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On mild mixing of special flows over irrational rotations under piecewise smooth functions

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Abstract. It is proved that all special flows over the rotation by an irrational α with bounded partial quotients and under f which is piecewise absolutely continuous with a non-zero sum of jumps are mildly mixing. Such flows are also shown to enjoy a condition which emulates the Ratner condition introduced in [**20**]. As a consequence we construct a smooth vector–field on $\mathbb{T}^2$ with one singularity point such that the corresponding flow $(\varphi_t)_{t\in\mathbb{R}}$ preserves a smooth measure, its set of ergodic components consists of a family of periodic orbits and one component of positive measure on which $(\varphi_t)_{t\in\mathbb{R}}$ is mildly mixing and is spectrally disjoint from all mixing flows.

1. *Introduction*

The property of mild mixing of a (finite) measure–preserving transformation has been introduced by Furstenberg and Weiss in [**7**]. By definition, a finite measure–preserving transformation is mildly mixing if its Cartesian product with an arbitrary ergodic (finite or infinite not of type I) measure-preserving transformation remains ergodic. It is also proved in [**7**] that a probability measure–preserving transformation $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is mildly mixing iff T has no non-trivial rigid factor, i.e. $\liminf_{n\to+\infty} \mu(T^{-n}B \triangle B) > 0$ for every $B \in \mathcal{B}$, $0 < \mu(B) < 1$. For importance and naturality of the notion of mild mixing see e.g. [**1, 4, 6, 15, 16, 22**].

It is immediate from the definition that the (strong) mixing property of an action implies its mild mixing which in turn implies the weak mixing property. In case of Abelian non–compact group actions, Schmidt in [**21**] constructed examples (using Gaussian processes) of mildly mixing actions that are not mixing. A famous example of a mildly mixing but not mixing system is the well–known Chacon transformation T (mild mixing of T follows directly from the minimal self-joining property of T, [**10**]). However, none of known examples of mild but not mixing dynamical system was proved to be of smooth origin (see [**12**] and [**8**] - discussions about the three paradigms of smooth ergodic theory).

Special flows built over an ergodic rotation on the circle and under a piecewise C^1–function with non–zero sum of jumps were introduced and studied by J. von Neumann in [**18**]$*$. He proved that such flows are weakly mixing for each irrational rotation. The weak mixing property was then proved for the von Neumann class of functions but over ergodic interval exchange transformations by Katok in [**11**], while in [**9**] the weak mixing property was shown for the von Neumann class of functions where the C^1–condition is replaced by the absolute continuity, however in [**9**], T is again an arbitrary irrational rotation. The absence of mixing of T^f is well–known; it has been proved by Kočergin in [**14**]. In fact, from the spectral point of view special flows in this paper have no spectral measure which is Rajchman, i.e. they are spectrally disjoint from mixing flows (see [**5**]).

The aim of this paper is to show that the class of special flows built from a piecewise absolutely continuous function $f : \mathbb{T} \to \mathbb{R}$ with a non–zero sum of jumps and over a rotation by α with bounded partial quotients is mildly mixing (Theorem 7.2). One of the main tools, which yet can be considered as another motivation of this paper, is Theorem 6.1 in which we prove that T^f satisfies a property similar to the famous Ratner property† introduced in [**20**] (see also [**23**]). It will follow that any ergodic joining of T^f with any ergodic flow (S_t) is either the product joining or a finite extension of (S_t). In Section 7, the absence of partial rigidity for T^f will be shown. Finally these two properties combined will yield mild mixing (see Lemma 4.1).

As a consequence of our measure–theoretic results we will construct a mildly mixing (but not mixing) C^∞–flow $(\varphi_t)_{t\in\mathbb{R}}$ whose corresponding vector–field has one singular point (of a simple pole type). More precisely, we will construct $(\varphi_t)_{t\in\mathbb{R}}$ on the two–dimensional torus, such that (φ_t) preserves a positive C^∞–measure and the family of ergodic components of (φ_t) consists of a family of periodic orbits and one non–trivial component of positive measure which is mildly mixing but not mixing. More precisely, the non–trivial component of (φ_t) is measure–theoretically isomorphic to a special flow T^f which is built over an irrational rotation $Tx = x+\alpha$ on the circle and under a piecewise C^∞–function $f : \mathbb{T} \to \mathbb{R}$ with a non–zero sum of jumps. In these circumstances T^f lies in the parabolic paradigm (see [**8**]).

Some minor changes in the construction of the C^∞–flow $(\varphi)_{t\in\mathbb{R}}$ (which uses some ideas descended from Blokhin [**2**]) yield an ergodic C^∞–flow which is mildly mixing but not mixing and lives on the torus with attached Möbius strip. This flow will enjoy the Ratner property in the sense introduced in Section 5.

$*$ We thank A. Katok for turning our attention to this article.

† The possibility of having the Ratner property for some special flows over irrational rotations was suggested to us by B. Fayad and J.-P. Thouvenot.

2. *Basic definitions and notation*

Assume that T is an ergodic automorphism of a standard probability space $(X,\mathcal{B},\mu)$. A measurable function $f:X\to\mathbb{R}$ determines a cocycle $f^{(\,\cdot\,)}(\,\cdot\,):\mathbb{Z}\times X\to\mathbb{R}$ given by

$$f^{(m)}(x)=\begin{cases} f(x)+f(Tx)+\ldots+f(T^{m-1}x) & \text{if} \quad m>0\\ 0 & \text{if} \quad m=0\\ -\big(f(T^m x)+\ldots+f(T^{-1}x)\big) & \text{if} \quad m<0.\end{cases}$$

Denote by λ Lebesgue measure on $\mathbb{R}$. If $f:X\to\mathbb{R}$ is a strictly positive L^1-function, then by $T^f=(T^f_t)_{t\in\mathbb{R}}$ we will mean the corresponding special flow under f (see e.g. [**3**], Chapter 11) acting on $(X^f,\mathcal{B}^f,\mu^f)$, where $X^f=\{(x,s)\in X\times\mathbb{R}: 0\le s<f(x)\}$ and $\mathcal{B}^f$ (μ^f) is the restriction of $\mathcal{B}\otimes\mathcal{B}(\mathbb{R})$ $(\mu\otimes\lambda)$ to X^f. Under the action of the flow T^f each point in X^f moves vertically at unit speed, and we identify the point $(x,f(x))$ with $(Tx,0)$. More precisely, if $(x,s)\in X^f$ then

$$T^f_t(x,s)=(T^n x, s+t-f^{(n)}(x)),$$

where $n\in\mathbb{Z}$ is a unique number such that

$$f^{(n)}(x)\le s+t<f^{(n+1)}(x).$$

We denote by $\mathbb{T}$ the circle group $\mathbb{R}/\mathbb{Z}$ which we will constantly identify with the interval $[0,1)$ with addition mod 1. For a real number t denote by $\{t\}$ its fractional part and by $\|t\|$ its distance to the nearest integer number. For an irrational $\alpha\in\mathbb{T}$ denote by (q_n) its sequence of denominators (see e.g. [**13**]), that is we have

$$\frac{1}{2q_nq_{n+1}}<\left|\alpha-\frac{p_n}{q_n}\right|<\frac{1}{q_nq_{n+1}}, \tag{1}$$

where

$$\begin{aligned} q_0&=1, \quad q_1=a_1, \quad q_{n+1}=a_{n+1}q_n+q_{n-1}\\ p_0&=0, \quad p_1=1, \quad p_{n+1}=a_{n+1}p_n+p_{n-1}\end{aligned}$$

and $[0;a_1,a_2,\ldots]$ stands for the continued fraction expansion of α. We say that α has *bounded partial quotients* if the sequence (a_n) is bounded. If $C=\sup\{a_n:n\in\mathbb{N}\}+1$ then

$$\frac{1}{2Cq_n}<\frac{1}{2q_{n+1}}<\|q_n\alpha\|<\frac{1}{q_{n+1}}<\frac{1}{q_n}$$

for each $n\in\mathbb{N}$.

3. *Construction*

In this section, using the procedure of gluing of flows which was described by Blokhin in [**2**], we will construct the flow $(\varphi_t)_{t\in\mathbb{R}}$ that was announced in Introduction.

Let $\alpha \in \mathbb{R}$ be an irrational number. We denote by $(\psi_t)_{t\in\mathbb{R}}$ the linear flow $\psi_t(x_1, x_2) = (x_1 + t\alpha, x_2 + t)$ of the torus $\mathbb{T}^2$.

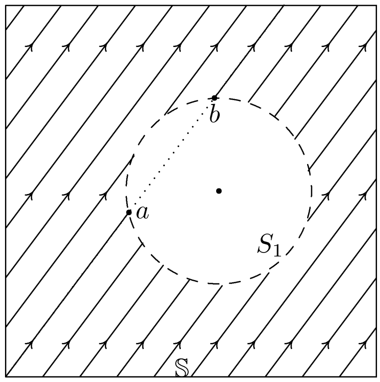

Figure 1

Let us cut out (from the torus) a disk D which is disjoint from the circle $\mathbb{S} = \{(x, 0) \in \mathbb{T}^2 : x \in [0, 1)\}$ and intersects the segment $O_{[0,1]} = \{(\alpha t, t) : t \in [0, 1]\}$. We will denote by S_1 the circle which bounds D. Let a and b be points of S_1 that lie on the segment $O_{[0,1]}$ (see Fig.1).

