied by Forni (Ann. of Math. 1997, 2001) for typical $(\phi_t)_{t\in\mathbb{R}}$ with no saddle connections.

Recurrence and ergodicity

The aim of my talk is to discuss recurrence and ergodicity of the flow $(\Phi_t^f)_{t\in\mathbb{R}}$ on $\mathcal{T}\times\mathbb{R}^\ell$, where $\mathcal{T}$ is a transitive component of multivalued Hamiltonian flow.

Let us consider the Poincaré map corresponding to the transversal submanifold $I\times\mathbb{R}^\ell\subset\overline{\mathcal{T}}\times\mathbb{R}^\ell$. This map is isomorphic to the skew product

$$T_\varphi : I\times\mathbb{R}^\ell\to I\times\mathbb{R}^\ell,\qquad T_\varphi(x,y)=(Tx,y+\varphi(x)),$$

where

$$\varphi(x)=\varphi^f(x):=F(\tau(x),x)=\int_0^{\tau(x)} f(\phi_s x)\,ds.$$

$$\int_I \varphi^f(x)\,dx=\int_{\mathcal{T}} f\,\omega$$

Therefore, the flow $(\Phi^f_t)_{t\in\mathbb{R}}$ on $\mathcal{T}\times\mathbb{R}^\ell$ is isomorphic to a special flows built over T_φ.

$$\text{the recurrence of } (\Phi^f_t)_{t\in\mathbb{R}} \iff \text{the recurrence of } T_\varphi$$

$$\text{the ergodicity of } (\Phi^f_t)_{t\in\mathbb{R}} \iff \text{the ergodicity of } T_\varphi$$

Corollary (after Schmidt)

If $\ell = 1$ then $(\Phi^f_t)_{t\in\mathbb{R}}$ on $\mathcal{T}\times\mathbb{R}$ is recurrent if and only if $\int_{\mathcal{T}} f\,\omega = 0$.

Corollary (after Conze or Schmidt)

if $\int_{\mathcal{T}} f\,\omega = \int_I \varphi(x)\,dx = 0$ and $\|\varphi^{(n)}\| = o(1/\sqrt[\ell]{n})$ then the skew product T_φ, and hence the flow $(\Phi^f_t)_{t\in\mathbb{R}}$ on $\mathcal{T}\times\mathbb{R}^\ell$ are recurrent.

Properties of φ

Theorem

The cocycle φ is piecewise C^∞ (over exchanged intervals) and

- *if $f(x) \neq 0$ for some $x \in \mathcal{F}(\beta) \cap \overline{\mathcal{T}}$ then φ has singularities of logarithmic type;*
- *if $f(x) = 0$ for all $x \in \mathcal{F}(\beta) \cap \overline{\mathcal{T}}$ then φ is of bounded variation and $S(\varphi) = \int_I \varphi'(x)\,dx = \int_{\partial\mathcal{T}} f\,\theta^\beta$;*
- *if additionally $f'(x) = f''(x) = 0$ for all $x \in \mathcal{F}(\beta) \cap \overline{\mathcal{T}}$ then φ and its derivative are piecewise continuous.*

The space of functions satisfying the last condition we will denote by $C_0^2(M, \beta)$.

Skew products over rotations

If $M = \mathbb{T}^2$ then T is an irrational rotation the study of $(\Phi_t^f)_{t\in\mathbb{R}}$ leads to the well explored world of cylindrical transformations for piecewise smooth cocycles.

- If $\varphi : \mathbb{T} \to \mathbb{R}^\ell$ is of bounded variation and $\int \varphi(x)\,dx = 0$ then by Denjoy-Koksma inequality T_φ is recurrent for each $\ell \geqslant 1$.
- If $\varphi : \mathbb{T} \to \mathbb{R}$ is piecewise absolutely continuous with $S(\varphi) \neq 0$ then T_φ is ergodic (Pask, 1990).
- If $\varphi(x) = -\log x - \log(1-x) + ac(x)$ (for all irrational rotations) or $\varphi(x) = -\log x + ac(x)$ (for almost every - well approximated - rotations) then T_φ is ergodic (Frączek–Lemańczyk 2004; Fayad–Lemańczyk 2006).

IETs of periodic type

Definition

Let $T : [0,1) \to [0,1)$ be an IET exchanging intervals I_j, $j = 1, \ldots, d$. $T : [0,1) \to [0,1)$ is said be of **periodic type** if for some $0 < \rho < 1$ the induced transformation T_J, $J = [0, \rho)$ is an IET which is isomorphic to T via the rescaling $[0,1) \ni x \mapsto \rho x \in [0, \rho)$, and each interval $J_i \subset J$, $i = 1, \ldots, d$ (exchanged by T_J) before the first return to J visits all intervals I_j, $j = 1, \ldots, d$.

This notion is an counterpart to quadratic irrationals.

