

NON-REVERSIBILITY AND SELF-JOININGS OF HIGHER ORDERS FOR ERGODIC FLOWS

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ABSTRACT. By studying the weak closure of multidimensional off-diagonal self-joinings we provide a sufficient condition for non-isomorphism of a flow with its inverse, hence the non-reversibility of a flow. This is applied to special flows over rigid automorphisms. In particular, we apply the criterion to special flows over irrational rotations, providing a large class of non-reversible flows, including some analytic reparametrizations of linear flows on \mathbb{T}^2 , so called von Neumann's flows and some special flows with piecewise polynomial roof functions. A topological counterpart is also developed with the full solution of the problem of the topological self-similarity of continuous special flows over irrational rotations. This yields examples of continuous special flows over irrational rotations without non-trivial topological self-similarities and having all non-zero real numbers as scales of measure-theoretic self-similarities.

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1. INTRODUCTION

Given a (measurable) measure-preserving flow $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ on a probability standard Borel space (X, \mathcal{B}, μ) one says that it is *reversible* (or *time reversible*) if \mathcal{T} is isomorphic to its inverse with a conjugating automorphism $S : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$

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being an involution¹, i.e.:

$$(1.1) \quad T_t \circ S = S \circ T_{-t} \text{ for each } t \in \mathbb{R}$$

and

$$(1.2) \quad S^2 = Id.$$

As far as we know, in ergodic theory, this problem was not systematically studied for flows. In case of automorphisms first steps were taken up in [13]. In that paper it has been shown that for an arbitrary automorphism T with simple spectrum all isomorphisms (if there is any) between T and T^{-1} must be involutions. The same result holds for flows: a simple spectrum flow isomorphic to its inverse is reversible, in fact, (1.1) implies (1.2)². Another class of flows in which (1.1) puts some restrictions on the order of S is the class of flows having so called weak closure property: each element R of the centralizer $C(\mathcal{T})$ is a weak limit of time- t automorphisms, i.e. $R = \lim_{k \rightarrow \infty} T_{t_k}$ for some $t_k \rightarrow \infty$, namely, we must have $S^4 = Id$ ³. Moreover, if $S^2 \neq Id$ then \mathcal{T} is not reversible⁴.

It is easy to observe that isomorphisms between T and T^{-1} lift to isomorphisms of the corresponding suspension flow (see Section 2 for the definition of the suspension flow) and its inverse. Moreover, as observed e.g. in [5], each isomorphism between the suspension flow and its inverse must come from an isomorphism of T and T^{-1} . In [13], there is a construction of an automorphism T satisfying the weak closure property, isomorphic to its inverse and such that all conjugations between T and T^{-1} have order four. By taking the suspension flow over this example we obtain an ergodic flow having the weak closure property, being isomorphic to its inverse and such that all conjugations satisfying (1.1) are of order four, so this flow is not reversible.

The problem of reversibility is closely related to the self-similarity problem (see [6], [9]). Recall that $s \in \mathbb{R}^*$ is a scale of self-similarity for a measure-preserving flow $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ if \mathcal{T} is isomorphic to the flow $\mathcal{T} \circ s := (T_{st})_{t \in \mathbb{R}}$. We will denote the multiplicative subgroup of all scales of self-similarity by $I(\mathcal{T}) \subset \mathbb{R}^*$. The flow \mathcal{T} is called self-similar if $I(\mathcal{T}) \not\subseteq \{-1, 1\}$. Of course, if \mathcal{T} is reversible then $-1 \in I(\mathcal{T})$.

¹It should be noticed that, in general, even if (1.1) and (1.2) are satisfied for some S then we can find S' which is not an involution but satisfies (1.1) [13]. For example, take $T(x, y) = (x + \alpha, x + y)$ on \mathbb{T}^2 . Then $T^{-1}(x, y) = (x - \alpha, -(x - \alpha) + y)$ and $S(x, y) = (-x, x + y)$ settles an isomorphism of T and its inverse. Of course $S^2 = Id$. On the other hand if we set $\sigma_\gamma(x, y) = (x, y + \gamma)$ then $\sigma_\gamma T = T \sigma_\gamma$ and $\sigma_\gamma S = S \sigma_\gamma$. Hence $(S \sigma_\gamma) T = T^{-1} (S \sigma_\gamma)$. But $(S \sigma_\gamma)^n = S^n \text{ (mod } 2) \sigma_{n\gamma}$, so we obtain a conjugation which is of infinite order (if γ is irrational).

Another example can be given by taking first a weakly mixing flow $(S_t)_{t \in \mathbb{R}}$ and then considering $T_t = S_t \times S_{-t}$ in which $(x, y) \mapsto (y, x)$ yields reversibility of \mathcal{T} . On the other hand, $W(x, y) = (S_1 y, x)$ also settles an isomorphism of \mathcal{T} and its inverse and since $W^2 = S_1 \times S_1$, W is even weakly mixing.

²The proof from [13] goes through for flows.

One more natural case when isomorphism of \mathcal{T} and its inverse implies reversibility arises if we assume that the centralizer $C((T_t))$ is trivial, i.e. equal to $\{T_t : t \in \mathbb{R}\}$ and the \mathbb{R} -action $t \mapsto T_t$ is free. Indeed, as in [13], we notice that whenever S satisfies (1.1) then S^2 belongs to $C(\mathcal{T})$, so $S^2 = T_{t_0}$. Now, clearly $T_{t_0} S = S T_{t_0}$ and since $T_{t_0} S = S T_{-t_0}$, we have $T_{-t_0} = T_{t_0}$ and hence $t_0 = 0$ by the freeness assumption.

³We borrow the argument from [13]: $C(\mathcal{T}) \ni S^2 = \lim_{k \rightarrow \infty} T_{t_k}$ and since $T_{t_k} S = S T_{-t_k}$, by passing to the limit, $S^3 = S^{-1}$.

⁴Again, borrowing the argument from [13], suppose that S' satisfies (1.1). Then $SS' \in C(\mathcal{T})$, so $SS' = \lim_{k \rightarrow \infty} T_{t_k}$. Since $T_{t_k} S = S T_{-t_k}$, $(SS') S = S(SS')^{-1}$, whence $(S')^2 = S^{-2}$, but $S^4 = Id$, so $S^2 = (S')^2$.

Note that it follows that if \mathcal{T} satisfies the weak closure theorem, is isomorphic to its inverse and is not reversible then it has a 2-point fiber factor, namely $\{B \in \mathcal{B} : S^2 B = B\}$, which is reversible.

The papers [40], [41] concerning some particular rank one automorphisms were historically the first to show that the non-isomorphism of an automorphism and its inverse can be detected on the level of 3-self-joinings by studying the weak closure of 3-off-diagonal self-joinings (see also [6]). By taking the suspensions of Ryzhikov's automorphisms we obtain flows non-isomorphic to their inverses.

One of the purposes of this paper is to generalize Ryzhikov's approach and present potential asymmetries in the weak closure of higher dimensional off-diagonal self-joinings when we change time in the suspension over a rigid automorphism. We recall that one of possibilities to show the absence of self-similarities for a non-rigid flow is to show that in the weak closure of its 2-off-diagonal self-joinings there is an integral of off-diagonal joinings, and this approach was fruitfully used in [9] to study special flows over irrational rotations.

In Section 3 we extend techniques introduced in [8] for 2-joinings⁵ to the class of higher order joinings, see Proposition 3.7. This is necessary because the 2-joining method breaks down when we try to prove the non-isomorphism of the flow and its inverse. Indeed, in our method (similarly to [41]) we seek integral type self-joinings which are sufficiently "asymmetric" and are in the weak closure of off-diagonal self-joinings. Then, the asymmetry should result in the non-isomorphism of a flow and its inverse. Such a method cannot work on the level of 2-self-joinings because it is spectral. Therefore, to distinguish between the flow and its inverse, we will apply r -joinings (for $r \geq 3$), see Proposition 3.9 and its consequences.

In Section 4, using Corollary 3.13 (i.e. the 3-joinings approach), we prove that any von Neumann flow is not isomorphic to its inverse for almost every rotation in the base.

In Section 5, the approach developed in Section 3 is applied to special flows T^f built over irrational rotations $Tx = x + \alpha$ on the circle and under C^{r-1} -roof functions (r is an odd natural number) which are polynomials of degree r on two complementary intervals $[0, \beta)$ and $[\beta, 1)$ ($0 < \beta < 1$). Here to show non-isomorphism of T^f with the inverse, we need to study $r + 1$ -joinings and we prove that for a.e. β the flow T^f is not isomorphic to its inverse, whenever α satisfies a Diophantine type condition (along a subsequence, see (5.2)).

In Section 6 the 3-joining approach turns out to be sufficient to construct an analytic area-preserving flow on the two torus which is not isomorphic to its inverse. In other words, we show that we can change time in an ergodic linear flow (which is always reversible) in an analytic way so that the resulting flow is weakly mixing and not reversible. We use the AACCP method introduced in [23]. Additionally, slightly modifying the construction, we prove that the resulting flow has no rational self-similarities. In fact, we obtain disjointness (in the Furstenberg sense) of any two different rational time automorphisms. This kind of investigations is partly motivated by Sarnak's conjecture on orthogonality of deterministic sequences from Möbius function through disjointness, see [3].

In Section 7, we come back to automorphisms, and as in [41], we show that the 3-joining method can be applied to a class of rank one automorphisms having a subsequence of towers of Chacon's type. We show that they are not reversible (we recall that rank one automorphisms have simple spectra).

In Section 8 we deal with topological self-similarities of continuous time changes of minimal linear flows on the two torus. Each such flow is topologically conjugate to the special flow T^f build over an irrational rotation $Tx = x + \alpha$ on the circle and under a continuous roof function $f : \mathbb{T} \rightarrow \mathbb{R}_+$. We show that if T^f is topologically

⁵We use the term 2-joinings for short, however only 2-self-joinings that are in the weak closure of off-diagonal joinings are used in this method. The same remark applies for joinings of higher order.

self-similar then α is a quadratic irrational and f is topologically cohomological to a constant function. It follows that if a continuous time change of a minimal linear flow on the two torus is topologically self-similar then it is topologically conjugate to a minimal linear flows as well. As a byproduct, we obtain an example of a continuous flow \mathcal{T} on the torus which has no topological self-similarities and the group of scales of self-similarity (for \mathcal{T} treated as a measure-preserving system) is equal to \mathbb{R}^* .

For the problem of topological reversibility and its generalizations in the class of topological Markov shifts, see [26].

First historical examples of automorphisms non-isomorphic to their inverses were provided by Anzai [2], Malkin [30] and Oseledets [35]. Moreover, the property of being isomorphic to its inverse (the more, reversibility) is not a typical property. As shown by del Junco [16] (for automorphisms) and by Danilenko and Ryzhikov [6] (for flows), typically we have disjointness with the inverse action. But there are quite a few natural classes of flows which are reversible. Let us go through a selection of known examples.

A) All ergodic flows with discrete spectrum are reversible. This follows easily from the Halmos-von Neumann theorem, see e.g. [4] (the fact that each isomorphism must be an involution is a consequence of the simplicity of the spectrum of such flows).

B) All Gaussian flows are reversible. Indeed, each Gaussian flow is determined by a one-parameter unitary group $\mathcal{U} = (U_t)_{t \in \mathbb{R}}$ acting on a separable Hilbert space H such that there is a spectral decomposition

$$(1.3) \quad H = \bigoplus_{n=1}^{\infty} \mathbb{R}(x_n) \quad \text{with } \sigma_{x_1} \gg \sigma_{x_2} \gg \dots \quad \text{and } \sigma_{x_n}(A) = \sigma_{x_n}(-A)$$

for each Borel subset $A \subset \mathbb{R}$ and $n \geq 1$ (and σ_{x_1} is assumed to be continuous), see [19], [27], [28]. Now, the action \mathcal{U} on $\mathbb{R}(x_n)$ is isomorphic to the action $\mathcal{V}^{(n)}$:

$$V_t^{(n)}(f)(x) = e^{2\pi i t x} f(x) \quad \text{for } f \in L^2(\mathbb{R}, \sigma_{x_n}),$$

so $I_n f(x) = f(-x)$ is an involution which settles an isomorphism of $\mathcal{V}^{(n)}$ and its inverse. Then, up to isomorphism, $I = \bigoplus_{n=1}^{\infty} I^{(n)}$ is an involution which settles an isomorphism of \mathcal{U} and its inverse⁶ and it extends to a measure-preserving isomorphism of the corresponding Gaussian flow (T_t) and its inverse, see e.g. [28].

C) Some horocycle flows are reversible. Let $\Gamma \subset PSL_2(\mathbb{R})$ be a discrete subgroup with finite covolume. Then the homogeneous space $X = \Gamma \backslash PSL_2(\mathbb{R})$ is the unit tangent bundle of a surface M of constant negative curvature. Let us consider the corresponding *horocycle flow*⁷ $(h_t)_{t \in \mathbb{R}}$ and *geodesic flow* $(g_s)_{s \in \mathbb{R}}$ on X . Since

$$(1.4) \quad g_s h_t g_s^{-1} = h_{e^{-2s}t} \quad \text{for all } s, t \in \mathbb{R},$$

the flows $(h_t)_{t \in \mathbb{R}}$ and $(h_{e^{-2s}t})_{t \in \mathbb{R}}$ are measure-theoretic isomorphic for each $s \in \mathbb{R}$, so all positive numbers are self-similarity scales for a horocycle flow.

We will now show that some horocycle flows are reversible. Let now J denote the matrix

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

⁶Notice that the same argument works for an arbitrary Koopman representation $U_t = U_{T_t}$. In other words, an arbitrary Koopman representation is unitarily reversible.

⁷We have $h_t(\Gamma x) = \Gamma \cdot \left(x \cdot \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \right)$ and $g_s(\Gamma x) = \Gamma \cdot \left(x \cdot \begin{bmatrix} e^{-s} & 0 \\ 0 & e^s \end{bmatrix} \right)$.