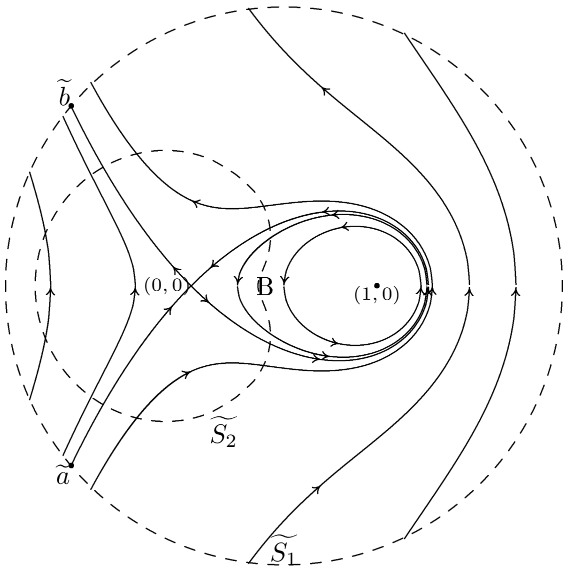

Figure 2. The phase portrait for the Hamiltonian system $H(x, y) = \frac{1}{2}e^{2x}(y^2 + (x - 1)^2)$; the portrait is the same as for (2)

Now let us consider the flow $(\widetilde{\psi}_t)_{t\in\mathbb{R}}$ on the disk $\widetilde{D} = \{(x, y) \in \mathbb{R}^2 : (x - 1/2)^2 + y^2 \leq (3/2)^2\}$ given by the system of equations (on $\mathbb{R}^2 \setminus \{(0, 0)\}$)

$$\begin{cases} \frac{dx}{dt} &= \frac{-y}{x^2+y^2} \\ \frac{dy}{dt} &= \frac{x(x-1)+y^2}{x^2+y^2}. \end{cases} \tag{2}$$

By the Liouville theorem, $(\widetilde{\psi}_t)_{t\in\mathbb{R}}$ preserves the measure $e^{2x}(x^2+y^2)\,dx\,dy$. Let $\widetilde{S_1}$ be the boundary of $\widetilde{D}$. The flow $(\widetilde{\psi}_t)$ has a singularity at $(0,0)$ and a fixed point $(1,0)$ which is a center (see Fig. 2). Moreover the set

$$B = \{(x,y)\in \widetilde{D} : e^{2x}(y^2+(x-1)^2)<1,\ x>0\}.$$

consists of periodic orbits. Let $\widetilde{a}$ and $\widetilde{b}$ be the points of intersection of $\widetilde{S_1}$ and the separatrices of $(0,0)$. By Lemma 1 in **[2]**, there exists a C^∞–diffeomorphism $g : S_1 \to \widetilde{S_1}$ such that $g(a)=\widetilde{a}$, $g(b)=\widetilde{b}$, and there exist a C^∞–flow $(\varphi_t)_{t\in\mathbb{R}}$ on $M=(\mathbb{T}\setminus D)\cup_g \widetilde{D}$ and a C^∞–measure μ on $(\mathbb{T}\setminus D)\cup_g \widetilde{D}$ such that

- $(\varphi_t)_{t\in\mathbb{R}}$ preserves μ,
- the flow $(\varphi_t)_{t\in\mathbb{R}}$ restricted to $\mathbb{T}\setminus D$ is equal to $(\psi_t)_{t\in\mathbb{R}}$,
- the flow $(\varphi_t)_{t\in\mathbb{R}}$ restricted to $\widetilde{D}$ is equal to $(\widetilde{\psi}_t)_{t\in\mathbb{R}}$.

M splits into two $(\varphi_t)_{t\in\mathbb{R}}$–invariant sets B and $A=M\setminus B$ such that B consists of periodic orbits and A is an ergodic component of positive measure. Moreover, the flow $(\varphi_t)_{t\in\mathbb{R}}$ on A can be represented as the special flow built over the rotation $Tx=x+\alpha$ and under a function $f:\mathbb{T}\to\mathbb{R}$ which is of class C^∞ on $\mathbb{T}\setminus\{0\}$. Of course, $f(x)$ is the first return time to $\mathbb{S}$ of the point $x\in\mathbb{S}\cong\mathbb{T}$. We will prove that $f:(0,1)\to\mathbb{R}$ can be extended to a C^∞ function on $[0,1]$, i.e. D^nf possesses limits at 0 and 1 for any $n\ge 0$. Moreover, we will show that $\lim_{x\to 0^+} f(x) > \lim_{x\to 1^-} f(x)$. To prove it we will need an auxiliary simple lemma.

LEMMA 3.1. *Let $U\subset\mathbb{C}$ be an open disk with center at 0 and $h:U\to\mathbb{C}$ be an analytic function such that $h(z)\neq 0$ for $z\in U$. Let us consider the differential equation*

$$\frac{dz}{dt}=\frac{i}{zh(z)}$$

on $U\setminus\{0\}$. Then there exists an open disk $\widetilde{U}\subset U$ containing 0 and a biholomorphic map $\xi:\widetilde{U}\to\xi(\widetilde{U})$ such that $\xi(0)=0$ and

$$\frac{d\omega}{dt}=1/\omega$$

on $\xi(\widetilde{U})\setminus\{0\}$, where $\omega=\sqrt{2}\xi(z)$.

Proof. Let $H:U\to\mathbb{C}$ be an analytic function such that $H'(z)=-izh(z)$ and $H(0)=0$. Since $H'(0)=0$ and $H''(0)=-ih(0)\neq 0$, there exists an open disk $\widetilde{U}\subset U$ containing 0 and a biholomorphic map $\xi:\widetilde{U}\to\xi(\widetilde{U})$ such that $H(z)=(\xi(z))^2$ for $z\in\tilde{U}$. Put $\omega=\sqrt{2}\xi(z)$, $z\in\tilde{U}$. Then $\omega^2/2=(\xi(z))^2=H(z)$, and consequently

$$\frac{d\omega}{dt}\omega = H'(z)\frac{dz}{dt}=\frac{iH'(z)}{zh(z)}=1.$$

□

Of course, the equation (2) can be written as $\frac{dz}{dt} = i(z-1)/z$. By Lemma 1 (with $h(z) = \frac{1}{z-1}$), there exist an open disk $0 \in V \subset \mathbb{C}$ and a biholomorphic map $F : V \to F(V)$ such that the flow $(F^{-1} \circ \varphi_t \circ F)_{t\in\mathbb{R}}$ on V is determined by the equation $\frac{d\omega}{dt} = 1/\omega$, i.e. by

$$\begin{cases} \frac{dx}{dt} &= \frac{x}{x^2+y^2} \\ \frac{dy}{dt} &= \frac{-y}{x^2+y^2}. \end{cases} \tag{3}$$

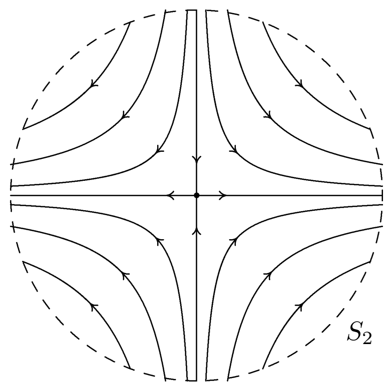

Figure 3

The trajectories of this flow are presented on Fig. 3. Denote by $S_2 = \{re^{it} : t \in [0, 2\pi]\}$ a circle which is contained in V. Let $\tau : S_2 \to \mathbb{R}$ be the function of first return time (counted forward or backward and staying inside S_2) to S_2. It is easy to check that

$$\tau(re^{it}) = -r^2 \cos(2t)$$

which is of class C^∞ (indeed, $\frac{d}{dt}(\omega^2) = 2$ and the first return time satisfies $|\omega^2(t)|^2 = r^4$). Let $\widetilde{S_2} := F(S_2)$ and $\widetilde{\tau} : \widetilde{S_2} \to \mathbb{R}$ be the function of the first return time (counted forward or backward inside $F(V)$) to $\widetilde{S_2}$. Then $\widetilde{\tau}$ is also of class C^∞. Consequently, $f : (0,1) \to \mathbb{R}$ can be extended to a C^∞ function on $[0,1]$ and

$$\lim_{x\to 0^+} f(x) - \lim_{x\to 1^-} f(x) \geq \tau_0,$$

where τ_0 is the time of the first positive return of the point 0 to itself via the separatrice which starts and stops at 0. In this way we constructed a C^∞–flow with one singular point on the torus which

- preserves a C^∞–measure,
- possesses two invariant subsets A and B: A is an ergodic component of positive measure and B consists of periodic orbits,
- the flow on A is measure–theoretically isomorphic to a special flow T^f, where T is the rotation by α and $f : \mathbb{T} \to \mathbb{R}$ is C^∞ function on $\mathbb{T} \setminus \{0\}$ and $\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$.

By cutting out the disk with the center at $(1,0)$ and of radius $1/2$ (it intersects A but does not contain the point $(0,0)$) from the flow $(\varphi_t)_{t\in\mathbb{R}}$ and gluing a Möbius strip endowed with the flow considered by Blokhin in [**2**, §3] we can obtain a C^∞ flow $(u_t)_{t\in\mathbb{R}}$ with one singularity on a non-orientable surface of Euler characteristic -1 such that (u_t) is isomorphic to the action of (φ_t) on the component A.