Proposition

If $T : I \to I$ is of periodic type then all maximal subintervals of continuity of T^n have proportional length, i.e.

$$\frac{1}{cn} \leqslant |I'| \leqslant \frac{c}{n} \text{ for each such subinterval } I'.$$

Classical approach

Essential values of the cocycle $\varphi : X \to G$. $g \in E(\varphi)$ if

$$\forall_{0 \in V} \forall_{\mu(B)>0} \exists_{n \in \mathbb{Z}}\ \mu(B \cap T^{-n}B \cap (\varphi^{(n)} \in g + V)) > 0$$

$E(\varphi) \subset G$ is a subgroup and

$$T_\varphi \text{ is ergodic } \iff E(\varphi) = G.$$

If there exists (C_n), $\mu(C_n) \geqslant \alpha > 0$, $\mu(C_n \triangle T^{-1}C_n) \to 0$ and φ satisfies a Denjoy-Koksma type inequality on C_n, i.e.

$$(\varphi^{(q_n)}) \text{ is "bounded"},$$

then $(\varphi^{(q_n)})_*(\mu(\,\cdot\,|C_n)) \to \nu$ and supp $\nu \subset E(\varphi)$. This approach works for irrational rotations, but does not work for IETs, for which any appropriate Denjoy-Koksma inequality does not exist (Zorich, 1997). Here

$$|\varphi^{(h_n)}(x) - a_n| \leqslant Var\, f \text{ on } C_n,$$

but we lose control of the behaviour of the sequence (a_n).

Method working for cocycles with non-zero sum of jumps

Theorem

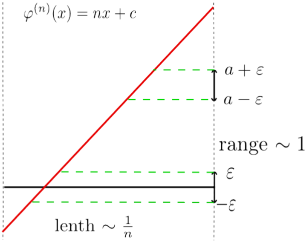
Let T be an IET of periodic type and let $\varphi : I \to \mathbb{R}$ be C^2–function on each exchanged interval. If φ has zero mean and $S(\varphi) \neq 0$ then the skew product is ergodic.

Corollary (from Marmi-Moussa-Yoccoz, 2005)

Any such cocycle is cohomologous to a piecewise linear cocycle with slope $S(\varphi)$.

Therefore we can assume that $\varphi^{(n)} = nx + c$ on each interval of continuity. Since $\int \varphi\, dx = 0$, φ is recurrent

$$\forall_{\varepsilon>0}\forall_{\mu(B)>0}\exists_{n>0}\ \mu(B \cap T^{-n}B \cap (\varphi^{(n)} \in (-\varepsilon, \varepsilon))) > 0$$



it follows that

$$\mu(B \cap T^{-n}B \cap (\varphi^{(n)} \in (a - \varepsilon, a + \varepsilon))) > 0$$

for each a from an interval. Consequently, $E(\varphi)$ contains an interval, and hence T_φ is ergodic.

Correction of cocycles

Instead of proving the ergodicity for the original cocycle we make a correction which kills the influence of the unstable bundle of, so called, Rauzy-Veech cocycle.

Theorem

If T has periodic type then every zero mean cocycle $\varphi : I \to \mathbb{R}$ of bounded variation there exists a piecewise constant (over exchanged intervals) function h such that for the corrected cocycle $\widehat{\varphi} = \varphi + h$ a Denjoy-Koksma type inequality holds.

Then we can use the classical approach.

Theorem

Let T be an IET of periodic type and $\varphi : I \to \mathbb{R}$ be a zero mean cocycle $\varphi : I \to \mathbb{R}$ with $S(\varphi) = 0$. If φ "has enough rationally independent jumps" (it is a typical property) then the corrected cocycle is ergodic.

The existence of the correction is based on ideas introduced by Marmi-Moussa-Yoccoz (2005). This correction can be naturally transported to the multivalued Hamiltonian setting. More precisely, there exists a finite-dimensional subspace $H \subset C_0^2(M,\beta)$ and a bounded operator $P : C_0^2(M,\beta) \to H$ such that

$$\varphi^{f+Pf} = \widehat{\varphi}.$$

The operator $P : C_0^2(M,\beta) \to H$ is closely related to the space of invariant distributions used by Forni (2001) in order to prove the deviation spectrum property. More precisely, if f has zero mean on $\mathcal{T}$ and $Pf = 0$ then

$$|\int_0^T f(\phi_s x)\, ds| \leqslant C_f \log T.$$

Ergodicity of extensions of multivalued Hamiltonians

Theorem

Suppose that $(\phi_t)_{t\in\mathbb{R}}$ is a multivalued Hamiltonian flow such that $(\phi_t)_{t\in\mathbb{R}}$ on $\mathcal{T}$ has special representation over an IET of periodic type. If $f \in C_0^2(M,\beta)$ is a function such that $\int_{\mathcal{T}} f\,\omega = 0$ and

- $\int_{\partial\mathcal{T}} f\,\theta^\beta \neq 0$ *then the extension $(\Phi_t^f)_{t\in\mathbb{R}}$ is ergodic on $\mathcal{T}\times\mathbb{R}$;*
- $\int_{\partial\mathcal{T}} f\,\theta^\beta = 0$ *and we "control" $\int f\,\theta^\beta$ for connected components of $\partial\mathcal{T}$ then the corrected extension $(\Phi_t^{f+Pf})_{t\in\mathbb{R}}$ is ergodic on $\mathcal{T}\times\mathbb{R}$.*

Using both methods of proving ergodicity we can also construct functions f taking values in $\mathbb{R}^\ell$ such that the flow $(\Phi_t^f)_{t\in\mathbb{R}}$ is ergodic on $\mathcal{T}\times\mathbb{R}^\ell$. Here we have to prove recurrence at first.

Theorem

Let $T: I\to I$ be an IET of periodic type and let $\theta_1>\theta_2\geqslant 1$ be the greatest Lyapunov exponents of, so called periodic matrix of T. If $\varphi: I\to\mathbb{R}$ is a function of bounded variation and zero mean then

$$|\varphi^{(n)}(x)|\leqslant Cn^{\theta_2/\theta_1}.$$

In particular, if $\theta_2/\theta_1<1/\ell$ then each cocycle $\varphi: I\to\mathbb{R}^\ell$ of bounded variation and zero mean is recurrent.

THE END!