Clearly, $J \notin SL_2(\mathbb{R})$. However, if Γ satisfies $\Gamma = J^{-1}\Gamma J$ then J will also act on $\Gamma \backslash PSL_2(\mathbb{R})$:

$$J(\Gamma x) := J^{-1}\Gamma x J = (J^{-1}\Gamma J)J^{-1}xJ = \Gamma J^{-1}xJ.$$

Moreover,

$$J^{-1}h_t J = h_{-t}$$

and since J yields an order two map, we obtain that in this case the horocycle flow is reversible. It follows that $I((h_t)_{t \in \mathbb{R}}) = \mathbb{R}^*$.

Corollary 1.1. *In the modular case $\Gamma := PSL_2(\mathbb{Z}) \subset PSL_2(\mathbb{R})$, the horocycle flow $(h_t)_{t \in \mathbb{R}}$ is reversible.*

There are even cocompact lattices Γ which are not “compatible” with the matrix J . In this case a deep theory of Ratner [38] implies that in particular $(h_t)_{t \in \mathbb{R}}$ is not measure-theoretically isomorphic to its inverse.

Let us come back to the horocycle flow $(h_t)_{t \in \mathbb{R}}$ on the modular space $\Gamma \backslash PSL_2(\mathbb{R})$, $\Gamma = PSL_2(\mathbb{Z})$. By Corollary 1.1, this flow is reversible. Moreover, $C((h_t)_{t \in \mathbb{R}}) = \{h_t : t \in \mathbb{R}\}$. Indeed, first note that

$$\{\alpha \in PSL_2(\mathbb{R}) : \alpha \Gamma \alpha^{-1} = \Gamma\} = \Gamma.$$

In view of the celebrated Ratner’s Rigidity Theorem (see Corollary 2 in [37]), it follows that $C((h_t)_{t \in \mathbb{R}})$ is indeed trivial⁸. Hence, we obtain the following more precise version of Corollary 1.1 (cf. footnote 2).

Corollary 1.2. *In the modular case $\Gamma := PSL_2(\mathbb{Z}) \subset PSL_2(\mathbb{R})$ we have $C((h_t)_{t \in \mathbb{R}}) = \{h_t : t \in \mathbb{R}\}$. Then, each S establishing isomorphism of $(h_t)_{t \in \mathbb{R}}$ with its inverse is an involution. Moreover, $S = h_{t_0} \circ J$ for some $t_0 \in \mathbb{R}$.*

D) All Bernoulli flows are reversible. This is done in two steps. If the entropy is infinite then (via Ornstein’s isomorphisms theorem [32]) we have a Gaussian realization of such a flow and we use **B**). If the entropy is finite then (again via [32]) we can consider the geodesic flow on $\Gamma \backslash PSL_2(\mathbb{R})$. Then

$$K^{-1}g_t K = g_{-t} \text{ for all } t \in \mathbb{R}, \quad \text{where} \quad K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This establishes an isomorphism between $(g_t)_{t \in \mathbb{R}}$ and $(g_{-t})_{t \in \mathbb{R}}$ via an involution ($K^2 = Id$ as an element of $PSL_2(\mathbb{R})$)⁹ and hence the isomorphism of $(g_{st})_{t \in \mathbb{R}}$ and $(g_{-st})_{t \in \mathbb{R}}$ for each $s \in \mathbb{R} \setminus \{0\}$.

E) Geodesic flow revisited, Hamiltonian dynamics.¹⁰ In this case we obtain always reversibility, because each such flow acts on a tangent space following geodesics: the configuration space consists of pairs (x, v) (x – placement, v – speed) and the involution is simply given by

$$(x, v) \mapsto (x, -v).$$

⁸The general result of Ratner states that elements of the centralizer are the composition of h_{t_0} with the automorphism given by $\Gamma x \mapsto \Gamma(\alpha x)$, where $\alpha \Gamma \alpha^{-1} = \Gamma$.

⁹An alternative proof of reversibility of Bernoulli was pointed to us by J.-P. Thouvenot. Indeed, consider the shift $T: \{0, 1\}^{\mathbb{Z}} \rightarrow \{0, 1\}^{\mathbb{Z}}$ given by $T((x_n)_{n \in \mathbb{Z}}) = (x_{n+1})_{n \in \mathbb{Z}}$, where $\{0, 1\}^{\mathbb{Z}}$ is equipped with the product measure $\mu = P^{\otimes \mathbb{Z}}$ with $P(\{0\}) = P(\{1\}) = 1/2$. Then the map $I: (x_n)_{n \in \mathbb{Z}} \mapsto (x_{-n})_{n \in \mathbb{Z}}$ is an involution conjugating T with T^{-1} . Moreover, there is a roof function f constant on each of the cylinder sets $\{(x_n)_{n \in \mathbb{Z}}: x_0 = 0\}$, $\{(x_n)_{n \in \mathbb{Z}}: x_0 = 1\}$ such that the special flow T^f is Bernoulli [33]. Now, it suffices to apply Remark 2.3 below to conclude that T^f (as well as $T^f \circ s$ for each $s \in \mathbb{R} \setminus \{0\}$) is reversible. For the infinite entropy case it suffices to consider the infinite product $T^f \times T^f \times \dots$

¹⁰This was pointed out to us by E. Gutkin.

The authors would like to thank the referee for the comments improving the presentation of the results of the paper and also for suggesting us one more natural class of reversible flows given by special flows over reversible interval exchange transformations (see Remark 2.4).

2. SPECIAL FLOWS

Assume that T is an ergodic automorphism of a standard probability Borel space (X, \mathcal{B}, μ) . We let $\mathcal{B}(\mathbb{R})$ and $\lambda_{\mathbb{R}}$ stand for the Borel σ -algebra and Lebesgue measure on \mathbb{R} respectively.

Assume $f : X \rightarrow \mathbb{R}$ is an L^1 strictly positive function. Denote by $\mathcal{T}^f = (T_t^f)_{t \in \mathbb{R}}$ the corresponding special flow under f (see e.g. [4], Chapter 11). Recall that such a flow acts on $(X^f, \mathcal{B}^f, \mu^f)$, where $X^f = \{(x, s) \in X \times \mathbb{R} : 0 \leq s < f(x)\}$ and \mathcal{B}^f (μ^f) is the restriction of $\mathcal{B} \otimes \mathcal{B}(\mathbb{R})$ ($\mu \otimes \lambda_{\mathbb{R}}$) to X^f . Under the action of \mathcal{T}^f a point in X^f moves vertically at unit speed, and we identify the point $(x, f(x))$ with $(Tx, 0)$. Clearly, \mathcal{T}^f is ergodic as T is ergodic. To describe this \mathbb{R} -action formally set

$$f^{(k)}(x) = \begin{cases} f(x) + f(Tx) + \dots + f(T^{k-1}x) & \text{if } k > 0 \\ 0 & \text{if } k = 0 \\ -(f(T^k x) + \dots + f(T^{-1}x)) & \text{if } k < 0. \end{cases}$$

Let us consider the skew product $T_{-f} : X \times \mathbb{R} \rightarrow X \times \mathbb{R}$,

$$T_{-f}(x, r) = (Tx, r - f(x))$$

and the flow $(\sigma_t)_{t \in \mathbb{R}}$ on $(X \times \mathbb{R}, \mathcal{B} \otimes \mathcal{B}(\mathbb{R}), \mu \otimes \lambda_{\mathbb{R}})$

$$\sigma_t(x, r) = (x, r + t).$$

Then for every $(x, r) \in X^f$ we have

$$(2.1) \quad T_t^f(x, r) = T_{-f}^n \circ \sigma_t(x, r) = (T^n x, r + t - f^{(n)}(x)),$$

where $n \in \mathbb{Z}$ is unique for which $f^{(n)}(x) \leq r + t < f^{(n+1)}(x)$.

Remark 2.1. Recall that if T is an ergodic automorphism of a standard probability Borel space (X, \mathcal{B}, μ) is aperiodic. Moreover, any special flow \mathcal{T}^f is also aperiodic, i.e. for every $t \neq 0$ we have $\mu^f(\{(x, s) \in X^f : T_t^f(x, s) = (x, s)\}) = 0$.

Remark 2.2. The special flow \mathcal{T}^f can also be seen as the quotient \mathbb{R} -action $(\sigma_t)_{t \in \mathbb{R}}$, $\sigma_t(x, r) = (x, r + t)$ on the space $X \times \mathbb{R} / \equiv$, where \equiv is the T_{-f} -orbital equivalence relation, $T_{-f}(x, r) = (Tx, -f(x) + r)$. Indeed, $\sigma_t \circ T_{-f} = T_{-f} \circ \sigma_t$, so σ_t acts on the quotient space. Moreover, the quotient space $X \times \mathbb{R} / \equiv$ is naturally isomorphic with X^f by choosing the unique point from the T_{-f} -orbit of (x, r) belonging to X^f . Finally,

$$(T^f)_t(x, r) = (T_{-f})^k \sigma_t(x, r)$$

for a unique $k \in \mathbb{Z}$.

Using Remark 2.2 we will now provide a criterion for a special flow to be isomorphic to its inverse.

Remark 2.3. Assume that T is isomorphic to its inverse: $ST = T^{-1}S$. Assume moreover that

$$(2.2) \quad f(Sx) - f(x) = h(x) - h(Tx)$$

for a measurable $h : X \rightarrow \mathbb{R}$. We claim that the special flow \mathcal{T}^f is isomorphic to its inverse and is reversible if $S^2 = Id$ and $h(TSx) = h(x)$. Indeed, first notice that

$$(2.3) \quad (T_{-f})^{-1} = T_{f \circ T^{-1}}^{-1}.$$

Set

$$g(x) = f(x) - h(Tx)$$

and consider $S_{g,-1} : X \times \mathbb{R} \rightarrow X \times \mathbb{R}$,

$$S_{g,-1}(x, r) = (Sx, g(x) - r).$$

Note that $S_{g,-1}$ is measurable and preserves the measure $\mu \otimes \lambda_{\mathbb{R}}$. It follows immediately that

$$(2.4) \quad S_{g,-1} \circ \sigma_t = \sigma_{-t} \circ S_{g,-1} \quad \text{for each } t \in \mathbb{R}.$$

All we need to show is that $S_{g,-1}$ acts on the space of orbits, that is, it sends a T_{-f} -orbit into another T_{-f} -orbit. For that, it is enough to show that

$$(2.5) \quad S_{g,-1} \circ T_{-f} \circ (S_{g,-1})^{-1} = (T_{-f})^{-1}.$$

Now, in view of (2.3), the equation (2.5) is equivalent to showing that

$$f(T^{-1}Sx) + g(x) = g(Tx) + f(x)$$

which indeed holds as by (2.2) (replacing x by Tx) we have

$$f(STx) - f(Tx) = h(Tx) - h(T^2x),$$

so $f(T^{-1}Sx) - f(Tx) = h(Tx) - h(T^2x)$, whence

$$f(T^{-1}Sx) - f(x) = f(Tx) - f(x) + h(Tx) - h(T^2x) = g(Tx) - g(x),$$

so indeed $S_{g,-1}$ settles an isomorphism of T^f and its inverse.

For the second part, we simply check that under the assumption $S^2 = Id$, we have $g(Sx) = g(x)$ if and only if $h(x) = h(TSx)$.

Finally, notice that in the original functional equation (2.2) we can consider RS instead of S with $R \in C(T)$ (note however that even if $S^2 = Id$ we may now have $(RS)^2 \neq Id$).

To illustrate Remark 2.3 first consider the special flow over irrational rotation $Tx = x + \alpha$ on $\mathbb{T} = [0, 1)$ with the roof function f of the form

$$f(x) = \begin{cases} b & \text{for } x \in [0, a) \\ c & \text{for } x \in [a, 1), \end{cases}$$

where $0 < a < 1$ and $b, c > 0$. Then take $Rx = x + a$ and $Sx = -x$. Note that RS is involution and check that $f \circ R \circ S = f$, which by Remark 2.3 means that T^f is reversible.

Here is another application of Remark 2.3.

Remark 2.4. Let $T : [0, 1) \rightarrow [0, 1)$ be an m -interval exchange transformation (see e.g. [42]) based on the symmetric permutation, i.e. T transforms the i -th interval to an $(m + 1 - i)$ -th interval for $i = 1, \dots, m$. Denote by $\lambda_i > 0$ the length of the i -th interval. Let us consider the roof function $f : [0, 1) \rightarrow \mathbb{R}_+$ which is constant over each exchanged interval. Then the special flow T^f is reversible. Indeed, remark that T^{-1} is also an m -interval exchange transformation based on the symmetric permutation for which the length of the i -th interval is λ_{m+1-i} for $i = 1, \dots, m$. It follows that T^{-1} is metrically isomorphic to T via the involution $Sx = 1 - x$. Moreover, $f \circ T^{-1} = f \circ S$. Therefore, $f(Sx) - f(x) = h(x) - h(Tx)$ with $h(x) = f(T^{-1}x) = f(Sx)$ and $h(TSx) = f(Sx) = h(x)$. In view of Remark 2.3, this yields the reversibility of T^f .

This gives rise to a class of reversible linear flows on a translation space of each genus ≥ 1 (see [42], Lemma 2.12).

If we take $f = 1$ then the resulting special flow is called the *suspension flow* of T . Note also that special flows are obtained by so called (measurable) change of time of the suspension flow (see [4]). It is easy to see that

$$T_t^1(x, r) = (T^{\lceil t+r \rceil}x, \{t+r\}).^{11}$$

Recall that a sequence (q_n) of integers, $q_n \rightarrow \infty$, is called a *rigidity sequence* for T if $T^{q_n} \rightarrow Id$ (similarly we define a real-valued rigidity sequence for flows). Note that whenever (q_n) is

(2.6) a rigidity sequence for T , it is a rigidity sequence for the suspension.

Directly from Remark 2.3, it follows that the suspension of the reversible automorphism yields a reversible flow.

Remark 2.5. Similarly as the functional equation (2.2) defines an isomorphism of T^f with its inverse, if $S \in C(T)$ in (2.2) then

$$(2.7) \quad f(Sx) - f(x) = g(x) - g(Tx)$$

with $g : X \rightarrow \mathbb{R}$ measurable, yields an element of $C(T^f)$. Indeed, consider the skew product

$$S_g : X \times \mathbb{R} \rightarrow X \times \mathbb{R}, \quad S_g(x, r) = (Sx, r + g(x)).$$

Then S_g commutes with the flow $(\sigma_t)_{t \in \mathbb{R}}$ and, by (2.8), with the skew product T_{-f} . It follows that S_g can be considered as an automorphism on $X^f = (X \times \mathbb{R})/ \equiv$ with commutes with the special flow T^f .