4. *Joinings*

Assume that $\mathcal{S}=(S_t)_{t\in\mathbb{R}}$ is a flow on a standard probability space $(X,\mathcal{B},\mu)$. By that we mean always a so called *measurable flow*, i.e. we require in particular that the map $\mathbb{R}\ni t\to\langle f\circ S_t,g\rangle\in\mathbb{C}$ is continuous for each $f,g\in L^2(X,\mathcal{B},\mu)$. Assume moreover that $\mathcal{S}$ is ergodic and let $\mathcal{T}=(T_t)_{t\in\mathbb{R}}$ be another ergodic flow defined on $(Y,\mathcal{C},\nu)$. By a *joining* between $\mathcal{S}$ and $\mathcal{T}$ we mean any probability $(S_t\times T_t)_{t\in\mathbb{R}}$–invariant measure on $(X\times Y,\mathcal{B}\otimes\mathcal{C})$ whose projections on X and Y are equal to μ and ν respectively. The set of joinings between $\mathcal{S}$ and $\mathcal{T}$ is denoted by $J(\mathcal{S},\mathcal{T})$. The subset of ergodic joinings is denoted by $J^e(\mathcal{S},\mathcal{T})$. Ergodic joinings are exactly extremal points in the simplex $J(\mathcal{S},\mathcal{T})$. Let $\{A_n:n\in\mathbb{N}\}$ and $\{B_n:n\in\mathbb{N}\}$ be two countable families in $\mathcal{B}$ and $\mathcal{C}$ respectively which are dense in $\mathcal{B}$ and $\mathcal{C}$ for the (pseudo–)metrics $d_\mu(A,B)=\mu(A\triangle B)$ and $d_\nu(A,B)=\nu(A\triangle B)$ respectively. Let us consider the metric d on $J(\mathcal{S},\mathcal{T})$ defined by

$$d(\rho,\rho')=\sum_{m,n\in\mathbb{N}}\frac{1}{2^{m+n}}|\rho(A_n\times B_m)-\rho'(A_n\times B_m)|.$$

Endowed with corresponding to d topology, to which we will refer as the weak topology, the set $J(\mathcal{S},\mathcal{T})$ is compact.

Suppose that $\mathcal{A}\subset\mathcal{B}$ is a *factor* of $\mathcal{S}$, i.e. $\mathcal{A}$ is an $\mathcal{S}$–invariant sub–σ–algebra. Denote by $\mu\otimes_{\mathcal{A}}\mu\in J(\mathcal{S},\mathcal{S})$ the *relatively independent joining* of the measure μ over the factor $\mathcal{A}$, i.e. $\mu\otimes_{\mathcal{A}}\mu\in J(\mathcal{S},\mathcal{S})$ is defined by

$$(\mu\otimes_{\mathcal{A}}\mu)(D)=\int_{X/\mathcal{A}}(\mu_{\overline{x}}\otimes\mu_{\overline{x}})(D)\,d\overline{\mu}(\overline{x})$$

for $D\in\mathcal{B}\otimes\mathcal{C}$, where $\{\mu_{\overline{x}}:\overline{x}\in X/\mathcal{A}\}$ is the disintegration of the measure μ over the factor $\mathcal{A}$ and $\overline{\mu}$ is the image of μ by the factor map $X\to X/\mathcal{A}$.

For every $t\in\mathbb{R}$ by $\mu_{S_t}\in J(\mathcal{S},\mathcal{S})$ we will denote the graph joining determined by $\mu_{S_t}(A\times B)=\mu(A\cap S_{-t}B)$ for $A,B\in\mathcal{B}$. Then μ_{S_t} is concentrated on the graph of S_t and $\mu_{S_t}\in J^e(\mathcal{S},\mathcal{S})$.

Let $(t_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that $t_n\to+\infty$. We say that a flow $\mathcal{S}$ on $(X,\mathcal{B},\mu)$ is *rigid* along (t_n) if

$$\mu(A\cap S_{-t_n}A)\to\mu(A) \tag{4}$$

for every $A\in\mathcal{B}$, or, equivalently, $\mu_{S_{t_n}}\to\mu_{Id}$ weakly in $J(\mathcal{S},\mathcal{S})$. In particular, a factor $\mathcal{A}\subset\mathcal{B}$ of $\mathcal{S}$ is rigid along (t_n) if the convergence (4) holds for every $A\in\mathcal{A}$. It is well known that a flow is mildly mixing iff it has no non–trivial rigid factor (see [**7**, **22**]).

Definition. A flow $\mathcal{S}$ on $(X, \mathcal{B}, \mu)$ is called *partially rigid* along (t_n) if there exists $0 < u \leq 1$ such that

$$\liminf_{n\to\infty} \mu(A \cap S_{-t_n}A) \geq u\mu(A) \quad \text{for every} \quad A \in \mathcal{B},$$

or, equivalently, every weak limit point ρ of the sequence $(\mu_{S_{t_n}})_{n\in\mathbb{N}}$ in $J(\mathcal{S}, \mathcal{S})$ satisfies $\rho(\Delta) \geq u$, where $\Delta = \{(x, x) \in X \times X : x \in X\}$.

The proof of the following proposition is the same as in the case of measure–preserving transformations and can be found in [**17**].

PROPOSITION 4.1. *Let $\mathcal{S}$ be an ergodic flow on $(X, \mathcal{B}, \mu)$. Suppose that $\mathcal{A} \subset \mathcal{B}$ is a non–trivial rigid factor of $\mathcal{S}$. Then there exist a factor $\mathcal{A}' \supset \mathcal{A}$ of $\mathcal{S}$ and a rigidity sequence (t_n) for $\mathcal{A}'$ such that $\mu_{S_{t_n}} \to \mu \otimes_{\mathcal{A}'} \mu$ weakly in $J(\mathcal{S}, \mathcal{S})$.*

Recall that in general the notions of (absence of) partial rigidity and mild mixing are not related. For example, the Chacon transformation is partially rigid (see e.g. [**19**]) and mildly mixing. On the other hand the Cartesian product of a mixing transformation and a rigid transformation is not mildly mixing and has no partial rigidity. Under some additional strong assumption we have however the following.

LEMMA 4.1. *Let $\mathcal{S}$ be an ergodic flow on $(X, \mathcal{B}, \mu)$ which is a finite extension of each of its non–trivial factors. Then if the flow $\mathcal{S}$ is not partially rigid then it is mildly mixing.*

Proof. Suppose, contrary to our claim, that there exists a non–trivial factor $\mathcal{A}$ of $\mathcal{S}$ which is rigid. By Proposition 4.1 there exist a factor $\mathcal{A}' \supset \mathcal{A}$ and a rigidity sequence (t_n) for $\mathcal{A}'$ such that

$$\mu_{S_{t_n}} \to \mu \otimes_{\mathcal{A}'} \mu \text{ weakly in } J(\mathcal{S}, \mathcal{S}). \tag{5}$$

Since $\mathcal{S}$ is ergodic and it is a finite extension of $\mathcal{S}|_{\mathcal{A}'}$, there exists a natural number k such that every fiber measure $\mu_{\overline{x}}$, $\overline{x} \in X/\mathcal{A}'$ is atomic with k atoms each of measure $1/k$, where $\{\mu_{\overline{x}} : \overline{x} \in X/\mathcal{A}'\}$ is the disintegration of the measure μ over the factor $\mathcal{A}'$. Then

$$(\mu \otimes_{\mathcal{A}'} \mu)(\Delta) = \int_{X/\mathcal{A}} (\mu_{\overline{x}} \otimes \mu_{\overline{x}})(\Delta)\, d\overline{\mu}(\overline{x}) = \int_{X/\mathcal{A}} \frac{1}{k}\, d\overline{\mu}(\overline{x}) = \frac{1}{k},$$

which, in view of (5), gives the partial rigidity of $\mathcal{S}$ and we obtain a contradiction. □

5. *Ratner property*

In this section we introduce and study a condition which emulates the Ratner condition from [**20**].

Definition. (cf. [**20, 23**]) Let (X, d) be a σ–compact metric space, $\mathcal{B}$ be the σ–algebra of Borel subsets of X, μ a Borel probability measure on (X, d) and let $(S_t)_{t\in\mathbb{R}}$ be a flow on the space $(X, \mathcal{B}, \mu)$. Let $P \subset \mathbb{R} \setminus \{0\}$ be a finite subset and

$t_0 \in \mathbb{R}\setminus\{0\}$. The flow $(S_t)_{t\in\mathbb{R}}$ is said to have *the property* $\mathrm{R}(t_0, P)$ if for every $\varepsilon > 0$ and $N \in \mathbb{N}$ there exist $\kappa = \kappa(\varepsilon) > 0$, $\delta = \delta(\varepsilon, N) > 0$ and a subset $Z = Z(\varepsilon, N) \in \mathcal{B}$ with $\mu(Z) > 1 - \varepsilon$ such that if $x, x' \in Z$, x' is not in the orbit x and $d(x, x') < \delta$, then there are $M = M(x, x')$, $L = L(x, x') \geq N$ such that $L/M \geq \kappa$ and there exists $p = p(x, x') \in P$ such that

$$\frac{\#\{n \in \mathbb{Z} \cap [M, M + L] : d(S_{nt_0}(x), S_{nt_0+p}(x')) < \varepsilon\}}{L} > 1 - \varepsilon.$$

Moreover, we say that $(S_t)_{t\in\mathbb{R}}$ has *the property* $\mathrm{R}(P)$ if the set of all $s \in \mathbb{R}$ such that the flow $(S_t)_{t\in\mathbb{R}}$ has the $\mathrm{R}(s, P)$–property is uncountable.

Remark 1. In the original definition of M. Ratner, for $t_0 \neq 0$, $P = \{-t_0, t_0\}$. In our situation a priori there is no relation between t_0 and P. Analysis similar to that in the proof of Theorem 2 in [**20**] shows that $\mathrm{R}(t_0, P)$ and $\mathrm{R}(P)$ properties are invariant under measure–theoretic isomorphism.

We now prove an extension of Theorem 3 in [**20**] that brings important information about ergodic joinings with flows satisfying the $\mathrm{R}(P)$–property.

THEOREM 5.1. *Let (X, d) be a σ–compact metric space, $\mathcal{B}$ be the σ–algebra of Borel subsets of X and μ a probability measure on $(X, \mathcal{B})$. Let $P \subset \mathbb{R}\setminus\{0\}$ be a nonempty finite set. Assume that $(S_t)_{t\in\mathbb{R}}$ is an ergodic flow on $(X, \mathcal{B}, \mu)$ such that every automorphism $S_p : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ for $p \in P$ is ergodic. Suppose that $(S_t)_{t\in\mathbb{R}}$ satisfies the $\mathrm{R}(P)$–property. Let $(T_t)_{t\in\mathbb{R}}$ be an ergodic flow on $(Y, \mathcal{C}, \nu)$ and let ρ be an ergodic joining of $(S_t)_{t\in\mathbb{R}}$ and $(T_t)_{t\in\mathbb{R}}$. Then either $\rho = \mu \otimes \nu$, or ρ is a finite extension of ν.*

Remark 2. Let $\mathcal{S}$ be an ergodic flow on $(X, \mathcal{B}, \mu)$. Assume that for each ergodic flow $\mathcal{T}$ acting on $(Y, \mathcal{C}, \nu)$ an arbitrary ergodic joining ρ of $\mathcal{S}$ with $\mathcal{T}$ is either the product measure or ρ is a finite extension of μ. Then $\mathcal{S}$ is a finite extension of each of its non–trivial factors. Indeed, suppose that $\mathcal{A} \subset \mathcal{B}$ is a non–trivial factor. Let us consider the factor action $\mathcal{S}|_{\mathcal{A}}$ on $(X/\mathcal{A}, \mathcal{A}, \overline{\mu})$ and the natural joining $\mu_{\mathcal{A}} \in J(\mathcal{S}, \mathcal{S}|_{\mathcal{A}})$ determined by $\mu_{\mathcal{A}}(B \times A) = \mu(B \cap A)$ for all $B \in \mathcal{B}$ and $A \in \mathcal{A}$. Clearly, the action $\mathcal{S} \times (\mathcal{S}|_{\mathcal{A}})$ on $(X \times (X/\mathcal{A}), \mathcal{B} \otimes \mathcal{A}, \mu_{\mathcal{A}})$ is isomorphic (via the projection on $(X, \mathcal{B}, \mu)$) to the action of $\mathcal{S}$. Since the measure $\mu_{\mathcal{A}}$ is not the product measure, by assumptions, the action $\mathcal{S} \times (\mathcal{S}|_{\mathcal{A}})$ on $(X \times (X/\mathcal{A}), \mathcal{B} \otimes \mathcal{A}, \mu_{\mathcal{A}})$ is a finite extension of $\mathcal{S}|_{\mathcal{A}}$.

To prove Theorem 5.1 we will need two ingredients. The proof of the following lemma is contained in the proof of Theorem 3 in [**20**].

LEMMA 5.1. *Let $(S_t)_{t\in\mathbb{R}}$ and $(T_t)_{t\in\mathbb{R}}$ be ergodic flows acting on $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$ respectively and let $\rho \in J^e(\mathcal{S}, \mathcal{T})$. Suppose that there exists $U \in \mathcal{B} \otimes \mathcal{C}$ with $\rho(U) > 0$ and $\delta > 0$ such that if $(x, y) \in U$, $(x', y) \in U$ then either x and x' are in the same orbit or $d(x, x') \geq \delta$. Then ρ is a finite extension of ν.*

The following simple fact will be used in the proof of Theorem 5.1 and in the remainder of the paper.

Remark 3. Notice that if

$$\left|\frac{1}{M}\sum_{n=0}^{M-1}\chi_A(T^nx)-\mu(A)\right|<\varepsilon \text{ and } \left|\frac{1}{M+L+1}\sum_{n=0}^{M+L}\chi_A(T^nx)-\mu(A)\right|<\varepsilon$$

then

$$\left|\frac{1}{L}\sum_{n=M}^{M+L}\chi_A(T^nx)-\mu(A)\right|<2\varepsilon\left(1+\frac{M}{L}\right).$$

The proof of Theorem 5.1, presented below, is much the same as the proof of Theorem 10 in [**23**].

Proof of Theorem 5.1. Suppose that $\rho\in J^e(\mathcal{S},\mathcal{T})$ and $\rho\neq\mu\otimes\nu$. Since the flow $(S_t\times T_t)_{t\in\mathbb{R}}$ is ergodic on $(X\times Y,\rho)$, we can find $t_0\neq 0$ such that the automorphism $S_{t_0}\times T_{t_0}:(X\times Y,\rho)\to(X\times Y,\rho)$ is ergodic and the flow $(S_t)_{t\in\mathbb{R}}$ has the $\mathrm{R}(t_0,P)$–property. To simplify notation we assume that $t_0=1$.

Since the ergodicity of S_p implies disjointness of S_p from the identity, for every $p\in P$ there exist closed subsets $A_p\subset X$, $B_p\subset Y$ such that

$$\rho(S_{-p}A_p\times B_p)\neq\rho(A_p\times B_p).$$

Let

$$0<\varepsilon:=\min\{|\rho(S_{-p}A_p\times B_p)-\rho(A_p\times B_p)|:p\in P\}. \tag{6}$$

Next choose $0<\varepsilon_1<\varepsilon/8$ such that $\mu(A_p^{\varepsilon_1}\setminus A_p)<\varepsilon/2$ for $p\in P$, where $A^{\varepsilon_1}=\{z\in X:d(z,A)<\varepsilon_1\}$. We have

$$\begin{aligned}|\rho(A_p\times B_p)-\rho(A_p^{\varepsilon_1}\times B_p)| = {}& \rho(A_p^{\varepsilon_1}\times B_p\setminus A_p\times B_p)\\ \leq {}& \rho((A_p^{\varepsilon_1}\setminus A_p)\times Y)=\mu(A_p^{\varepsilon_1}\setminus A_p)<\varepsilon/2\end{aligned} \tag{7}$$

and similarly

$$|\rho(S_{-p}A_p\times B_p)-\rho(S_{-p}(A_p^{\varepsilon_1})\times B_p)|<\varepsilon/2$$

for any $p\in P$.

Let $\kappa:=\kappa(\varepsilon_1)(>0)$. By the ergodic theorem together with Remark 3, there exist a measurable set $U\subset X\times Y$ with $\rho(U)>3/4$ and $N\in\mathbb{N}$ such that if $(x,y)\in U$, $p\in P$, $m\geq N$ and $l/m\geq\kappa$ then

$$\left|\frac{1}{l}\sum_{k=m}^{m+l}\chi_{A_p^{\varepsilon_1}\times B_p}(S_kx,T_ky)-\rho(A_p^{\varepsilon_1}\times B_p)\right|<\frac{\varepsilon}{8}, \tag{8}$$

$$\left|\frac{1}{l}\sum_{k=m}^{m+l}\chi_{S_{-p}A_p\times B_p}(S_kx,T_ky)-\rho(S_{-p}A_p\times B_p)\right|<\frac{\varepsilon}{8} \tag{9}$$

and similar inequalities hold for $A_p\times B_p$ and $S_{-p}(A_p^{\varepsilon_1})\times B_p$.

Next, by the property $\mathrm{R}(1,P)$, we obtain relevant $\delta=\delta(\varepsilon_1,N)>0$ and $Z=Z(\varepsilon_1,N)\in\mathcal{B}$, $\mu(Z)>1-\varepsilon_1$.