The following lemma tells us that whenever the centralizer of T^f is trivial, we can solve the functional equation (2.7) only in a trivial way.

Lemma 2.6. *Assume that T is ergodic and $C(T^f) = \{T_t^f : t \in \mathbb{R}\}$. Suppose that $S \in C(T)$ and $g : X \rightarrow \mathbb{R}$ is a measurable function such that*

$$(2.8) \quad f \circ S - f = g - g \circ T.$$

Then there exist $k \in \mathbb{Z}$ and $t_0 \in \mathbb{R}$ such that

$$S = T^k \quad \text{and} \quad g = t_0 - f^{(k)}.$$

Proof. By Remark 2.5, the automorphism

$$S_g : X \times \mathbb{R} \rightarrow X \times \mathbb{R}, \quad S_g(x, r) = (Sx, r + g(x))$$

can be considered as an element of $C(T^f)$. By assumption, there exists $t_0 \in \mathbb{R}$ such that $S_g = T_{t_0}^f$ on X^f . Therefore, there exists a measurable function $k : X \times \mathbb{R} \rightarrow \mathbb{Z}$ with

$$(Sx, r + g(x)) = S_g(x, r) = T_{-f}^{k(x,r)}(x, r + t_0) = (T^{k(x,r)}x, r + t_0 - f^{(k(x,r))}(x)),$$

so

$$Sx = T^{k(x,r)} \quad \text{and} \quad g(x) = t_0 - f^{(k(x,r))}(x).$$

It follows that k does not depend on the second coordinate, i.e. $k(x, r) = k(x)$ (indeed, $f^{(k_1)}(x) \neq f^{(k_2)}(x)$ whenever $k_1 \neq k_2$) and $Sx = T^{k(x)}x$. Thus

$$T^{1+k(x)}x = TSx = STx = T^{k(Tx)}(Tx) = T^{k(Tx)+1}x.$$

By the ergodicity of T , $k \circ T = k$ and hence k is constant, which proves our claim. \square

3. JOININGS AND NON-ISOMORPHISM OF A FLOW WITH ITS INVERSE

In this section we will present a method of proving non-reversibility by studying the weak closure of off diagonal self-joinings (of order at least 3).

¹¹[.] stands for the integer part of a real number.

3.1. Self-joinings for ergodic flows. Assume that $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ is an ergodic flow on (X, \mathcal{B}, μ) . For any $k \geq 2$ by a *k-self-joining* of \mathcal{T} we mean any probability $(T_t \times \dots \times T_t)_{t \in \mathbb{R}}$ -invariant measure λ on $(X^k, \mathcal{B}^{\otimes k})$ whose projections on all coordinates are equal to μ , i.e.

$$\lambda(X \times \dots \times X \times A_i \times X \times \dots \times X) = \mu(A_i) \quad \text{for any } 1 \leq i \leq k \text{ and } A_i \in \mathcal{B}.$$

We will denote by $J_k(\mathcal{T})$ the set of all k -self-joinings for \mathcal{T} . If the flow $(T_t \times \dots \times T_t)_{t \in \mathbb{R}}$ on (X^k, λ) is ergodic then λ is called an ergodic k -joining.

Let $\{B_n : n \in \mathbb{N}\}$ be a countable family in \mathcal{B} which is dense in \mathcal{B} for the (pseudo-) metric $d_\mu(A, B) = \mu(A \Delta B)$. Let us consider the metric d on $J_k(\mathcal{T})$ defined by

$$d(\lambda, \lambda') = \sum_{n_1, \dots, n_k \in \mathbb{N}} \frac{1}{2^{n_1 + \dots + n_k}} |\lambda(B_{n_1} \times \dots \times B_{n_k}) - \lambda'(B_{n_1} \times \dots \times B_{n_k})|.$$

Endowed with corresponding to d topology the set $J_k(\mathcal{T})$ is compact.

For any family S_1, \dots, S_{k-1} of elements of the centralizer $C(\mathcal{T})$ we will denote by $\mu_{S_1, \dots, S_{k-1}}$ the k -joining determined by

$$\mu_{S_1, \dots, S_{k-1}}(A_1 \times \dots \times A_{k-1} \times A_k) = \mu(S_1^{-1}A_1 \cap \dots \cap S_{k-1}^{-1}A_{k-1} \cap A_k)$$

for all $A_1, \dots, A_k \in \mathcal{B}$. Since $\mu_{S_1, \dots, S_{k-1}}$ is the image of the measure μ via the map $x \mapsto (S_1x, \dots, S_{k-1}x, x)$, this joining is ergodic. When all S_i are time t_i -automorphisms of the flow, then $\mu_{S_1, \dots, S_{k-1}}$ is called an *off-diagonal* self-joining.

For any probability Borel measure $P \in \mathcal{P}(\mathbb{R}^{k-1})$ we will deal with the measure $\int_{\mathbb{R}^{k-1}} \mu_{T_{t_1}, \dots, T_{t_{k-1}}} dP(t_1, \dots, t_{k-1})$ defined by

$$\int_{\mathbb{R}^{k-1}} \mu_{T_{t_1}, \dots, T_{t_{k-1}}} dP(t_1, \dots, t_{k-1})(A) := \int_{\mathbb{R}^{k-1}} \mu_{T_{t_1}, \dots, T_{t_{k-1}}}(A) dP(t_1, \dots, t_{k-1})$$

for any $A \in \mathcal{B}^{\otimes k}$. Then $\int_{\mathbb{R}^{k-1}} \mu_{T_{t_1}, \dots, T_{t_{k-1}}} dP(t_1, \dots, t_{k-1}) \in J_k(\mathcal{T})$. In the following section we will provide a criterion of having such an integral self-joining in the weak closure of off-diagonal joinings for some special flows.

Similarly, we also consider joinings between different (ergodic) flows, say $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ and $\mathcal{S} = (S_t)_{t \in \mathbb{R}}$. Following [10], we say that \mathcal{T} and \mathcal{S} are *disjoint* if $J(\mathcal{T}, \mathcal{S}) = \{\mu \otimes \nu\}$. We write $\mathcal{T} \perp \mathcal{S}$.

3.2. A sufficient condition of existence of integral joinings in the weak closure. Let G be a locally compact Abelian Polish group. Assume that $\|\cdot\|$ is an F-norm inducing a translation invariant metric d on G . Denote by \overline{G} the one-point compactification of G . Assume moreover that $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ is an ergodic automorphism and $F_n : X \rightarrow G$, $n \geq 1$, is a sequence of measurable functions such that

$$(3.1) \quad (F_n)_* \mu \rightarrow P \in \mathcal{P}(G)$$

*-weakly; $\mathcal{P}(G)$ stands for the set of probability Borel measures on G . The following result is a natural generalization of Lemma 4.1 from [8].

Proposition 3.1. *Under the above assumptions, suppose moreover that*

$$(3.2) \quad F_n \circ T - F_n \rightarrow 0 \text{ in measure.}$$

Then for each $\phi \in C(\overline{G})$, $h : X \rightarrow G$ measurable and $j \in L^1(X, \mathcal{B}, \mu)$ we have

$$\lim_{n \rightarrow \infty} \int_X \phi(F_n(x) + h(x)) j(x) d\mu(x) = \int_X \int_G \phi(g + h(x)) j(x) dP(g) d\mu(x).$$

Proof. We will first assume that $h = 0$. In order to prove the above weak convergence we need to show that

$$(3.3) \quad \lim_{n \rightarrow \infty} \int_X \phi(F_n(x))j(x) d\mu(x) = 0$$

for each j whose mean is zero. Now, since the functions of the form $k \circ T - k$ with $k \in L^1(X, \mathcal{B}, \mu)$ are dense in the latter subspace we need to show that $\lim_{n \rightarrow \infty} \int_X \phi(F_n(x))(k(Tx) - k(x)) d\mu(x) = 0$. We have

$$\begin{aligned} \int_X \phi(F_n(x))j(x) d\mu(x) &= \int_X \phi(F_n(x))k(Tx) d\mu(x) - \int_X \phi(F_n(Tx))k(Tx) d\mu(x) \\ &= \int_X (\phi(F_n(x)) - \phi(F_n(Tx)))k(Tx) d\mu(x). \end{aligned}$$

Now, since ϕ is uniformly continuous and bounded and (3.2) holds, (3.3) follows.

Suppose now that $h = \sum_{i=1}^m h_i \cdot \mathbf{1}_{A_i}$ is a simple function ($h_i \in G$ and the sets A_i , $1 \leq i \leq m$ form a measurable partition of X). We have

$$\begin{aligned} \int_X \phi(F_n(x) + h(x))j(x) d\mu(x) &= \sum_{i=1}^m \int_X \phi(F_n(x) + h_i)j(x)\mathbf{1}_{A_i}(x) d\mu(x) \\ &\rightarrow \sum_{i=1}^m \int_G \phi(g + h_i) dP(g) \int_X (j \cdot \mathbf{1}_{A_i})(x) d\mu(x) \\ &= \int_X \int_G \phi(g + h(x))j(x) dP(g)d\mu(x). \end{aligned}$$

All we need to show now is that for each $\varepsilon > 0$ we can find a measurable $h_\varepsilon : X \rightarrow G$ taking only finitely many values so that

$$(3.4) \quad \left| \int_X \phi(F_n(x) + h(x))j(x) d\mu(x) - \int_X \phi(F_n(x) + h_\varepsilon(x))j(x) d\mu(x) \right| < \varepsilon$$

and

$$(3.5) \quad \left| \int_X \phi(g + h(x))j(x) d\mu(x) - \int_X \phi(g + h_\varepsilon(x))j(x) d\mu(x) \right| < \varepsilon.$$

Given $\varepsilon > 0$ we select $\delta > 0$ so that

$$\|g_1 - g_2\| < \delta \Rightarrow |\phi(g_1) - \phi(g_2)| < \varepsilon/(2\|j\|_{L^1}).$$

Then select $\eta > 0$ so that whenever $\mu(A) < \eta$

$$\int_A |j(x)| d\mu(x) < \varepsilon/(4\|\phi\|_\infty).$$

Finally choose $h_\varepsilon : X \rightarrow G$ measurable so that h_ε takes only finitely many values and

$$\mu(\{x \in X : |h_\varepsilon(x) - h(x)| \geq \delta\}) < \eta.$$

We have

$$\begin{aligned} &\left| \int_X \phi(F_n(x) + h(x))j(x) d\mu(x) - \int_X \phi(F_n(x) + h_\varepsilon(x))j(x) d\mu(x) \right| \\ &\leq 2 \int_{\{x \in X : \|h_\varepsilon(x) - h(x)\| \geq \delta\}} \|\phi\|_\infty |j(x)| d\mu(x) \\ &\quad + \int_{\{x \in X : \|h_\varepsilon(x) - h(x)\| < \delta\}} \frac{\varepsilon}{2\|j\|_{L^1}} |j(x)| d\mu(x) < \varepsilon. \end{aligned}$$

We established (3.4) and (3.5) follows in the same manner. \square

Lemma 3.2. For all $t_0, \dots, t_{d-1} \in \mathbb{R}$ and all measurable sets $A_0, \dots, A_d \subset X^f$ we have

$$\mu^f \left(\bigcap_{i=0}^{d-1} (T^f)_{t_i} A_i \cap A_d \right) = \sum_{k_0, k_1, \dots, k_{d-1} \in \mathbb{Z}} \mu \otimes \lambda_{\mathbb{R}} \left(\bigcap_{i=0}^{d-1} ((T_{-f})^{k_i} \sigma_{t_i} A_i) \cap A_d \right).$$

Moreover, the sets $\bigcap_{i=0}^{d-1} ((T_{-f})^{k_i} \sigma_{t_i} A_i) \cap A_d$, $(k_0, \dots, k_{d-1}) \in \mathbb{Z}^d$ are pairwise disjoint.

Proof. Given $(t_0, \dots, t_{d-1}) \in \mathbb{R}^d$ and $(x, r) \in X^f$,

$$(T^f)_{t_i}(x, r) = (T_{-f})^{k_i} \sigma_{t_i}(x, r) \text{ for a unique } k_i \in \mathbb{Z} \text{ for } 0 \leq i \leq d-1.$$

Hence if we fix $i \in \{0, \dots, d-1\}$ then

$$T_{t_i}^f(A_i) = \bigcup_{k \in \mathbb{Z}} T_{-f}^k \sigma_{t_i}(A_i) \cap X^f.$$

The sets on the RHS¹² of the above equality are pairwise disjoint and they correspond to the images via $T_{t_i}^f$ of the partition of X^f into pairwise disjoint sets on which the action of $T_{t_i}^f$ corresponds to $T_{-f}^k \sigma_{t_i}$, $k \in \mathbb{Z}$. Therefore (remembering that $A_d \subset X^f$)

$$\begin{aligned} \bigcap_{i=0}^{d-1} T_{t_i}^f(A_i) \cap A_d &= \bigcap_{i=0}^{d-1} \bigcup_{k_i \in \mathbb{Z}} T_{-f}^{k_i} \sigma_{t_i}(A_i) \cap A_d \\ &= \bigcup_{k_0, k_1, \dots, k_{d-1} \in \mathbb{Z}} \left(\bigcap_{i=0}^{d-1} ((T_{-f})^{k_i} \sigma_{t_i} A_i) \cap A_d \right). \end{aligned}$$

It follows that the above representation corresponds to the partition of the space X^f into countably many subsets $X_{k_0, \dots, k_{d-1}}^f$, $(k_0, \dots, k_{d-1}) \in \mathbb{Z}^d$, on which, for each $i = 0, \dots, d-1$, $(T^f)_{t_i}$ acts as $(T_{-f})^{k_i} \sigma_{t_i}$. Moreover, since $(T^f)_{t_i}$ is an automorphism, the images $(T_{-f})^{k_i} \sigma_{t_i} \left(X_{k_0, \dots, k_{d-1}}^f \right)$ are pairwise disjoint for $(k_0, \dots, k_{d-1}) \in \mathbb{Z}^d$ and the result follows. \square

Lemma 3.3. Suppose that $A_0, \dots, A_d \subset X \times \mathbb{R}$ are measurable rectangles of the form $A_i = B_i \times C_i$ for $0 \leq i \leq d$. Then

$$\begin{aligned} \mu \otimes \lambda_{\mathbb{R}} \left(\bigcap_{i=0}^{d-1} ((T_{-f})^{k_i} A_i) \cap A_d \right) \\ = \int_{\bigcap_{i=0}^{d-1} T^{k_i} B_i \cap B_d} \lambda_{\mathbb{R}} \left(\bigcap_{i=0}^{d-1} (C_i + f^{(-k_i)}(x)) \cap C_d \right) d\mu(x). \end{aligned}$$

Proof. We have $(x, r) \in \bigcap_{i=0}^{d-1} (T_{-f})^{k_i} (B_i \times C_i) \cap B_d \times C_d$ if and only if

$$(x, r) = (T^{k_i} y_i, r_i - f^{(k_i)}(y_i)) \quad \text{with} \quad (y_i, r_i) \in B_i \times C_i$$

for $0 \leq i \leq d-1$ and $(x, r) \in B_d \times C_d$. Thus

$$(x, r) \in \bigcap_{i=0}^{d-1} (T_{-f})^{k_i} (B_i \times C_i) \cap B_d \times C_d$$

if and only if

$$x \in \bigcap_{i=0}^{d-1} T^{k_i} B_i \cap B_d \quad \text{and} \quad r \in \bigcap_{i=0}^{d-1} (C_i - f^{(k_i)}(T^{-k_i} x)) \cap C_d.$$

¹²This is the abbreviation of *right hand side*; similarly we use LHS.