Now assume that $(x,y)\in U$, $(x',y)\in U$, $x,x'\in Z$ and x' is not in the orbit of x. We claim that $d(x,x')\geq\delta$. Suppose that, on the contrary, $d(x,x')<\delta$. Then, by the property $\mathrm{R}(1,P)$, there exist $M = M(x,x')$, $L = L(x,x') \geq N$ with $L/M \geq \kappa$ and $p = p(x,x') \in P$ such that $(\#K_p)/L > 1-\varepsilon_1$, where

$$K_p = \{n \in \mathbb{Z} \cap [M, M+L] : d(S_n(x), S_{n+p}(x')) < \varepsilon_1\}.$$

If $k \in K_p$ and $S_{k+p}x' \in A_p$, then $S_k x \in A_p^{\varepsilon_1}$. Hence

$$\begin{aligned} &\frac{1}{L}\sum_{k=M}^{M+L} \chi_{S_{-p}A_p \times B_p}(S_k x', T_k y) \\ &\leq \frac{\#(\mathbb{Z} \cap [M, M+L] \setminus K_p)}{L} + \frac{1}{L}\sum_{k \in K_p} \chi_{A_p \times B_p}(S_{k+p}x', T_k y) \\ &\leq \varepsilon/8 + \frac{1}{L}\sum_{k=M}^{M+L} \chi_{A_p^{\varepsilon_1} \times B_p}(S_k x, T_k y). \end{aligned} \tag{10}$$

Now from (9), (10), (8) and (7) it follows that

$$\begin{aligned} \rho(S_{-p}A_p \times B_p) &\leq \frac{1}{L}\sum_{k=M}^{M+L} \chi_{S_{-p}A_p \times B_p}(S_k x', T_k y) + \varepsilon/8 \\ &\leq \varepsilon/4 + \frac{1}{L}\sum_{k=M}^{M+L} \chi_{A_p^{\varepsilon_1} \times B_p}(S_k x, T_k y) \\ &< \varepsilon/2 + \rho(A_p^{\varepsilon_1} \times B_p) \leq \varepsilon + \rho(A_p \times B_p). \end{aligned}$$

Applying similar arguments we get

$$\rho(A_p \times B_p) < \varepsilon + \rho(S_{-p}A_p \times B_p).$$

Consequently,

$$|\rho(A_p \times B_p) - \rho(S_{-p}A_p \times B_p)| < \varepsilon,$$

contrary to (6).

In summary, we found a measurable set $U_1 = U \cap (Z(\varepsilon_1, N) \times Y)$ and $\delta(\varepsilon_1, N) > 0$ such that $\rho(U_1) > 3/4 - \varepsilon_1 > 1/2$ and if $(x,y) \in U_1$, $(x',y) \in U_1$ then either x and x' are in the same orbit or $d(x,x') \geq \delta(\varepsilon_1, N)$. Now an application of Lemma 5.1 completes the proof. □

We end up this section with a general lemma that gives a criterion that allows one to prove the $\mathrm{R}(P)$–property for special flows built over irrational rotations on the circle and under bounded and bounded away from zero measurable functions.

While dealing with special flows over irrational rotations on $\mathbb{T}^f$ we will always consider the induced metric from the metric defined on $\mathbb{T} \times \mathbb{R}$ by $d((x,s),(y,t)) = \|x-y\| + |s-t|$.

LEMMA 5.2. *Let $P \subset \mathbb{R} \setminus \{0\}$ be a nonempty finite subset. Let $T : \mathbb{T} \to \mathbb{T}$ be an ergodic rotation and let $f : \mathbb{T} \to \mathbb{R}$ be a bounded positive measurable function which is bounded away from zero. Assume that for every $\varepsilon > 0$ and $N \in \mathbb{N}$ there exist $\kappa = \kappa(\varepsilon) > 0$ and $0 < \delta = \delta(\varepsilon, N) < \varepsilon$ such that if $x, y \in \mathbb{T}$, $0 < \|x-y\| < \delta$, then* *there are natural numbers* $M = M(x, y) \geq N$, $L = L(x, y) \geq N$ *such that* $L/M \geq \kappa$ *and there exists* $p = p(x, y) \in P$ *such that*

$$\frac{1}{L+1}\#\left\{M \leq n \leq M + L : |f^{(n)}(x) - f^{(n)}(y) + p| < \varepsilon\right\} > 1 - \varepsilon.$$

Suppose that $\gamma \in \mathbb{R}$ *is a positive number such that the instance automorphism* $T^f_\gamma : \mathbb{T}^f \to \mathbb{T}^f$ *is ergodic. Then the special flow* T^f *has the* $\mathrm{R}(\gamma, P)$*–property.*

Proof. Let c, C be positive numbers such that $0 < c \leq f(x) \leq C$ for every $x \in \mathbb{T}$. Let μ stand for Lebesgue measure on $\mathbb{T}$.

Fix $0 < \varepsilon$ and $N \in \mathbb{N}$. Put

$$\varepsilon_1 = \min\left(\frac{c\varepsilon}{8(\gamma + C)}, \frac{\varepsilon}{16}\right).$$

Take $\kappa' = \kappa(\varepsilon_1)$ and let $\kappa := \frac{c}{C}\kappa'$. Let us consider the set

$$X(\varepsilon) := \left\{(x, s) \in \mathbb{T}^f : \frac{\varepsilon}{8} < s < f(x) - \frac{\varepsilon}{8}\right\}.$$

Since $\mu^f(X(\varepsilon)^c) = \varepsilon/4$ and T^f_γ is ergodic, there exists $N(\varepsilon) \in \mathbb{N}$ such that $\mu^f(Z(\varepsilon)^c) < \varepsilon$, where $Z(\varepsilon)$ is the set of all $(x, s) \in \mathbb{T}^f$ such that

$$\left|\frac{1}{n}\#\{0 \leq k < n : T^f_{k\gamma}(x, s) \notin X(\varepsilon)\} - \frac{\varepsilon}{4}\right| < \frac{\kappa}{1+\kappa}\frac{\varepsilon}{8} \tag{11}$$

for each $n \geq N(\varepsilon)$. Take $\delta = \delta(\varepsilon_1, 2\gamma\max(N(\varepsilon), N)/c) < \varepsilon_1$. Let us consider a pair of points $(x, s), (y, s') \in Z(\varepsilon)$ such that $0 < d((x, s), (y, s')) < \delta$ and $x \neq y$. By assumption, there are natural numbers $M' = M(x, y), L' = L(x, y) \geq 2\gamma\max(N(\varepsilon), N)/c$ such that $L'/M' \geq \kappa'$ and there exists $p = p(x, y) \in P$ such that

$$\frac{\#A'}{L'+1} > 1 - \varepsilon_1,$$

where $A' = \left\{M' \leq n \leq M' + L' : |f^{(n)}(x) - f^{(n)}(y) + p| < \varepsilon_1\right\}$. Then

$$\frac{\#A''}{L'} > 1 - 4\varepsilon_1, \quad \text{where} \quad A'' = \{M' \leq n < M' + L' : n \in A', n + 1 \in A'\}. \tag{12}$$

Put

$$M := \frac{f^{(M')}(x) - s}{\gamma} \quad \text{and} \quad L := \frac{f^{(L')}(T^{M'}x)}{\gamma}.$$

Then

$$\frac{L}{M} = \frac{f^{(L')}(T^{M'}x)}{f^{(M')}(x) - s} \geq \frac{c}{C}\frac{L'}{M'} \geq \kappa.$$

But $s \leq f(x)$ and $M', L' \geq 2\gamma\max(N(\varepsilon), N)/c$, so

$$M = \frac{f^{(M')}(x) - s}{\gamma} \geq \frac{f^{(M'-1)}(Tx)}{\gamma} \geq \frac{c(M'-1)}{\gamma} \geq \frac{cM'}{2\gamma} \geq \max(N(\varepsilon), N)$$

and

$$L = \frac{f^{(L')}(T^{M'}x)}{L'}\frac{L'}{\gamma} \geq c\frac{L'}{\gamma} > N. \tag{13}$$

Now $M \geq N(\varepsilon)$, $L/M \geq \kappa$, $(x,s) \in Z(\varepsilon)$ (that is (x,s) satisfies (11)) so, by Remark 3, we have

$$\frac{1}{L}\#\{M \leq k < M+L : T^f_{k\gamma}(x,s) \notin X(\varepsilon)\} < \frac{\varepsilon}{2}. \tag{14}$$

Suppose that $M \leq k < M+L$. Then $k\gamma + s \in [f^{(M')}(x), f^{(M'+L')}(x))$ and there exists a unique $M' \leq m_k < M'+L'$ such that $k\gamma + s \in [f^{(m_k)}(x), f^{(m_k+1)}(x))$. Suppose that

$$k \in B := \{M \leq j < M+L : T^f_{j\gamma}(x,s) \in X(\varepsilon) \text{ and } m_j \in A''\}.$$

Then

$$f^{(m_k)}(x) + \varepsilon/8 < s + k\gamma < f^{(m_k+1)}(x) - \varepsilon/8.$$

Since $m_k \in A''$ and $|s'-s| < \delta < \varepsilon_1$, we have

$$\begin{aligned} s' + k\gamma + p &= (s+k\gamma) + (s'-s) + p < f^{(m_k+1)}(x) + p - \varepsilon/8 + \delta \\ &= f^{(m_k+1)}(y) + (f^{(m_k+1)}(x) - f^{(m_k+1)}(y) + p) - \varepsilon/8 + \varepsilon_1 \\ &< f^{(m_k+1)}(y) - \varepsilon/8 + 2\varepsilon_1 \leq f^{(m_k+1)}(y) \end{aligned}$$

and

$$\begin{aligned} s' + k\gamma + p &= (s+k\gamma) + (s'-s) + p > f^{(m_k)}(x) + p + \varepsilon/8 - \delta \\ &= f^{(m_k)}(y) + (f^{(m_k)}(x) - f^{(m_k)}(y) + p) + \varepsilon/8 - \varepsilon_1 \\ &> f^{(m_k)}(y) + \varepsilon/8 - 2\varepsilon_1 \geq f^{(m_k)}(y). \end{aligned}$$