Since $f^{(m)}(T^{-m}x) = -f^{(-m)}(x)$ for any $m \in \mathbb{Z}$, the result follows. \square

As an immediate consequence of (the second part of) Lemma 3.2 and Lemma 3.3 we obtain the following result.

Remark 3.4. For fixed $t_0, \dots, t_{d-1} \in \mathbb{R}$ and $k_{i_0} \in \mathbb{Z}$ with $0 \leq i_0 \leq d-1$ and for all measurable sets $A_i \subset X^f$ of the form $A_i = B_i \times C_i$ where $0 \leq i \leq d$ we have¹³

$$\begin{aligned} \sum_{k_j \in \mathbb{Z}, j \neq i_0} \mu \otimes \lambda_{\mathbb{R}} \left(\bigcap_{i=0}^{d-1} (T_{-f})^{k_i} \sigma_{t_i} A_i \cap A_d \right) &\leq \mu \otimes \lambda_{\mathbb{R}} \left((T_{-f})^{k_{i_0}} \sigma_{t_{i_0}} A_{i_0} \cap A_d \right) \\ &= \int_{T^{k_{i_0}} B_{i_0} \cap B_d} \lambda_{\mathbb{R}} \left((C_{i_0} + t_{i_0} + f^{(-k_{i_0})}(x)) \cap C_d \right) d\mu(x) \\ &\leq \int_X \lambda_{\mathbb{R}} \left((C_{i_0} + t_{i_0} + f^{(-k_{i_0})}(x)) \cap C_d \right) d\mu(x). \end{aligned}$$

Suppose that $f \in L^2(X, \mu)$ and $(q_n)_{n \in \mathbb{N}}$ is a sequence of integer numbers such that the sequence $(f_0^{(q_n)})_{n \in \mathbb{N}}$ is bounded in $L^2(X, \mu)$, where $f_0 := f - \int f d\mu$.

Lemma 3.5 (Lemma 4.4 in [8]). *For every pair of bounded sets $D, E \subset \mathbb{R}$ there exists a sequence $(a_k)_{k \in \mathbb{Z}}$ of positive numbers such that*

- $\sum_{k \in \mathbb{Z}} a_k < +\infty$,
- $\int_X \lambda_{\mathbb{R}} \left((D - f_0^{(q_n)}(x) + f^{(k)}(x)) \cap E \right) d\mu \leq a_k$ for each $n \in \mathbb{N}$ and $k \in \mathbb{Z}$.

Remark 3.6. For any $l_1, l_2 \in \mathbb{Z}$ we have

$$f^{(l_1+l_2)}(x) - l_1 = f^{(l_1)}(x) - l_1 + f^{(l_2)}(T^{l_1}x) = f_0^{(l_1)}(x) + f^{(l_2)}(T^{l_1}x).$$

Proposition 3.7. *Suppose that $f \in L^2(X, \mu)$ is a positive function with $\int_X f d\mu = 1$ and there exists $c > 0$ such that $f^{(k)} \geq ck$ for a.a. $x \in X$ and for all $k \in \mathbb{N}$ large enough. Let $(q_n^i)_{n \geq 1}$ be rigidity sequences for T for $0 \leq i \leq d-1$. Moreover, suppose that the sequences $(f_0^{(q_n^i)})_{n \geq 1}$ are bounded in $L^2(X, \mu)$ for $0 \leq i \leq d-1$ and*

$$(3.6) \quad \left(f_0^{(q_n^0)}, \dots, f_0^{(q_n^{d-1})} \right)_* (\mu) \rightarrow P \text{ weakly in } \mathcal{P}(\mathbb{R}^d).$$

Then

$$(3.7) \quad (\mu^f)_{T_{q_n^0}^f, \dots, T_{q_n^{d-1}}^f} \rightarrow \int_{\mathbb{R}} (\mu^f)_{T_{-t_0}^f, \dots, T_{-t_{d-1}}^f} dP(t_0, \dots, t_{d-1}).$$

Remark 3.8. Before we pass to the proof let us see the assertion of the proposition in case of the suspension flow, i.e. $f = 1$, that is, $f_0 = 0$. In this case P is the Dirac measure at zero of \mathbb{R}^d , so in (3.7) we have a convergence to the diagonal $(d+1)$ -self-joining Δ_{d+1} . This can be seen directly in view of (2.6); indeed, all sequences (q_n^i) are rigidity sequences for the suspension flow and hence yield convergence of the LHS in (3.7) to Δ_{d+1} . It follows that Proposition 3.7 provides a class of (measurable) change of times of the suspension flow, so that the LHS in (3.7) weakly converges to the integral of off-diagonal $(d+1)$ -self-joinings given by the limit distribution in (3.6).

If T is rigid and reversible, then so is its suspension. We will see later that the changes of time described in Proposition 3.7 may lead to non-reversible flows.

¹³Here and in what follows $\sum_{k_j \in \mathbb{Z}, j \neq i_0}$ means $\sum_{k_0, \dots, k_{i_0-1}, k_{i_0+1}, \dots, k_{d-1}}$.

Proof. First notice that all we need to show is that (3.7) holds for all measurable rectangles $A_i \subset X^f$ of the form $A_i = B_i \times C_i$ ($0 \leq i \leq d$) such that C_i are bounded for $0 \leq i \leq d$.

Setting

$$a_{k_0, \dots, k_{d-1}}^n := \mu \otimes \lambda_{\mathbb{R}} \left(\bigcap_{i=0}^{d-1} \left((T_{-f})^{-k_i} (T_{-f})^{-q_n^i} \sigma_{-q_n^i} A_i \right) \cap A_d \right)$$

for $n \in \mathbb{N}$, $k_0, \dots, k_{d-1} \in \mathbb{Z}$, by Lemma 3.2, we have

$$(3.8) \quad \mu^f \left(\bigcap_{i=0}^{d-1} (T^f)_{-q_n^i} A_i \cap A_d \right) = \sum_{k_0, \dots, k_{d-1} \in \mathbb{Z}} a_{k_0, \dots, k_{d-1}}^n.$$

Since $\sigma_{-q_n^i}(A_i) = B_i \times (C_i - q_n^i)$, using Lemma 3.3 and Remark 3.6, we obtain

$$(3.9) \quad \begin{aligned} a_{k_0, \dots, k_{d-1}}^n &= \int_{\bigcap_{i=0}^{d-1} T^{-k_i - q_n^i} B_i \cap B_d} \lambda_{\mathbb{R}} \left(\bigcap_{i=0}^{d-1} (C_i - q_n^i + f^{(k_i + q_n^i)}(x)) \cap C_d \right) d\mu(x) \\ &= \int_{\bigcap_{i=0}^{d-1} T^{-k_i - q_n^i} B_i \cap B_d} \lambda_{\mathbb{R}} \left(\bigcap_{i=0}^{d-1} (C_i + f_0^{(q_n^i)}(x) + f^{(k_i)}(T^{q_n^i} x)) \cap C_d \right) d\mu(x). \end{aligned}$$

Using again Remark 3.6, for all $n \in \mathbb{N}$, $k_0, \dots, k_{d-1} \in \mathbb{Z}$ we have

$$(3.10) \quad \begin{aligned} b_{k_0, \dots, k_{d-1}}^n &:= \int_{\bigcap_{i=0}^{d-1} T^{-k_i} B_i \cap B_d} \lambda_{\mathbb{R}} \left(\bigcap_{i=0}^{d-1} (C_i - q_n^i + f^{(k_i + q_n^i)}(x)) \cap C_d \right) d\mu(x) \\ &= \int_{\bigcap_{i=0}^{d-1} T^{-k_i} B_i \cap B_d} \lambda_{\mathbb{R}} \left(\bigcap_{i=0}^{d-1} (C_i + f_0^{(q_n^i)}(x) + f^{(k_i)}(T^{q_n^i} x)) \cap C_d \right) d\mu(x). \end{aligned}$$

We claim that

$$(3.11) \quad \lim_{n \rightarrow \infty} \left(a_{k_0, \dots, k_{d-1}}^n - b_{k_0, \dots, k_{d-1}}^n \right) = 0 \quad \text{for all } k_0, \dots, k_{d-1} \in \mathbb{Z}.$$

Notice that in formulas (3.9) and (3.10) describing $a_{k_0, \dots, k_{d-1}}^n$ and $b_{k_0, \dots, k_{d-1}}^n$ respectively we have

$$\psi_n(x) := \lambda_{\mathbb{R}} \left(\bigcap_{i=0}^{d-1} (C_i - q_n^i + f^{(k_i + q_n^i)}(x)) \cap C_d \right) \leq \lambda_{\mathbb{R}}(C_d).$$

Therefore,

$$\begin{aligned} |a_{k_0, \dots, k_{d-1}}^n - b_{k_0, \dots, k_{d-1}}^n| &= \left| \int_{\bigcap_{i=0}^{d-1} T^{-k_i - q_n^i} B_i \cap B_d} \psi_n d\mu - \int_{\bigcap_{i=0}^{d-1} T^{-k_i} B_i \cap B_d} \psi_n d\mu \right| \\ &\leq \lambda_{\mathbb{R}}(C_d) \mu \left(\left(\bigcap_{i=0}^{d-1} T^{-k_i - q_n^i} B_i \cap B_d \right) \Delta \left(\bigcap_{i=0}^{d-1} T^{-k_i} B_i \cap B_d \right) \right) \\ &\leq \lambda_{\mathbb{R}}(C_d) \sum_{i=0}^{d-1} \mu(T^{q_n^i} B_i \Delta B_i). \end{aligned}$$

and $(q_n^i)_{n \geq 1}$ for $0 \leq i \leq d-1$ are rigidity sequences for T , this gives (3.11).

Let $\varepsilon > 0$ and fix $0 \leq i_0 \leq d-1$ and $k_{i_0} \in \mathbb{Z}$. By Remark 3.4 and Remark 3.6, for any $n \in \mathbb{N}$ we have

$$\begin{aligned} \sum_{k_j \in \mathbb{Z}, j \neq i_0} a_{k_0, \dots, k_{d-1}}^n &\leq \int_X \lambda_{\mathbb{R}} \left((C_{i_0} - q_n^{i_0} + f^{(k_{i_0} + q_n^{i_0})}(x)) \cap C_d \right) d\mu(x) \\ &= \int_X \lambda_{\mathbb{R}} \left((C_{i_0} + f_0^{(q_n^{i_0})}(x) + f^{(k_{i_0})}(T^{q_n^{i_0}} x)) \cap C_d \right) d\mu(x) \\ &= \int_X \lambda_{\mathbb{R}} \left((C_{i_0} - f_0^{(-q_n^{i_0})}(x) + f^{(k_{i_0})}(x)) \cap C_d \right) d\mu(x). \end{aligned}$$

Therefore, by Lemma 3.5, there exists $M > 0$ such that for any $0 \leq i_0 \leq d-1$ and $n \in \mathbb{N}$

$$\sum_{|k_{i_0}| > M} \sum_{k_j \in \mathbb{Z}, j \neq i_0} a_{k_0, \dots, k_{d-1}}^n < \frac{\varepsilon}{4d}.$$

It follows that

$$(3.12) \quad \sum_{\max(|k_0|, \dots, |k_{d-1}|) > M} a_{k_0, \dots, k_{d-1}}^n \leq \sum_{0 \leq i_0 \leq d-1} \sum_{|k_{i_0}| > M} \sum_{k_j \in \mathbb{Z}, j \neq i_0} a_{k_0, \dots, k_{d-1}}^n \leq \varepsilon/4.$$