Thus

$$T^f_{k\gamma}(x,s) = (T^{m_k}x, s + k\gamma - f^{(m_k)}(x))$$

and

$$T^f_{k\gamma+p}(y,s') = (T^{m_k}y, s' + k\gamma + p - f^{(m_k)}(y)).$$

Hence

$$\begin{aligned} &d(T^f_{k\gamma}(x,s), T^f_{k\gamma+p}(y,s')) \\ &\quad = \|y-x\| + |(s'-s) + (f^{(m_k)}(x) - f^{(m_k)}(y) + p)| < \delta + \varepsilon_1 < 2\varepsilon_1 < \varepsilon. \end{aligned}$$

It follows that

$$B \subset \{k \in \mathbb{Z} \cap [M, M+L) : d(T^f_{k\gamma}(x,s), T^f_{k\gamma+p}(y,s')) < \varepsilon\}. \tag{15}$$

If $k \in (\mathbb{Z} \cap [M, M+L]) \setminus B$ then either $T^f_{k\gamma}(x,s) \notin X(\varepsilon)$ or $m_k \notin A''$. Since for every $m \in \mathbb{N}$ the set $\{k \in \mathbb{N} : m_k = m\}$ has at most $C/\gamma + 1$ elements, we have

$$L - \#B \leq \#\left\{M \leq k < M+L : T^f_{k\gamma}(x,s) \notin X(\varepsilon)\right\} + \left(\frac{C}{\gamma} + 1\right)(L' - \#A'').$$

Hence by (14), (12) and (13) we obtain

$$L - \#B \leq \frac{\varepsilon}{2}L + \left(\frac{C}{\gamma} + 1\right)4\varepsilon_1 L' \leq \left(\frac{\varepsilon}{2} + 4\frac{C+\gamma}{c}\varepsilon_1\right)L \leq \left(\frac{\varepsilon}{2} + \frac{\varepsilon}{2}\right)L \leq \varepsilon L.$$

Consequently $(\#B)/L > 1 - \varepsilon$, and (15) completes the proof. □

6. *Ratner property for the von Neumann class of functions*
We call a function $f:\mathbb{T}\to\mathbb{R}$ *piecewise absolutely continuous* if there exist $\beta_1,\ldots,\beta_k\in\mathbb{T}$ such that $f|_{(\beta_j,\beta_{j+1})}$ is an absolutely continuous function for $j=1,\ldots,k$ ($\beta_{k+1}=\beta_1$). Let $d_j:=f_-(\beta_j)-f_+(\beta_j)$, where $f_\pm(\beta)=\lim_{y\to\beta^\pm}f(y)$. Then the number

$$S(f):=\sum_{j=1}^{k}d_j=\int_{\mathbb{T}}f'(x)dx \tag{16}$$

is the *sum of jumps* of f.

Let $T:\mathbb{T}\to\mathbb{T}$ be the rotation by an irrational number α which has bounded partial quotients. We will prove that if f is a positive piecewise absolutely continuous function with a non–zero sum of jumps, then the special flow T^f satisfies the R(t_0,P)–property for every $t_0\neq 0$, where $P\subset\mathbb{R}\setminus\{0\}$ is a non–empty finite set.

Lemma 6.1. *Let $T:\mathbb{T}\to\mathbb{T}$ be the rotation by an irrational number α which has bounded partial quotients and let $f:\mathbb{T}\to\mathbb{R}$ be an absolutely continuous function. Then*

$$\sup_{0\le n\le q_{s+1}}\sup_{\|y-x\|<1/q_s}|f^{(n)}(y)-f^{(n)}(x)|\to 0$$

as $s\to\infty$.

Proof. We first prove that if T is an irrational rotation by α then

$$\sup_{0\le n\le q_s}\sup_{\|y-x\|<1/q_s}|f^{(n)}(y)-f^{(n)}(x)|\to 0\quad\text{as}\quad s\to\infty \tag{17}$$

for every absolutely continuous $f:\mathbb{T}\to\mathbb{R}$. We recall that (17) was already proved to hold in **[3]** (see Lemma 2 Ch.16, §3) for C^1–functions.

Let $f:\mathbb{T}\to\mathbb{R}$ be an absolutely continuous function. Then for every $\varepsilon>0$ there exists a C^1–function $f_\varepsilon:\mathbb{T}\to\mathbb{R}$ such that

$$\sup_{x\in\mathbb{T}}|f(x)-f_\varepsilon(x)|+\operatorname{Var}(f-f_\varepsilon)<\varepsilon/2.$$

Suppose that $0\le n\le q_s$ and $0<y-x<1/q_s$. Let us consider the family of intervals $\mathcal{I}=\{[x,y],[Tx,Ty],\ldots,[T^{n-1}x,T^{n-1}y]\}$. For every $0\le i\neq j<n$ we have

$$\|T^ix-T^jx\|\ge\|q_{s-1}\alpha\|>\frac{1}{2q_s},$$

by (1). It follows that a point from $\mathbb{T}$ belongs to at most two intervals from the family $\mathcal{I}$. Therefore

$$\begin{aligned}&|(f^{(n)}(y)-f^{(n)}(x))-(f_\varepsilon^{(n)}(y)-f_\varepsilon^{(n)}(x))|\\ \le\;&\sum_{i=0}^{n-1}|(f-f_\varepsilon)(T^iy)-(f-f_\varepsilon)(T^ix)|\\ \le\;&\sum_{i=0}^{n-1}\operatorname{Var}_{[T^ix,T^iy]}(f-f_\varepsilon)\le 2\operatorname{Var}(f-f_\varepsilon)<\varepsilon.\end{aligned}$$

Since this inequality holds for every $\varepsilon > 0$ and the convergence in (17) holds for f_ε, by a standard argument, we obtain (17) for f.

Suppose that α has bounded partial quotients and let $C = \sup\{a_n : n \in \mathbb{N}\} + 1$. Since every $0 \leq n \leq q_{s+1}$ can be represented as $n = bq_s + d$, where $b \leq a_{s+1}$ and $d \leq q_{s-1}$, we have

$$\sup_{0 \leq n \leq q_{s+1}} \sup_{\|y-x\|<1/q_s} |f^{(n)}(y) - f^{(n)}(x)| \leq C \sup_{0 \leq n \leq q_s} \sup_{\|y-x\|<1/q_s} |f^{(n)}(y) - f^{(n)}(x)|,$$

which completes the proof. □

Let $T : \mathbb{T} \to \mathbb{T}$ be the rotation by an irrational number α which has bounded partial quotients and let $C = \sup\{a_n : n \in \mathbb{N}\} + 1$. Suppose that $f : \mathbb{T} \to \mathbb{R}$ is a positive piecewise absolutely continuous function with a non–zero sum of jumps $S = S(f)$. Put

$$D := \{n_1 d_1 + \ldots + n_k d_k : 0 \leq n_1, \ldots, n_k \leq 2C + 1\}.$$

Since D is finite, we can choose $p \in (0, |S|) \setminus (D \cup (-D))$. Then $0 \notin \operatorname{sgn}(S)p - D$.

THEOREM 6.1. *Suppose that $T : \mathbb{T} \to \mathbb{T}$ is the rotation by an irrational number α with bounded partial quotients and $f : \mathbb{T} \to \mathbb{R}$ a positive and bounded away from zero piecewise absolutely continuous function with a non–zero sum of jumps. Then the special flow T^f has the property* $\mathrm{R}(\gamma, (\operatorname{sgn}(S)p - D) \cup (-\operatorname{sgn}(S)p + D))$ *for every $\gamma > 0$.*

Proof. Without loss of generality we can assume that f is continuous from the right. A consequence of (16) is that we can represent f as the sum of two functions f_{pl} and f_{ac}, where $f_{ac} : \mathbb{T} \to \mathbb{R}$ is an absolutely continuous function with zero mean and $f_{pl} : \mathbb{T} \to \mathbb{R}$ is piecewise linear with $f'_{pl}(x) = S$ for all $x \in \mathbb{T} \setminus \{\beta_1, \ldots, \beta_k\}$. The discontinuity points and size of jumps of f and f_{pl} are the same. Explicitly,

$$f_{pl}(x) = \sum_{i=1}^{k} d_i \{x - \beta_i\} + c$$

for some $c \in \mathbb{R}$.