Let us consider $F_n : X \rightarrow \mathbb{R}^d$, $F_n(x) = (F_n^0(x), \dots, F_n^{d-1}(x))$ with

$$F_n^i(x) = f_0^{(q_n^i)}(x) + f^{(k_i)}(T^{q_n^i} x) - f^{(k_i)}(x) \quad \text{for } i = 0, \dots, d-1$$

and (k_0, \dots, k_{d-1}) fixed. Since $(q_n^i)_{n \geq 1}$ is a rigidity sequence for T , $f^{(k_i)} \circ T^{q_n^i} - f^{(k_i)}$ tends to zero in measure when $n \rightarrow \infty$ for every $i = 0, \dots, d-1$. Therefore, (3.6) implies $(F_n)_* \mu \rightarrow P$ weakly in $\mathcal{P}(\mathbb{R}^d)$. Moreover,

$$F_n^i \circ T - F_n^i = (f \circ T^{k_i}) \circ T^{q_n^i} - (f \circ T^{k_i}) \quad \text{for } i = 0, \dots, d-1,$$

so $F_n \circ T - F_n \rightarrow 0$ in measure. Now using Proposition 3.1 with $G = \mathbb{R}^d$ and (3.10) we obtain

$$\begin{aligned} (3.13) \quad &b_{k_0, \dots, k_{d-1}}^n \\ &= \int_{\bigcap_{i=0}^{d-1} T^{-k_i} B_i \cap B_d} \lambda_{\mathbb{R}} \left(\bigcap_{i=0}^{d-1} (C_i + f_0^{(q_n^i)}(x) + f^{(k_i)}(T^{q_n^i} x)) \cap C_d \right) d\mu(x) \\ &= \int_{\bigcap_{i=0}^{d-1} T^{-k_i} B_i \cap B_d} \lambda_{\mathbb{R}} \left(\bigcap_{i=0}^{d-1} (C_i + F_n^i(x) + f^{(k_i)}(x)) \cap C_d \right) d\mu(x) \\ &\rightarrow \int_{\bigcap_{i=0}^{d-1} T^{-k_i} B_i \cap B_d} \int_{\mathbb{R}^d} \lambda_{\mathbb{R}} \left(\bigcap_{i=0}^{d-1} (C_i + t_i + f^{(k_i)}(x)) \cap C_d \right) dP(t_0, \dots, t_{d-1}) d\mu(x) \\ &=: c_{k_0, \dots, k_{d-1}} \end{aligned}$$

for each $k_0, \dots, k_{d-1} \in \mathbb{Z}$. By Fubini's theorem and Lemma 3.3 we have

$$\begin{aligned}
(3.14) \quad & c_{k_0, \dots, k_{d-1}} \\
&= \int_{\mathbb{R}^d} \int_{\bigcap_{i=0}^{d-1} T^{k_i} B_i \cap B_d} \lambda_{\mathbb{R}} \left(\bigcap_{i=0}^{d-1} (C_i + t_i + f^{(k_i)}(x)) \cap C_d \right) d\mu(x) dP(t_0, \dots, t_{d-1}) \\
&= \int_{\mathbb{R}^d} \mu \otimes \lambda_{\mathbb{R}} \left(\bigcap_{i=0}^{d-1} (T_{-f})^{-k_i} \sigma_{t_i} (B_i \times C_i) \cap (B_d \times C_d) \right) dP(t_0, \dots, t_{d-1}) \\
&= \int_{\mathbb{R}^d} \mu \otimes \lambda_{\mathbb{R}} \left(\bigcap_{i=0}^{d-1} (T_{-f})^{-k_i} \sigma_{t_i} A_i \cap A_d \right) dP(t_0, \dots, t_{d-1}).
\end{aligned}$$

Moreover, by Lemma 3.2,

$$\begin{aligned}
(3.15) \quad & \sum_{k_0, \dots, k_{d-1} \in \mathbb{Z}} c_{k_0, \dots, k_{d-1}} \\
&= \sum_{k_0, \dots, k_{d-1} \in \mathbb{Z}} \int_{\mathbb{R}^d} \mu \otimes \lambda_{\mathbb{R}} \left(\bigcap_{i=0}^{d-1} (T_{-f})^{-k_i} \sigma_{t_i} A_i \cap A_d \right) dP(t_0, \dots, t_{d-1}) \\
&= \int_{\mathbb{R}^d} \mu^f \left(\bigcap_{i=0}^{d-1} (T^f)_{t_i} A_i \cap A_d \right) dP(t_0, \dots, t_{d-1}).
\end{aligned}$$

Increasing M , if necessary, we can assume that

$$(3.16) \quad \sum_{\max(|k_0|, \dots, |k_{d-1}|) > M} c_{k_0, \dots, k_{d-1}} \leq \varepsilon/4.$$

Combining (3.11) with (3.13) we get

$$a_{k_0, \dots, k_{d-1}}^n \rightarrow c_{k_0, \dots, k_{d-1}} \quad \text{for all } k_0, \dots, k_{d-1} \in \mathbb{Z}.$$

Therefore, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and $k_0, \dots, k_{d-1} \in \mathbb{Z}$ with $\max(|k_0|, \dots, |k_{d-1}|) \leq M$

$$|a_{k_0, \dots, k_{d-1}}^n - c_{k_0, \dots, k_{d-1}}| < \frac{\varepsilon}{2(2M+1)^d}.$$

In view of (3.12) and (3.16), it follows that

$$\begin{aligned}
& \left| \sum_{k_0, \dots, k_{d-1} \in \mathbb{Z}} a_{k_0, \dots, k_{d-1}}^n - \sum_{k_0, \dots, k_{d-1} \in \mathbb{Z}} c_{k_0, \dots, k_{d-1}} \right| \\
& \leq \sum_{\max(|k_0|, \dots, |k_{d-1}|) > M} a_{k_0, \dots, k_{d-1}}^n + \sum_{\max(|k_0|, \dots, |k_{d-1}|) > M} c_{k_0, \dots, k_{d-1}} \\
& \quad + \sum_{\max(|k_0|, \dots, |k_{d-1}|) \leq M} |a_{k_0, \dots, k_{d-1}}^n - c_{k_0, \dots, k_{d-1}}| < \varepsilon.
\end{aligned}$$

By (3.8) and (3.15), this completes the proof. \square

3.3. FS-type joinings and non-isomorphism of a flow with its inverse.

From now on we assume that all flows under consideration are ergodic and aperiodic.

For any $\bar{\varepsilon} = (\varepsilon_0, \dots, \varepsilon_{d-1}) \in \{0, 1\}^d := \{0, 1\}^d \setminus \{(0, \dots, 0)\}$ and for any vector $\bar{x} = (x_0, \dots, x_{d-1}) \in \mathbb{R}^d$ let

$$\bar{x}(\varepsilon) = \varepsilon_0 x_0 + \varepsilon_1 x_1 + \dots + \varepsilon_{d-1} x_{d-1}.$$

If we look at the assumptions of Proposition 3.7 we see that for any choice of $\bar{\varepsilon} = (\varepsilon_0, \dots, \varepsilon_{d-1}) \in \{0, 1\}'^d$ setting $\bar{q}_n := (q_n^0, q_n^1, \dots, q_n^{d-1})$ we have

$$(\bar{q}_n(\bar{\varepsilon}))_{n \geq 1} \text{ is a rigidity sequence for } T \text{ and } \left(f_0^{(\bar{q}_n(\bar{\varepsilon}))} \right)_{n \geq 1} \text{ is bounded in } L^2.$$

we can assume that

$$\left(\left(f_0^{(\bar{q}_n(\bar{\varepsilon}))} \right)_{\bar{\varepsilon} \in \{0, 1\}'^d} \right)_* \rightarrow Q \in \mathcal{P}(\mathbb{R}^{\{0, 1\}'^d}) \quad \text{when } n \rightarrow \infty.$$

For any $\bar{t} \in \mathbb{R}^{\{0, 1\}'^d}$ denote by $\mu_{\bar{t}}^f \in J_{2^d}(T^f)$ the off-diagonal 2^d -self-joining defined the family of elements of the centralizer $\{T_{\bar{t}\bar{\varepsilon}}^f : \bar{\varepsilon} \in \{0, 1\}'^d\}$, this is

$$\mu_{\bar{t}}^f \left(\prod_{\bar{\varepsilon} \in \{0, 1\}'^d} A_{\bar{\varepsilon}} \right) = \mu^f \left(\bigcap_{\bar{\varepsilon} \in \{0, 1\}'^d} T_{-\bar{t}\bar{\varepsilon}}^f A_{\bar{\varepsilon}} \right),$$

we make the convention that $\bar{t}_{(0, \dots, 0)} = 0$ for any $\bar{t} \in \mathbb{R}^{\{0, 1\}'^d}$. Hence, in view of Proposition 3.7

$$(3.17) \quad \mu_{(\bar{q}_n(\bar{\varepsilon}))_{\bar{\varepsilon} \in \{0, 1\}'^d}}^f \rightarrow \int_{\mathbb{R}^{\{0, 1\}'^d}} \mu_{-\bar{t}}^f dQ(\bar{t}).$$

Recall that given $\bar{a} = (a_0, \dots, a_{d-1}) \in \mathbb{R}^d$, by the finite sum set $FS(\bar{a})$ of \bar{a} we mean

$$\begin{aligned} FS(\bar{a}) &= \{a_0, a_1, \dots, a_{d-1}, a_0 + a_1, a_0 + a_2, \dots, a_0 + a_1 + \dots + a_{d-1}\} \\ &= \{\bar{a}(\bar{\varepsilon}) : \bar{\varepsilon} \in \{0, 1\}'^d\}. \end{aligned}$$

The off-diagonal joinings on the LHS of (3.17) have certain symmetry property (explored below) which, when assuming isomorphism of the flow with its inverse, should result in a certain symmetry property of the limit measure Q . Hence, if the expected symmetry of Q does not take place we obtain that the flow is not isomorphic to its inverse. We now pass to a precise description of the symmetry of Q in a more general situation.

Assume that $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ is an ergodic and aperiodic flow on (X, \mathcal{B}, μ) . Suppose that there exists a sequence $(\bar{q}_n)_{n \geq 1}$ in \mathbb{R}^d , and a probability Borel measure $Q \in \mathcal{P}(\mathbb{R}^{\{0, 1\}'^d})$ such that

$$(3.18) \quad \mu_{(\bar{q}_n(\bar{\varepsilon}))_{\bar{\varepsilon} \in \{0, 1\}'^d}} \rightarrow \int_{\mathbb{R}^{\{0, 1\}'^d}} \mu_{-\bar{t}} dQ(\bar{t}) \quad \text{in } J_{2^d}(\mathcal{T}).$$

Note that, because of the aperiodicity of \mathcal{T} , for distinct $\bar{t}, \bar{s} \in \mathbb{R}^{\{0, 1\}'^d}$ the measures $\mu_{\bar{t}}, \mu_{\bar{s}}$ are orthogonal. Therefore, the integral in (3.18) represents the ergodic decomposition of the limit measure.

We also assume that \mathcal{T} and $\mathcal{T} \circ (-1)$ are isomorphic, i.e. for some invertible $S : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$

$$(3.19) \quad S \circ T_t \circ S^{-1} = T_{-t} \quad \text{for each } t \in \mathbb{R}.$$

The map $S : X \rightarrow X$ induces a continuous (affine) invertible map $S_* : J_{2^d}(\mathcal{T}) \rightarrow J_{2^d}(\mathcal{T})$ such that

$$S_*(\rho) \left(\prod_{\bar{\varepsilon} \in \{0, 1\}'^d} A_{\bar{\varepsilon}} \right) := \rho \left(\prod_{\bar{\varepsilon} \in \{0, 1\}'^d} S^{-1} A_{\bar{\varepsilon}} \right) \quad \text{for } A_{\bar{\varepsilon}} \in \mathcal{B}, \bar{\varepsilon} \in \{0, 1\}'^d.$$

Moreover, for any $\bar{t} \in \mathbb{R}^{\{0,1\}^d}$

$$\begin{aligned} S_*(\mu_{\bar{t}}) \left(\prod_{\bar{\varepsilon} \in \{0,1\}^d} A_{\bar{\varepsilon}} \right) &= \mu_{\bar{t}} \left(\prod_{\bar{\varepsilon} \in \{0,1\}^d} S^{-1} A_{\bar{\varepsilon}} \right) = \mu \left(\bigcap_{\bar{\varepsilon} \in \{0,1\}^d} T_{-\bar{t}_{\bar{\varepsilon}}} S^{-1} A_{\bar{\varepsilon}} \right) \\ &= \mu \left(S^{-1} \bigcap_{\bar{\varepsilon} \in \{0,1\}^d} T_{\bar{t}_{\bar{\varepsilon}}} A_{\bar{\varepsilon}} \right) = \mu \left(\bigcap_{\bar{\varepsilon} \in \{0,1\}^d} T_{\bar{t}_{\bar{\varepsilon}}} A_{\bar{\varepsilon}} \right) = \mu_{-\bar{t}} \left(\prod_{\bar{\varepsilon} \in \{0,1\}^d} A_{\bar{\varepsilon}} \right) \end{aligned}$$

Thus

$$(3.20) \quad S_*(\mu_{\bar{t}}) = \mu_{-\bar{t}}.$$

By the continuity of S_*

$$S_*(\mu_{(\bar{q}_n(\bar{\varepsilon}))_{\bar{\varepsilon} \in \{0,1\}^d}}) \rightarrow S_* \left(\int_{\mathbb{R}^{\{0,1\}^d}} \mu_{-\bar{t}} dQ(\bar{t}) \right) = \int_{\mathbb{R}^{\{0,1\}^d}} S_*(\mu_{-\bar{t}}) dQ(\bar{t}).$$

In view of (3.20), it follows that

$$(3.21) \quad \mu_{(-\bar{q}_n(\bar{\varepsilon}))_{\bar{\varepsilon} \in \{0,1\}^d}} \rightarrow \int_{\mathbb{R}^{\{0,1\}^d}} \mu_{\bar{t}} dQ(\bar{t}).$$

Let us consider the involution

$$\begin{aligned} I : \{0,1\}^d &\rightarrow \{0,1\}^d, & I(\varepsilon_0, \dots, \varepsilon_{d-1}) &= (1 - \varepsilon_0, \dots, 1 - \varepsilon_{d-1}), \\ \bar{\theta} : \mathbb{R}^{\{0,1\}^d} &\rightarrow \mathbb{R}^{\{0,1\}^d}, & \bar{\theta} \left((\bar{t}_{\bar{\varepsilon}})_{\bar{\varepsilon} \in \{0,1\}^d} \right) &= \left((\bar{t}_{(1, \dots, 1)} - \bar{t}_{I(\bar{\varepsilon})})_{\bar{\varepsilon} \in \{0,1\}^d} \right). \end{aligned}$$

Thus, by (3.18)

$$\begin{aligned} \mu_{(-\bar{q}_n(\bar{\varepsilon}))_{\bar{\varepsilon} \in \{0,1\}^d}} \left(\prod_{\bar{\varepsilon} \in \{0,1\}^d} A_{\bar{\varepsilon}} \right) &= \mu \left(\bigcap_{\bar{\varepsilon} \in \{0,1\}^d} T_{\bar{q}_n(\bar{\varepsilon})} A_{\bar{\varepsilon}} \right) \\ &= \mu \left(\bigcap_{\bar{\varepsilon} \in \{0,1\}^d} T_{\bar{q}_n(\bar{\varepsilon}) - \bar{q}_n(1, \dots, 1)} A_{\bar{\varepsilon}} \right) = \mu \left(\bigcap_{\bar{\varepsilon} \in \{0,1\}^d} T_{-\bar{q}_n(I(\bar{\varepsilon}))} A_{\bar{\varepsilon}} \right) \\ &= \mu \left(\bigcap_{\bar{\varepsilon} \in \{0,1\}^d} T_{-\bar{q}_n(\bar{\varepsilon})} A_{I(\bar{\varepsilon})} \right) = \mu_{(\bar{q}_n(\bar{\varepsilon}))_{\bar{\varepsilon} \in \{0,1\}^d}} \left(\prod_{\bar{\varepsilon} \in \{0,1\}^d} A_{I(\bar{\varepsilon})} \right) \\ &\rightarrow \int_{\mathbb{R}^{\{0,1\}^d}} \mu_{-\bar{t}} \left(\prod_{\bar{\varepsilon} \in \{0,1\}^d} A_{I(\bar{\varepsilon})} \right) dQ(\bar{t}) = \int_{\mathbb{R}^{\{0,1\}^d}} \mu \left(\bigcap_{\bar{\varepsilon} \in \{0,1\}^d} T_{\bar{t}_{\bar{\varepsilon}}} A_{I(\bar{\varepsilon})} \right) dQ(\bar{t}) \\ &= \int_{\mathbb{R}^{\{0,1\}^d}} \mu \left(\bigcap_{\bar{\varepsilon} \in \{0,1\}^d} T_{\bar{t}_{I(\bar{\varepsilon})}} A_{\bar{\varepsilon}} \right) dQ(\bar{t}) \\ &= \int_{\mathbb{R}^{\{0,1\}^d}} \mu \left(\bigcap_{\bar{\varepsilon} \in \{0,1\}^d} T_{-(\bar{t}_{(1, \dots, 1)} - \bar{t}_{I(\bar{\varepsilon})})} A_{\bar{\varepsilon}} \right) dQ(\bar{t}) \\ &= \int_{\mathbb{R}^{\{0,1\}^d}} \mu_{\bar{\theta}(\bar{t})} \left(\prod_{\bar{\varepsilon} \in \{0,1\}^d} A_{\bar{\varepsilon}} \right) dQ(\bar{t}); \end{aligned}$$

in the last line we use the fact that $\bar{t}_{(1, \dots, 1)} - \bar{t}_{I(\bar{\varepsilon})} = 0$ for $\bar{\varepsilon} = (0, \dots, 0)$. In view of (3.21), it follows that

$$\int_{\mathbb{R}^{\{0,1\}^d}} \mu_{T_{\bar{t}}} dQ(\bar{t}) = \int_{\mathbb{R}^{\{0,1\}^d}} \mu_{T_{\bar{t}}} d\bar{\theta}_* Q(\bar{t}).$$

By the uniqueness of ergodic decomposition, we get $\bar{\theta}_*(Q) = Q$. In this way we have proved the following result.

Proposition 3.9. *Assume that $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ is an ergodic and aperiodic flow on (X, \mathcal{B}, μ) . Assume that \mathcal{T} satisfies (3.18). If the measure Q is not invariant under the map $\bar{\theta} : \mathbb{R}^{\{0,1\}^d} \rightarrow \mathbb{R}^{\{0,1\}^d}$ then $(T_t)_{t \in \mathbb{R}}$ is not isomorphic to its inverse. In particular, $(T_t)_{t \in \mathbb{R}}$ is not reversible.*

Remark 3.10. Suppose additionally that the flow \mathcal{T} is weakly mixing. Then each its non-trivial factor is also weakly mixing, so it is ergodic and aperiodic. For each such factor (3.18) is evidently valid. It follows that the absence of isomorphism to the inverse is inherited by non-trivial factors of \mathcal{T} .

Two particular cases follows. First, consider the case $d = 2$. Then the space $\mathbb{R}^{\{0,1\}^2}$ is identified with \mathbb{R}^3 by the map $\mathbb{R}^{\{0,1\}^2} \ni t \mapsto (t_{(1,1)}, t_{(1,0)}, t_{(0,1)}) \in \mathbb{R}^3$. The the map $\bar{\theta}$ is identified with $\theta : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $\theta(t, u, v) = (t, t - v, t - u)$.

Corollary 3.11. *Assume that $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ is an ergodic and aperiodic flow on (X, \mathcal{B}, μ) . Assume moreover that*

$$\mu_{T_{r_n+q_n}, T_{r_n}, T_{q_n}} \mapsto \int_{\mathbb{R}^3} \mu_{T_{-t}, T_{-u}, T_{-v}} dQ(t, u, v)$$

for some probability measure $Q \in \mathcal{P}(\mathbb{R}^3)$. If the measure Q is not invariant under the map $(t, u, v) \mapsto (t, t - v, t - u)$ then \mathcal{T} is not isomorphic to its inverse.

Now suppose that $\bar{q}_n = (q_n, \dots, q_n)$. Then $\bar{q}_n(\bar{\varepsilon}) = |\bar{\varepsilon}|q_n$, where $|\bar{\varepsilon}| = \varepsilon_1 + \dots + \varepsilon_d$. Let us consider the maps

$$\begin{aligned} \varrho : \mathbb{R}^d &\rightarrow \mathbb{R}^{\{0,1\}^d}, & \varrho((x_j)_{j=0}^{d-1}) &= (x_{d-|\bar{\varepsilon}|})_{\bar{\varepsilon} \in \{0,1\}^d}, \\ \theta : \mathbb{R}^d &\rightarrow \mathbb{R}^d, & \theta(t_0, t_1, \dots, t_{d-1}) &= (t_0, t_0 - t_{d-1}, \dots, t_0 - t_1). \end{aligned}$$

Then $\varrho \circ \theta = \bar{\theta} \circ \varrho$. Moreover, if

$$(3.22) \quad \mu_{T_{d q_n}, T_{(d-1)q_n}, \dots, T_{q_n}} \mapsto \int_{\mathbb{R}^d} \mu_{T_{-t_0}, T_{-t_1}, \dots, T_{-t_{d-1}}} dP(t_0, \dots, t_{d-1})$$

for some $P \in \mathcal{P}(\mathbb{R}^d)$ then (3.18) holds for a measure $Q = \varrho_*(P) \in \mathcal{P}(\mathbb{R}^{\{0,1\}^d})$. Moreover, $\bar{\theta}_*(Q) = Q$ implies $\varrho_*\theta_*(P) = \varrho_*(P)$, and hence $\theta_*(P) = P$. As a conclusion from Proposition 3.9 we obtain the following.

Corollary 3.12. *Assume that $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ is an ergodic and aperiodic flow on (X, \mathcal{B}, μ) . Assume that (3.22) is valid for a measure $P \in \mathcal{P}(\mathbb{R}^d)$. If the measure P is not invariant under the map $\theta : \mathbb{R}^d \rightarrow \mathbb{R}^d$ then \mathcal{T} is not isomorphic to its inverse.*

Finally consider $d = 2$.

Corollary 3.13. *Assume that $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ is an ergodic and aperiodic flow on (X, \mathcal{B}, μ) . Assume also that*

$$\mu_{T_{2q_n}, T_{q_n}} \mapsto \int_{\mathbb{R}^2} \mu_{T_{-t}, T_{-u}} dQ(t, u)$$

for some probability measure Q on \mathbb{R}^2 . If the measure Q is not invariant under $\theta(t, u) = (t, t - u)$ then $(T_t)_{t \in \mathbb{R}}$ is not isomorphic to its inverse. In particular, if $(0, x) \in \mathbb{R}^2$ is an atom of Q but $(0, -x)$ is not then \mathcal{T} is not isomorphic to its inverse.

In next three sections we will deal with special flows built over irrational rotations on the circle. Such flows are always ergodic and aperiodic (see Remark 2.1), so we can apply the results of this section for proving the absence of isomorphism with their inverses.

4. SPECIAL FLOWS OVER IRRATIONAL ROTATIONS NON-ISOMORPHIC TO THEIR INVERSES

In this section we will discuss non-reversibility property for special flows built over irrational rotations on the circle and under piecewise absolutely continuous

roof functions. For a real number t denote by $\{t\}$ its fractional part and by $\|t\|$ its distance to the nearest integer number.

We call a function $f : \mathbb{T} \rightarrow \mathbb{R}$ *piecewise absolutely continuous* if there exist $\beta_1, \dots, \beta_K \in \mathbb{T}$ such that $f|_{(\beta_j, \beta_{j+1})}$ is an absolutely continuous function for $j = 1, \dots, K$ ($\beta_{K+1} = \beta_1$). Let $d_j := f_-(\beta_j) - f_+(\beta_j)$, where $f_{\pm}(\beta) = \lim_{y \rightarrow \beta_{\pm}} f(y)$. Then the number

$$(4.1) \quad S(f) := \sum_{j=1}^K d_j = \int_{\mathbb{T}} f'(x) dx$$

is the *sum of jumps* of f . Without loss of generality we can restrict ourselves to functions continuous on the right. Each such function can be represented as $f = f_{pl} + f_{ac}$, where $f_{ac} : \mathbb{T} \rightarrow \mathbb{R}$ is an absolutely continuous function with zero mean and

$$f_{pl}(x) = \sum_{i=1}^K d_i \{x - \beta_i\} + d.$$

In this section we will prove non-reversibility for special flows T^f built over almost every irrational rotation $Tx = x + \alpha$ and under roof functions f with $S(f) \neq 0$. Such flows are called von Neumann flows.

We need some auxiliary simple lemmas.

Lemma 4.1. *Let (X_n) be a sequence of random variables (each one defined on a probability space $(\Omega, \mathcal{F}, \mu)$) with values on \mathbb{R}^d . Assume that for $n \geq 1$ we have a partition $\{A_k^n : k = 1, \dots, K\}$ of Ω such that $\mu(A_k^n) \rightarrow \delta_k$ when $n \rightarrow \infty$ for each $k = 1, \dots, K$. Assume moreover that for each $k = 1, \dots, K$*

$$(X_n)_*(\mu_{A_k^n}) \rightarrow P_k \text{ when } n \rightarrow \infty$$

weakly in the space of probability measures on \mathbb{R}^d (μ_C stands for the relevant conditional measure: $\mu_C(A) := \mu(A \cap C)/\mu(C)$). Then

$$(X_n)_*(\mu) \rightarrow \sum_{k=1}^K \delta_k P_k.$$

Proof. Assume that $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is continuous and bounded. Then

$$\begin{aligned} \int_{\mathbb{R}^d} \phi(t) d((X_n)_*(\mu))(t) &= \int_{\Omega} \phi(X_n) d\mu = \sum_{k=1}^K \mu(A_k^n) \int_{\Omega} \phi(X_n) d\mu_{A_k^n} \\ &\rightarrow \sum_{k=1}^K \delta_k \int_{\mathbb{R}^d} \phi(t) dP_k(t). \end{aligned}$$

□

Lemma 4.2. *Let (X_n) and (C_n) be sequences of random variables (each one defined on a probability space $(\Omega_n, \mathcal{F}_n, \mu_n)$) with values on \mathbb{R}^d . Assume that $(X_n)_*(\mu_n) \rightarrow P$ and C_n tends uniformly to the constant function $c \in \mathbb{R}^d$. Then*

$$(X_n + C_n)_*(\mu_n) \rightarrow (T_c)_*(P),$$

where $T_c(x) = x + c$ in \mathbb{R}^d .

Proof. Fix $s \in \mathbb{R}^d$. By assumption

$$\int_{\Omega} e^{2\pi i s \cdot X_n} d\mu_n \rightarrow \int_{\mathbb{R}^d} e^{2\pi i s \cdot t} dP(t).$$

Moreover,

$$\begin{aligned} & \left| \int_{\Omega} e^{2\pi i s \cdot (X_n + C_n)} d\mu_n - \int_{\mathbb{R}^d} e^{2\pi i s \cdot (t+c)} dP(t) \right| \\ &= \left| \int_{\Omega} e^{2\pi i s \cdot (X_n + C_n - c)} d\mu_n - \int_{\mathbb{R}^d} e^{2\pi i s \cdot t} dP(t) \right| \\ &\leq \left| \int_{\Omega} e^{2\pi i s \cdot X_n} d\mu_n - \int_{\mathbb{R}^d} e^{2\pi i s \cdot t} dP(t) \right| + 2\pi |s| \int_{\Omega} |C_n - c| dP(t). \end{aligned}$$

It follows that

$$\int_{\Omega} e^{2\pi i s \cdot (X_n + C_n)} d\mu_n \rightarrow \int_{\mathbb{R}^d} e^{2\pi i s \cdot (t+c)} dP(t) = \int_{\mathbb{R}^d} e^{2\pi i s \cdot t} d((T_c)_*(P))(t),$$

which completes the proof. \square

The following lemma holds.

Lemma 4.3. *Let (X_n) be a sequence of random variables (each one defined on a probability space $(\Omega_n, \mathcal{F}_n, \mu_n)$) with values on \mathbb{R}^d such that $(X_n)_*(\mu_n) \rightarrow P$. Assume that $A : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is continuous. Then*

$$(A(X_n))_*(\mu_n) \rightarrow A_*(P).$$

Remark 4.4. Directly from the definition it follows that $\{x + y\} = \{x + \{y\}\}$ for each $x, y \in \mathbb{R}$. Moreover, whenever $a, b \in \mathbb{T} = [0, 1)$, we have

$$\{x + a - b\} - \{x - b\} = a - \mathbf{1}_{\{[b-a], b\}}(x)$$

for $x \in \mathbb{T}$, where $\{[b-a], b\}$ is understood as an interval on the circle (if $d > e$ then $[d, e] = [d, 1) \cup [0, e)$). Indeed, $\{x + a - b\} - \{x - b\} = \{a + \{x - b\}\} - \{x - b\}$ and for $0 \leq t < 1$ we have $\{a + t\} - t = a$ if $0 \leq t < 1 - a$ and $a - 1$ for $1 - a \leq t < 1$.