Let $C = \sup\{a_n : n \in \mathbb{N}\} + 1$. Fix $\varepsilon > 0$ and $N \in \mathbb{N}$. Then put

$$\kappa(\varepsilon) = \frac{1}{k(2C+1)} \cdot \min\left(\frac{\varepsilon}{2pC}, \frac{1}{C^2}\right).$$

By Lemma 6.1, there exists s_0 such that for any $s \geq s_0$ we have

$$\sup_{0 \leq n \leq q_{s+1}} \sup_{\|y-x\|<1/q_s} |f_{ac}^{(n)}(y) - f_{ac}^{(n)}(x)| < \frac{\varepsilon}{2} \tag{18}$$

and

$$\min\left(\kappa(\varepsilon), 1\right) \cdot q_{s_0} > N. \tag{19}$$

Then put

$$\delta(\varepsilon, N) = \frac{p}{|S| q_{s_0+1}}.$$

Take $x, y \in \mathbb{T}$ such that $0 < \|x-y\| < \delta(\varepsilon, N)$. Let s be a (unique) natural number such that

$$\frac{p}{|S|q_{s+1}} < \|x-y\| \leq \frac{p}{|S|q_s}. \tag{20}$$

Then $s \geq s_0$. Without loss of generality we can assume that $x < y$. We will also assume that $S > 0$, in the case $S < 0$ the proof is similar. Let us consider the sequence $\left(f_{pl}^{(n)}(y) - f_{pl}^{(n)}(x)\right)_{n\in\mathbb{N}}$. We have

$$\begin{aligned} & f_{pl}^{(n+1)}(y) - f_{pl}^{(n+1)}(x) \\ = \;& f_{pl}^{(n)}(y) - f_{pl}^{(n)}(x) + \sum_{i=1}^{k} d_i(\{y + n\alpha - \beta_i\} - \{x + n\alpha - \beta_i\}) \\ = \;& f_{pl}^{(n)}(y) - f_{pl}^{(n)}(x) + \sum_{i=1}^{k} d_i(y - x - \chi_{(x,y]}(\{\beta_i - n\alpha\})). \end{aligned}$$

It follows that for every $n \geq 0$ we have

$$f_{pl}^{(n)}(y) - f_{pl}^{(n)}(x) = nS(y-x) - \overline{d}_n, \tag{21}$$

where

$$\overline{d}_n := \overline{d}_n(x,y) = \sum_{\{1\leq i\leq k, 0\leq j<n : \{\beta_i - j\alpha\}\in(x,y]\}} d_i.$$

Take $1 \leq i \leq k$. Suppose that $\{\beta_i - k\alpha\}, \{\beta_i - l\alpha\} \in (x,y]$, where $0 \leq k, l < q_{s+1}$ and $k \neq l$. Then

$$\|\{\beta_i - k\alpha\} - \{\beta_i - l\alpha\}\| \geq \|q_s\alpha\| > \frac{1}{2q_{s+1}} \geq \frac{1}{2Cq_s}.$$

It follows that the number of discontinuities of $f_{pl}^{(q_{s+1})}$ which are of the form $\beta_i - j\alpha$ and are in the interval $(x,y]$ is less than

$$2Cq_s|y-x| + 1 \leq 2C\frac{p}{|S|} + 1 \leq 2C + 1.$$

It follows that the elements of the sequence $(\overline{d}_n)_{n=1}^{q_{s+1}}$ belong to D. In view of (21) and (20) we have

$$f_{pl}^{(q_s)}(y) - f_{pl}^{(q_s)}(x) + \overline{d}_{q_s} = q_s S(y-x) \leq p$$

and

$$f_{pl}^{(q_{s+1})}(y) - f_{pl}^{(q_{s+1})}(x) + \overline{d}_{q_{s+1}} = q_{s+1} S(y-x) > p.$$

Moreover, for any natural n we have

$$0 < f_{pl}^{(n+1)}(x) - f_{pl}^{(n+1)}(y) + \overline{d}_{n+1} - (f_{pl}^{(n)}(x) - f_{pl}^{(n)}(y) + \overline{d}_n) = S(y-x) \leq \frac{p}{q_s}.$$

Hence, there exists an integer interval $I \subset [q_s, q_{s+1}]$ such that

$$|f_{pl}^{(n)}(x) - f_{pl}^{(n)}(y) + \overline{d}_n - p| < \frac{\varepsilon}{2} \text{ for } n \in I$$

and

$$|I| \geq \min\left(\frac{\varepsilon}{2p}\, q_s, q_{s+1} - q_s\right) \geq \min\left(\frac{\varepsilon}{2pC}, \frac{1}{C^2}\right) \cdot q_{s+1}.$$

Since $s \geq s_0$, by (18) we have

$$|f^{(n)}(x) - f^{(n)}(y) + \overline{d}_n - p| < \varepsilon \text{ for } n \in I.$$

Note that if $f^{(n)}$ and $f^{(n+1)}$ have the same points of discontinuity in the interval $(x, y]$ then $\overline{d}_n = \overline{d}_{n+1}$ and since $f^{(q_{s+1})}$ has at most $k(2C+1)$ discontinuities in $(x, y]$, we can split I into at most $k(2C+1)$ integer intervals on which the sequence $(\overline{d}_n)_{n\in I}$ is constant. Thus we can choose $d \in D$ and an integer subinterval $J \subset I$ such that $\overline{d}_n = d$ for $n \in J$ and

$$|J| \geq \frac{1}{k(2C+1)} \cdot \min\left(\frac{\varepsilon}{2pC}, \frac{1}{C^2}\right) \cdot q_{s+1} = \kappa(\varepsilon) \cdot q_{s+1}.$$

Therefore

$$|f^{(n)}(x) - f^{(n)}(y) - (p - d)| < \varepsilon \text{ for } n \in J.$$

Now let M, L be natural numbers such that $J = [M, M + L]$. Then

$$\frac{L}{M} \geq \frac{|J|}{q_{s+1}} \geq \kappa(\varepsilon),$$

$$M \geq q_s \geq q_{s_0} > N \quad \text{and} \quad L \geq |J| \geq \kappa(\varepsilon) q_{s+1} \geq \kappa(\varepsilon) q_{s_0} > N,$$

by (19). Since the special flow T^f is weakly mixing (see Proposition 2 in [**9**]), the automorphism T^f_γ is ergodic for all $\gamma \neq 0$, and hence an application of Lemma 5.2 completes the proof. □

Since special flows built over irrational rotations on the circle and under piecewise absolutely continuous roof functions with a non–zero sum of jumps are weakly mixing (see [**9**]), from Theorems 5.1 and 6.1 we obtain the following.

COROLLARY 6.1. *Suppose that $T : \mathbb{T} \to \mathbb{T}$ is the rotation by an irrational number α with bounded partial quotients and $f : \mathbb{T} \to \mathbb{R}$ is a positive and bounded away from zero piecewise absolutely continuous function with a non–zero sum of jumps. Then any ergodic joining ρ of the special flow $(T^f_t)_{t\in\mathbb{R}}$ and an ergodic flow $(T_t)_{t\in\mathbb{R}}$ acting on $(Y, \mathcal{C}, \nu)$ is either the product joining, or ρ is a finite extension of ν.*

Problem. It would be interesting to decide whether in the family of special flows over the rotation by an irrational α with bounded partial quotients and under f which is piecewise absolutely continuous with a non–zero sum of jumps we can find some with the minimal self–joining property.

7. *Absence of partial rigidity*

LEMMA 7.1. *Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be an ergodic automorphism and $f \in L^1(X, \mu)$ be a positive function such that $f \geq c > 0$. Suppose that the special flow T^f is partially rigid along a sequence (t_n), $t_n \to +\infty$. Then there exists $0 < u \leq 1$ such that for every $0 < \varepsilon < c$ we have*

$$\liminf_{n\to\infty} \mu\{x \in X : \exists_{j\in\mathbb{N}}\, |f^{(j)}(x) - t_n| < \varepsilon\} \geq u.$$

Proof. By assumption, there exists $0 < u \leq 1$ such that for any measurable set $D \subset X^f$ we have

$$\liminf_{n\to\infty} \mu^f(D \cap T^f_{-t_n} D) \geq u\mu^f(D).$$

Fix $0 < \varepsilon < c$. Let $A := X \times [0, \varepsilon)$ and for any natural n put

$$B_n = \{x \in X : \exists_{j\in\mathbb{N}}\ |f^{(j)}(x) - t_n| < \varepsilon\}.$$

Suppose that $(x, s) \in A \cap T^f_{-t_n} A$. Then $0 \leq s < \varepsilon$ and there exists $j \in \mathbb{Z}$ such that $0 \leq s + t_n - f^{(j)}(x) < \varepsilon$. It follows that $-\varepsilon < t_n - f^{(j)}(x) < \varepsilon$, hence $x \in B_n$. Therefore $A \cap T^f_{-t_n} A \subset B_n \times [0, \varepsilon)$ and

$$\varepsilon \liminf_{n\to\infty} \mu(B_n) = \liminf_{n\to\infty} \mu^f(B_n \times [0, \varepsilon)) \geq \liminf_{n\to\infty} \mu^f(A \cap T^f_{-t_n} A) \geq u\mu^f(A) = \varepsilon u$$

and the proof is complete. □

THEOREM 7.1. *Let $T : \mathbb{T} \to \mathbb{T}$ be an ergodic rotation. Suppose that $f : \mathbb{T} \to \mathbb{R}$ is a positive and bounded away from zero piecewise absolutely continuous function with $S(f) \neq 0$. Then the special flow T^f is not partially rigid.*

Proof. Let c, C be positive numbers such that $0 < c \leq f(x) \leq C$ for every $x \in \mathbb{T}$. Let μ stand for Lebesgue measure on $\mathbb{T}$. Assume, contrary to our claim, that (t_n), $t_n \to +\infty$, is a partial rigidity time for T^f. By Lemma 7.1, there exists $0 < u \leq 1$ such that for every $0 < \varepsilon < c$ we have

$$\liminf_{n\to\infty} \mu\{x \in \mathbb{T} : \exists_{j\in\mathbb{N}}\ |f^{(j)}(x) - t_n| < \varepsilon\} \geq u. \tag{22}$$

Without loss of generality we can assume that $S := S(f) > 0$, in the case $S < 0$ the proof is the same. Suppose that β_i, $i = 1, \ldots, k$ are all points of discontinuity of f. Fix

$$0 < \varepsilon < \min\left(\frac{Sc^2}{32kC(c + \operatorname{Var} f) + Sc^2)} u, \frac{c}{4}\right). \tag{23}$$

Since $f' \in L^1(\mathbb{T}, \mu)$, there exists $0 < \delta < \varepsilon$ such that $\mu(A) < \delta$ implies $\int_A |f'|\, d\mu < \varepsilon$. Moreover, by the ergodicity of T (and recalling that $S = \int f'\, d\mu$) there exist $A_\varepsilon \subset \mathbb{T}$ with $\mu(A_\varepsilon) > 1 - \delta$ and $m_0 \in \mathbb{N}$ such that

$$\frac{S}{2} \leq \frac{1}{m} f'^{(m)}(x) \tag{24}$$

for all $m \geq m_0$ and $x \in A_\varepsilon$.