For any irrational number $\alpha = [0; a_1, a_2, \dots] \in \mathbb{T}$ denote by $(p_n/q_n)_{n \geq 0}$ the sequence of convergents in continued fraction expansion of α (see e.g. [21] for basic properties of continued fraction expansion of α).

Lemma 4.5. *The set $\Lambda \subset [0, 1)$ of those α irrational for which for each $\varepsilon > 0$ there exists $0 < \delta < \varepsilon$ such that*

$$q_{n_k} \|q_{n_k} \alpha\| \rightarrow \delta$$

along a subsequence $n_k = n_k(\varepsilon)$ is of full Lebesgue measure.

Proof. We have

$$\frac{1}{2} \frac{1}{a_{n+1} + 1} < q_n \|q_n \alpha\| < \frac{1}{a_{n+1}}.$$

The result follows directly from the ergodicity of the Gauss map $G : [0, 1) \rightarrow [0, 1)$ (see e.g. [7]). \square

Assume that $f(x) = \sum_{i=1}^K d_i \{x - \beta_i\} + d$. Let $Tx = x + \alpha$ and suppose that $\{q_n \alpha\} = \|q_n \alpha\|$. The case where $\{q_n \alpha\} = 1 - \|q_n \alpha\|$ can be treated in a similar way. We have

$$f_0^{(q_n)}(T^{q_n} x) - f_0^{(q_n)}(x) = f^{(q_n)}(T^{q_n} x) - f^{(q_n)}(x) = \sum_{j=0}^{q_n-1} (f \circ T^{q_n} - f)(T^j x).$$

Moreover, in view of Remark 4.4,

$$\begin{aligned} f(T^{q_n} y) - f(y) &= \sum_{i=1}^K d_i (\{y + q_n \alpha - \beta_i\} - \{y - \beta_i\}) \\ &= \sum_{i=1}^K d_i (\{y + \|q_n \alpha\| - \beta_i\} - \{y - \beta_i\}) = \sum_{i=1}^K d_i (\|q_n \alpha\| - \mathbf{1}_{[\beta_i - \|q_n \alpha\|, \beta_i)}(y)). \end{aligned}$$

Thus

$$(4.2) \quad f^{(q_n)}(T^{q_n}x) - f^{(q_n)}(x) = q_n \|q_n \alpha\| \left(\sum_{i=1}^K d_i \right) - \sum_{i=1}^K d_i \sum_{j=0}^{q_n-1} \mathbf{1}_{[\beta_i - \|q_n \alpha\|, \beta_i)}(x + j\alpha).$$

Moreover, since $[s, s + \|q_n \alpha\|)$ is the base of a Rokhlin tower of height q_{n+1} , we have

$$(4.3) \quad \sum_{j=0}^{q_n-1} \mathbf{1}_{[\beta_i - \|q_n \alpha\|, \beta_i)}(x + j\alpha) = 0 \quad \text{or} \quad 1.$$

Given $n \geq 1$ and $\epsilon \in \{0, 1\}^K$, taking into account (4.3), set

$$(4.4) \quad A_\epsilon^n = \left\{ x \in \mathbb{T} : \sum_{j=0}^{q_n-1} \mathbf{1}_{[\beta_i - \|q_n \alpha\|, \beta_i)}(x + j\alpha) = \epsilon_i \text{ for } i = 1, \dots, K \right\}.$$

Then, in view of (4.2), for $x \in A_\epsilon^n$ we have

$$(4.5) \quad f^{(q_n)}(T^{q_n}x) - f^{(q_n)}(x) = q_n \|q_n \alpha\| S(f) - C_\epsilon,$$

where $C_\epsilon = \sum_{i=1}^K d_i \epsilon_i$ (note that $C_{\underline{0}} = C_{(0, \dots, 0)} = 0$).

Suppose that the roof function $f : \mathbb{T} \rightarrow \mathbb{R}$ is a piecewise absolutely continuous function and let us decompose $f = f_{pl} + f_{ac}$. Suppose that $q_n \|q_n \alpha\| \rightarrow \delta > 0$ and $\mu(A_\epsilon^n) \rightarrow p_\epsilon$ for $\epsilon \in \{0, 1\}^K$ (sets A_ϵ^n are defined accordingly to the function f_{pl}). By Koksma-Denjoy inequality (see e.g. [24]), $\|(f_{pl})_0^{(q_n)}\|_{\text{sup}} \leq \text{Var } f$, thus the sequence $((f_{pl})_0^{(q_n)})_*$ of distributions is uniformly tight. By passing to a further subsequence, if necessary, we can also assume that

$$(4.6) \quad ((f_{pl})_0^{(q_n)})_* (\mu_{A_\epsilon^n}) \rightarrow P_\epsilon \text{ when } n \rightarrow \infty.$$

Recall that (see e.g. [15])

$$(4.7) \quad \|f_{ac}^{(q_n)}\|_{\text{sup}} \rightarrow 0.$$

Set $X_n = (f_0^{(2q_n)}, f_0^{(q_n)}) : \mathbb{T} \rightarrow \mathbb{R}^2$. Then

$$(f_0^{(2q_n)}, f_0^{(q_n)}) = \left(2(f_{pl})_0^{(q_n)} + f_{pl}^{(q_n)} \circ T^{q_n} - f_{pl}^{(q_n)}, (f_{pl})_0^{(q_n)} \right) + \left(f_{ac}^{(2q_n)}, f_{ac}^{(q_n)} \right).$$

In view of (4.5), for $x \in A_\epsilon^n$

$$(4.8) \quad X_n = \left(2(f_{pl})_0^{(q_n)}, (f_{pl})_0^{(q_n)} \right) + (q_n \|q_n \alpha\| S(f) - C_\epsilon, 0) + \left(f_{ac}^{(2q_n)}, f_{ac}^{(q_n)} \right).$$

Let $A : \mathbb{R} \rightarrow \mathbb{R}^2$, $Ax = (2x, x)$. Thus

$$Y_n := \left(2(f_{pl})_0^{(q_n)}, (f_{pl})_0^{(q_n)} \right) = A \circ f_0^{(q_n)},$$

so by Lemma 4.3 and (4.6),

$$(4.9) \quad (Y_n)_* (\mu_{A_\epsilon^n}) \rightarrow A_* (P_\epsilon).$$

Since $X_n = Y_n + (q_n \|q_n \alpha\| S(f) - C_\epsilon, 0) + (f_{ac}^{(2q_n)}, f_{ac}^{(q_n)})$ on A_ϵ^n , $(f_{ac}^{(2q_n)}, f_{ac}^{(q_n)})$ uniformly tends to zero (see (4.7)) and $q_n \|q_n \alpha\| \rightarrow \delta$, in view of Lemma 4.2 ¹⁴,

$$(4.10) \quad (X_n)_* (\mu_{A_\epsilon^n}) \rightarrow (T_{(\delta S(f) - C_\epsilon, 0)})_* A_* (P_\epsilon).$$

Therefore, by Lemma 4.1,

$$(4.11) \quad (X_n)_* (\mu) \rightarrow \sum_{\epsilon \in \{0, 1\}^K} p_\epsilon (T_{(\delta S(f) - C_\epsilon, 0)})_* A_* (P_\epsilon) =: P.$$

¹⁴We apply the lemma for $\mu_n = \mu_{A_\epsilon^n}$, $X_n = Y_n$ and $C_n = (q_n \|q_n \alpha\| S(f) - C_\epsilon, 0) + (f_{ac}^{(2q_n)}, f_{ac}^{(q_n)})$.

On the other hand (see Proposition 3.7), $\lim_{n \rightarrow \infty} (X_n)_* (\mu) = P$, so

$$(4.12) \quad \sum_{\epsilon \in \{0,1\}^K} p_\epsilon (T_{(\delta S(f) - C_\epsilon, 0)})_* A_*(P_\epsilon) = P.$$

Theorem 4.6. *If $\alpha \in \Lambda$ (see Lemma 4.5) and $S(f) \neq 0$ then the special flow T^f is not isomorphic to its inverse.*

Proof. Take $\delta > 0$ so that $K\delta < 1$, $q_n \|q_n \alpha\| \rightarrow \delta$ (by passing to a subsequence, if necessary) and

$$\delta < \min\{|C_\epsilon| > 0, \epsilon \in \{0,1\}^K\} / (2|S(f)|).$$

Suppose now that the special flow T^f is reversible. By Proposition 3.7 and Corollary 3.13, $\theta_* P = P$, where $\theta(t, u) = (t, t - u)$. Using (4.12), since

$$\theta \circ T_{(c,0)} \circ A = T_{(-c,0)} \circ A \circ T_c,$$

we have

$$\theta_* P = \sum_{\epsilon \in \{0,1\}^K} p_\epsilon (T_{(-\delta S(f) + C_\epsilon, 0)})_* A_* (T_{\delta S(f) - C_\epsilon})_* (P_\epsilon).$$

Each measure of the form $(T_{(c,0)})_* A_* P'$ (with P' a probability on \mathbb{R}) is concentrated on the set

$$R_c := \{(2x + c, x) : x \in \mathbb{R}\}.$$

Clearly, $R_c \cap R_{c'} = \emptyset$ for $c \neq c'$. If for some $\epsilon \in \{0,1\}^K$, $p_\epsilon > 0$ and $\delta S(f) - C_\epsilon \neq 0$, since $\theta_* P = P$, there must exist $\epsilon' \in \{0,1\}^K$ such that

$$p_{\epsilon'} > 0 \quad \text{and} \quad -\delta S(f) + C_\epsilon = \delta S(f) - C_{\epsilon'},$$

whence

$$(4.13) \quad C_\epsilon + C_{\epsilon'} = 2\delta S(f).$$

Then

$$\begin{aligned} A_0^n &= \{x \in \mathbb{T} : \sum_{j=0}^{q_n-1} \mathbf{1}_{[\beta_i - \|q_n \alpha\|, \beta_i)}(x + j\alpha) = 0 \text{ for } i = 1, \dots, K\} \\ &= \{x \in \mathbb{T} : (\forall 0 \leq j < q_n)(\forall 1 \leq i \leq K) \ x + j\alpha \notin [\beta_i - \|q_n \alpha\|, \beta_i)\} \\ &= \bigcap_{j=0}^{q_n-1} \bigcap_{i=1}^K (\mathbb{T} \setminus T^{-j}[\beta_i - \|q_n \alpha\|, \beta_i)) = \mathbb{T} \setminus \bigcup_{j=0}^{q_n-1} \bigcup_{i=1}^K T^{-j}[\beta_i - \|q_n \alpha\|, \beta_i). \end{aligned}$$

It follows that

$$1 - \mu(A_0^n) = \mu \left(\bigcup_{j=0}^{q_n-1} \bigcup_{i=1}^K T^{-j}[\beta_i - \|q_n \alpha\|, \beta_i) \right) \leq K q_n \|q_n \alpha\|,$$

so $\mu(A_0^n) \geq 1 - K q_n \|q_n \alpha\|$ and therefore

$$\liminf \mu(A_0^n) \geq 1 - K\delta > 0.$$

Thus $p_0 > 0$ and $\delta S(f) - C_0 = \delta S(f) \neq 0$, it follows from (4.13) (applied to $\epsilon = 0$) that there exists $\epsilon \in \{0,1\}^K$ such that

$$C_\epsilon + C_0 = 2\delta S(f),$$

whence $\frac{|C_\epsilon|}{2|S(f)|} = \delta$ which yields a contradiction to the definition of δ . \square

4.1. **Non-isomorphism with the inverse in the affine case.** Given a special flow T^f for which $\int_X f d\mu = 1$, $T^{r_n}, T^{q_n} \rightarrow Id$, assume that

$$(f_0^{(r_n+q_n)}, f_0^{(r_n)}, f_0^{(q_n)}) \rightarrow P$$

with $\|f_0^{(q_n)}\|_{L^2}, \|f_0^{(r_n)}\|_{L^2} \leq C$. In view of Proposition 3.7 and Corollary 3.11, we have:

$$\mu_{T_{r_n+q_n}^f, T_{r_n}^f, T_{q_n}^f} \rightarrow \int_{\mathbb{R}^3} \mu_{T_{-t}^f, T_{-u}^f, T_{-v}^f} dP(t, u, v).$$

For each $(a, b, c) \in \mathbb{R}^3$ we have

$$\widehat{P}(a, b, c) = \lim_{n \rightarrow \infty} \int_X e^{2\pi i(a f_0^{(r_n+q_n)}(x) + b f_0^{(r_n)}(x) + c f_0^{(q_n)}(x))} d\mu(x).$$

Denote $\theta(t, u, v) = (t, t - v, t - u)$ and note that

$$(4.14) \quad \theta_*(P) = P \text{ if and only if } \widehat{P}(a, b, c) = \widehat{P}(a + b + c, -c, -a).$$

Moreover

$$(4.15) \quad \begin{aligned} \widehat{P}(a + b + c, -c, -b) &= \lim_{n \rightarrow \infty} \int_X e^{2\pi i((a+b+c)f_0^{(r_n+q_n)} - c f_0^{(r_n)} - b f_0^{(q_n)})} d\mu \\ &= \lim_{n \rightarrow \infty} \int_X e^{2\pi i(a f_0^{(r_n+q_n)} + b f_0^{(r_n)} + c f_0^{(q_n)} + (b+c)(f^{(r_n)} \circ T^{q_n} - f^{(r_n)}))} d\mu, \end{aligned}$$

note that $f^{(r_n)} \circ T^{q_n} - f^{(r_n)} = f^{(q_n)} \circ T^{r_n} - f^{(q_n)}$.

Consider now the affine case

$$f(x) = x + c, \quad Tx = x + \alpha$$

with $f_0(x) = x - \frac{1}{2}$ and $\alpha = [0; a_1, a_2, \dots]$. Our aim is to get a larger set of α 's than those resulting from Theorem 4.6 for which the special flow T^f is not isomorphic to its inverse..