Then take any $n \in \mathbb{N}$ such that $t_n/(2C) \geq m_0$ and $t_n > 2\varepsilon$. Now let us consider the set $J_{n,\varepsilon}$ of all natural j such that $|f^{(j)}(x) - t_n| < \varepsilon$ for some $x \in \mathbb{T}$. Then for such j and x we have

$$t_n + \varepsilon > f^{(j)}(x) \geq cj \quad \text{and} \quad t_n - \varepsilon < f^{(j)}(x) \leq Cj,$$

whence

$$t_n/(2C) \leq (t_n - \varepsilon)/C < j < (t_n + \varepsilon)/c \leq 2t_n/c \tag{25}$$

for any $j \in J_{n,\varepsilon}$; in particular, $j \in J_{n,\varepsilon}$ implies $j \geq m_0$.

Let $j_n = \max J_{n,\varepsilon}$. The points of discontinuity of $f^{(j_n)}$, i.e. $\{\beta_i - j\alpha\}, 1 \le i \le k, 0 \le j < j_n$, divide $\mathbb{T}$ into subintervals $I_1^{(n)}, \dots, I_{kj_n}^{(n)}$. Some of these intervals can be empty. Notice that for every $j \in J_{n,\varepsilon}$ the function $f^{(j)}$ is absolutely continuous on the interior of any interval $I_i^{(n)}$, $i = 1, \dots, kj_n$.

Fix $1 \le i \le kj_n$. For every $j \in J_{n,\varepsilon}$ let $I_{i,j}^{(n)}$ stand for the minimal closed subinterval of $\overline{I_i^{(n)}}$ which includes the set $\{x \in I_i^{(n)} : |f^{(j)}(x) - t_n| < \varepsilon\}$. Of course, $I_{i,j}^{(n)}$ may be empty. If $I_{i,j}^{(n)} = [z_1, z_2]$ is not empty then

$$\left|\int_{I_{i,j}^{(n)}} \frac{f'^{(j)}}{j}\, d\mu\right| = \frac{|(f^{(j)})_-(z_2) - (f^{(j)})_+(z_1)|}{j} \le \frac{2\varepsilon}{j} \le \frac{4C\varepsilon}{t_n}. \tag{26}$$

Now suppose that x is an end of $I_{i,j}^{(n)}$ and y is an end of $I_{i,j'}^{(n)}$ with $j \ne j'$. From (23) it follows that

$$\begin{aligned}\int_x^y |f'|^{(j_n)}\, d\mu \ge \left|\int_x^y f'^{(j)}\, d\mu\right| &= |f^{(j)}(y) - f^{(j)}(x)| \\ &\ge |f^{(j)}(y) - f^{(j')}(y)| - |f^{(j')}(y) - t_n| - |f^{(j)}(x) - t_n| \\ &\ge c - 2\varepsilon \ge \frac{c}{2}.\end{aligned} \tag{27}$$

Let $K_i = \{j \in J_{n,\varepsilon} : I_{i,j}^{(n)} \ne \emptyset\}$ and suppose that $s = \# K_i \ge 1$. Then there exist $s-1$ pairwise disjoint subintervals $H_l \subset I_i^{(n)}$, $l = 1, \dots, s-1$ that are disjoint from intervals $I_{i,j}^{(n)}$, $j \in K_i$ and fill up the space between those intervals. In view of (27) we have $\int_{H_l} |f'|^{(j_n)}\, d\mu \ge c/2$ for $l = 1, \dots, s-1$. Therefore, by (26) and (27), we obtain

$$\begin{aligned}\left|\sum_{j\in K_i} \int_{I_{i,j}^{(n)}} \frac{f'^{(j)}}{j}\, d\mu\right| &\le s\frac{4C\varepsilon}{t_n} = \frac{4C\varepsilon}{t_n} + \frac{8C\varepsilon}{ct_n}(s-1)\frac{c}{2} \\ &\le \frac{4C\varepsilon}{t_n} + \frac{8C\varepsilon}{ct_n}\sum_{l=1}^{s-1}\int_{H_l} |f'|^{(j_n)}\, d\mu \\ &\le \frac{4C\varepsilon}{t_n} + \frac{8C\varepsilon}{ct_n}\int_{I_i^{(n)}} |f'|^{(j_n)}\, d\mu.\end{aligned} \tag{28}$$

Since $\mu(A_\varepsilon^c) < \delta$, we have

$$\begin{aligned}\left|\sum_{i=1}^{kj_n}\sum_{j\in K_i}\int_{I_{i,j}^{(n)}\cap A_\varepsilon^c} \frac{f'^{(j)}}{j}\, d\mu\right| &\le \frac{2C}{t_n}\sum_{i=1}^{kj_n}\sum_{j\in K_i}\int_{I_{i,j}^{(n)}\cap A_\varepsilon^c} |f'|^{(j_n)}\, d\mu \\ &\le \frac{2C}{t_n}\int_{A_\varepsilon^c} |f'|^{(j_n)}\, d\mu \le \frac{2C}{t_n} j_n\varepsilon \le \frac{4C}{c}\varepsilon.\end{aligned} \tag{29}$$

As

$$B_n := \{x \in \mathbb{T} : \exists_{j\in\mathbb{N}}\, |f^{(j)}(x) - t_n| < \varepsilon\} \subset \bigcup_{i=1}^{kj_n}\bigcup_{j\in K_i} I_{i,j}^{(n)},$$

by (24), (28), (29) and (25) we have

$$\begin{aligned}\frac{S}{2}\mu(B_n\cap A_\varepsilon) &\leq \sum_{i=1}^{kj_n}\sum_{j\in K_i}\int_{I_{i,j}^{(n)}\cap A_\varepsilon}\frac{f'^{(j)}}{j}\,d\mu\\ &\leq \left|\sum_{i=1}^{kj_n}\sum_{j\in K_i}\int_{I_{i,j}^{(n)}}\frac{f'^{(j)}}{j}\,d\mu\right|+\left|\sum_{i=1}^{kj_n}\sum_{j\in K_i}\int_{I_{i,j}^{(n)}\cap A_\varepsilon^c}\frac{f'^{(j)}}{j}\,d\mu\right|\\ &\leq kj_n\frac{4C\varepsilon}{t_n}+\frac{8C\varepsilon}{t_nc}\int_{\mathbb{T}}|f'|^{(j_n)}\,d\mu+\frac{4C}{c}\varepsilon\\ &\leq \frac{8kC\varepsilon}{c}+\frac{4C\varepsilon}{c}+\frac{16C\varepsilon}{c^2}\|f'\|_{L^1}\leq\frac{16kC}{c^2}(c+\operatorname{Var}f)\varepsilon.\end{aligned}$$

Finally, from (23) we obtain

$$\mu(B_n)\leq\mu(B_n\cap A_\varepsilon)+\mu(A_\varepsilon^c)<\frac{32kC}{Sc^2}(c+\operatorname{Var}f)\varepsilon+\varepsilon<u,$$

contrary to (22). □

Collecting now Theorems 6.1, 7.1 and Lemma 4.1 together with Remark 2 we obtain the following.

THEOREM 7.2. *Suppose that $T:\mathbb{T}\to\mathbb{T}$ is the rotation by an irrational number α with bounded partial quotients and $f:\mathbb{T}\to\mathbb{R}$ is a positive and bounded away from zero piecewise absolutely continuous function with a non–zero sum of jumps. Then the special flow $(T_t^f)_{t\in\mathbb{R}}$ is mildly mixing.*

Applying now the construction from Section 3 we have the following.

COROLLARY 7.1. *On the two–dimensional torus there exists a C^∞–flow $(\varphi_t)_{t\in\mathbb{R}}$ with one singular point (of a simple pole type) such that (φ_t) preserves a positive C^∞–measure and the set of ergodic components of (φ_t) consists of a family of periodic orbits and one non–trivial component of positive measure on which the flow is mildly mixing but not mixing. Moreover, on that component (φ_t) has the Ratner property* R(P) *for some nonempty finite set $P\subseteq\mathbb{R}\setminus\{0\}$.*

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