Proposition 4.7. *If there exists a subsequence of denominators $(q_{k_n})_{n \geq 1}$ of α such that $q_{k_n+1} \|q_{k_n} \alpha\| \rightarrow \kappa \in (1/2, 1)$ then T^f is not isomorphic to its inverse.*

Proof. To simplify notation we will write n instead of k_n .

Suppose that T^f is isomorphic to its inverse. In view of Corollary 3.11 and (4.14), if $(f_0^{(q_{n+1}+q_n)}, f_0^{(q_{n+1})}, f_0^{(q_n)}) \rightarrow P$ then $\widehat{P}(a, b, c) = \widehat{P}(a + b + c, -c, -a)$ for each $a, b, c \in \mathbb{R}$. We have

$$f_0^{(q_{n+1})}(T^{q_n}x) - f_0^{(q_{n+1})}(x) = \sum_{j=0}^{q_{n+1}-1} (f(T^j x + q_n \alpha) - f(T^j x))$$

and, by Remark 4.4, $f(y + q_n \alpha) - f(y) \in \pm \|q_n \alpha\| + \mathbb{Z}$ for any $y \in \mathbb{T}$. Thus

$$f_0^{(q_{n+1})}(T^{q_n}x) - f_0^{(q_{n+1})}(x) = \pm q_{n+1} \|q_n \alpha\| + M_n(x) \text{ with } M_n(x) \in \mathbb{Z}.$$

It follows that

$$e^{2\pi i l (f_0^{(q_{n+1})}(T^{q_n}x) - f_0^{(q_{n+1})}(x))} = e^{\pm 2\pi i l q_{n+1} \|q_n \alpha\|} \rightarrow e^{\pm 2\pi i l \kappa}$$

for each integer l . By our standing assumption, $e^{4\pi i \kappa} \neq 1$. Taking into account (4.15) we obtain that

$$\widehat{P}(a, b, c) = \widehat{P}(a + b + c, -a, -c) = e^{\pm 2\pi i (b+c)\kappa} \widehat{P}(a, b, c) \text{ whenever } b + c \in \mathbb{Z},$$

hence

$$(4.16) \quad \widehat{P}(1, -1, -1) = 0.$$

On the other hand, the function

$$\begin{aligned} f_0^{(q_{n+1}+q_n)}(x) - f_0^{(q_{n+1})}(x) - f_0^{(q_n)}(x) &= f_0^{(q_{n+1})}(T^{q_n}x) - f_0^{(q_{n+1})}(x) \\ &= \pm q_{n+1} \|q_n \alpha\| + M_n(x), \end{aligned}$$

so

$$\begin{aligned} |\widehat{P}(1, -1, -1)| &= \lim_{n \rightarrow \infty} \left| \int_{\mathbb{T}} e^{2\pi i (f_0^{(q_{n+1}+q_n)}(x) - f_0^{(q_{n+1})}(x) - f_0^{(q_n)}(x))} dx \right| \\ &= \lim_{n \rightarrow \infty} |e^{\pm 2\pi i q_{n+1} \|q_n \alpha\|}| = 1. \end{aligned}$$

This implies $|\widehat{P}(1, -1, -1)| = 1$ which gives rise to a contradiction to (4.16). \square

Remark 4.8. Since $\alpha = \frac{p_{n+1}+p_n G^{n+1}(\alpha)}{q_{n+1}+q_n G^{n+1}(\alpha)}$ (see e.g. [7]), we have

$$\frac{1}{1 + \frac{1}{a_{n+1}} \frac{1}{a_{n+2}}} < q_{n+1} \|q_n \alpha\| < \frac{1}{1 + \frac{1}{a_{n+1}+1} \frac{1}{a_{n+2}+1}}$$

and

$$\frac{1}{1 + \frac{1}{a_{n+1} + \frac{1}{a_{n+1}}} \frac{1}{a_{n+2} + \frac{1}{a_{n+3}+1}}} < q_{n+1} \|q_n \alpha\| < \frac{1}{1 + \frac{1}{a_{n+1} + \frac{1}{a_n}} \frac{1}{a_{n+2} + \frac{1}{a_{n+3}+1}}}.$$

Therefore

$$q_{k_n+1} \|q_{k_n} \alpha\| \rightarrow 1 \quad \Leftrightarrow \quad a_{k_n+1} + a_{k_n+2} \rightarrow +\infty$$

and

$$q_{k_n+1} \|q_{k_n} \alpha\| \rightarrow 1/2 \quad \Leftrightarrow \quad a_{k_n+1} = a_{k_n+2} = 1 \text{ and } a_{k_n}, a_{k_n+3} \rightarrow +\infty.$$

The set of excluded irrational rotations $E \subset \mathbb{T}$ in Theorem 4.7 consists of all irrational α for which the set of limit points of the sequence $(q_{n+1} \|q_n \alpha\|)_{n \geq 1}$ is $\{1/2, 1\}$. Therefore $\alpha \in E$ if and only if the set of limit points of the sequence $(a_n + a_{n+1})_{n \geq 1}$ is $\{2, +\infty\}$ and if there exists a subsequence $(a_{k_n})_{n \geq 1}$ such that $a_{k_n} = a_{k_n+1} = 1$ then $a_{k_n-1}, a_{k_n+2} \rightarrow +\infty$.

Remark 4.9. A natural question arises whether we could apply (4.15) choosing a sequence of pairs of denominators, say we consider q_{l_n}, q_{k_n} , $n \geq 1$ when α is Liouville in the sense that the sequence of partial quotients tends to infinity. This approach seems to fail whenever f is of bounded variation. Indeed,

$$\left| f_0^{(q_{l_n})} \circ T^{q_{k_n}} - f_0^{(q_{l_n})} \right| \leq \|q_{k_n} \alpha\| \text{Var} f_0^{(q_{l_n})} \leq q_{k_n} \|q_{l_n} \alpha\| \text{Var} f_0$$

and $q_{k_n} \|q_{l_n} \alpha\| \rightarrow 0$ whenever α is a Liouville number.

5. PIECEWISE POLYNOMIAL ROOF FUNCTIONS

Let $r \geq 1$ be an odd number and let $0 < \beta < 1$. In this section we will study the problem of isomorphism to the inverse for special flows T^f built over irrational rotations $Tx = x + \alpha$ on the circle and under C^{r-1} -function which are polynomials after restriction to intervals $[0, \beta)$ and $[\beta, 1)$.

Let us consider a C^{r-1} -function $f : \mathbb{T} \rightarrow \mathbb{R}_+$ such that $D^{r-1}f$ is a function linear on both intervals $[0, \beta)$ and $[\beta, 1)$ with slopes $1 - \beta$ and $-\beta$ respectively. Therefore, $D^{r-1}f$ is an absolutely continuous function whose derivative is equal to

$$D^r f = (1 - \beta)1_{[0, \beta)} - \beta 1_{[\beta, 1)} = 1_{[0, \beta)} - \beta.$$

Thus f restricted to each interval $[0, \beta)$ and $[\beta, 1)$ is a polynomial of degree r with leading coefficients $(1 - \beta)/r!$ and $-\beta/r!$ respectively. Since $D^{r-1}f$ is absolutely continuous and $D^r f$ is of bounded variation, the Fourier coefficients satisfy $\widehat{f}(n) = O(1/|n|^{r+1})$.

If the irrational number α is slowly approximated by rationals, more precisely $\liminf_{n \rightarrow \infty} q_n^{r+1-\epsilon} \|q_n \alpha\| > 0$ for some $\epsilon > 0$, then f is cohomologous to a constant function, so the special flow T^f is isomorphic to its inverse. In this section we deal with rotations satisfying

$$0 < \limsup_{n \rightarrow \infty} q_n^{r+1} \|q_n \alpha\|.$$

Remark 5.1. Note that we can not expect the non-reversibility of T^f for any β . Indeed, suppose that $\beta = \{k\alpha + \beta'\}$, where $k \in \mathbb{Z}$ and $\beta' = 0$ or $1/2$. Then

$$D^r f(x) = 1_{[0, k\alpha)}(x - \beta') - \{k\alpha\} + 1_{[0, \beta')} (x) - \beta'$$

and

$$1_{[0, k\alpha)}(x) - \{k\alpha\} = \{x - k\alpha\} - \{x\},$$

hence, the cocycle $1_{[0, k\alpha)}(x - \beta') - \{k\alpha\}$ is a coboundary. Denote by $g : \mathbb{T} \rightarrow \mathbb{R}$, $s : \mathbb{T} \rightarrow \mathbb{R}$ the unique functions with zero mean such that $D^r g(x) = 1_{[0, k\alpha)}(x - \beta') - \{k\alpha\}$ and $D^r s(x) = 1_{[0, \beta')} (x) - \beta'$. Then $f(x) = g(x) + s(x)$. Let $Sx = 1 - x$. Since $D^r s(Sx) = -D^r s(x)$ and r is odd, we have $s(Sx) = s(x)$. As $D^r g$ is a coboundary, the cocycle g is a coboundary. It follows that the special flow T^f is isomorphic to T^s . By Remark 2.3, the latter flow is reversible and hence T^f is reversible.

The main result of this section (Theorem 5.3) establishes some technical conditions on α that gives non-isomorphism of T^f to its inverse for almost every choice of $\beta \in \mathbb{T}$.

Remark 5.2. In the proof of Theorem 5.3 we will use simple properties of the following standard difference operator. For any $h > 0$ let us consider the difference operator

$$\Delta_h : \mathbb{R}^{[a, b]} \rightarrow \mathbb{R}^{[a, b-h]}, \quad \Delta_h g(x) = g(x+h) - g(x).$$

For every natural r denote by $\Delta_h^r : \mathbb{R}^{[a, b]} \rightarrow \mathbb{R}^{[a, b-rh]}$ the r -th iteration of the operator Δ_h . By induction and using $\binom{r}{k-1} + \binom{r}{k} = \binom{r+1}{k}$, we have the following standard formula

$$(5.1) \quad \Delta_h^r g(x) = \sum_{k=0}^r (-1)^{r-k} \binom{r}{k} g(x+kh) \quad \text{for } x \in [a, b-rh].$$

Moreover, if g is a polynomial function of degree r with leading coefficient a_r then $\Delta_h^r g$ is a constant function equal to $r! a_r h^r$.

Theorem 5.3. *Suppose that α is an irrational number for which there exists a subsequence of denominators $(q_{k_n})_{n \geq 1}$ such that*

$$(5.2) \quad q_{k_n}^{r+1} \|q_{k_n} \alpha\| \rightarrow \kappa \in \left(0, \frac{1}{2(r+1)}\right).$$

Then for almost every $\beta \in \mathbb{T}$ the special flow T^f is not isomorphic to its inverse.

Proof. By Weyl's theorem (see Theorem 4.1 in [24]), for almost every $\beta \in \mathbb{T}$ the sequence $(\{q_{k_n} \beta\})_{n \geq 1}$ is uniformly distributed in $[0, 1)$. It follows that there exists $\gamma \in (0, 1) \setminus \{1/2\}$ and a subsequence $(q_{k_{l_n}})$ such that $\{q_{k_{l_n}} \beta\} \rightarrow \gamma$. To simplify notation we will write n instead of k_{l_n} . Assume also that $\{q_n \alpha\} = \|q_n \alpha\|$. The case where $\{q_n \alpha\} = 1 - \|q_n \alpha\|$ can be treated in a similar way.

Suppose that T^f is isomorphic to its inverse. Since $\hat{f}(n) = O(1/|n|^{r+1})$, in view of Corollary 3.1 in [1], the sequence $(f_0^{(q_n^{r+1})})_{n \geq 1}$ is bounded in L^2 . Therefore, by passing to a further subsequence, if necessary, we can assume that

$$\left(f_0^{((r+1)q_n^{r+1})}, f_0^{(rq_n^{r+1})}, \dots, f_0^{(q_n^{r+1})}\right)_* (\mu) \rightarrow P \quad \text{in } \mathcal{P}(\mathbb{R}^{r+1}).$$

Since $D^r f = (1 - \beta)1_{[0, \beta]} - \beta 1_{[\beta, 1]}$, by the Koksma-Denjoy inequality¹⁵ (see [24]),

$$|(D^r f)^{(q_n)}(x)| \leq q_n D_{q_n}^*(\alpha) \text{Var}_{[0, 1]}(D^r f) \leq 1 + \frac{q_n}{q_{n+1}}.$$

The function $(D^r f)^{(q_n)}$ takes values only in the set $\mathbb{Z} - q_n \beta$ and $\{q_n \beta\} \rightarrow \gamma \in (0, 1)$. Since $q_n/q_{n+1} \rightarrow 0$, it follows that for all n large enough $(D^r f)^{(q_n)}(x)$ is equal to $1 - \{q_n \beta\}$ or $-\{q_n \beta\}$ for every $x \in \mathbb{T}$.

Let

$$A_n := \mathbb{T} \setminus \left(\bigcup_{j=0}^{q_n-1} T^{-j}[1 - (r+1)q_n^r \|q_n \alpha\|, 1] \cup \bigcup_{j=0}^{q_n-1} T^{-j}[\beta - (r+1)q_n^r \|q_n \alpha\|, \beta] \right).$$

Thus, by (5.2),

$$(5.3) \quad \mu(A_n) \geq 1 - 2(r+1)q_n q_n^r \|q_n \alpha\| \rightarrow 1 - 2(r+1)\kappa > 1/2.$$

Moreover, for every $x \in A_n$ the point 0 and β do not belong to any interval $T^j[x, T^{(r+1)q_n^{r+1}}x]$ for all $0 \leq j < q_n$. It follows that $(D^r f)^{(q_n)}$ on $[x, T^{(r+1)q_n^{r+1}}x]$ is constant and equal $s - \{q_n \beta\}$ for some $s \in \{0, 1\}$. Therefore, for every $y \in [x, T^{r q_n^{r+1}}x]$ and $0 \leq j < q_n^r$ we have

$$T^{j q_n} y \in [T^{j q_n} x, T^{j q_n + r q_n^{r+1}} x] \subset [x, T^{(r+1)q_n^{r+1}} x],$$

so

$$(D^r f)^{(q_n^{r+1})}(y) = \sum_{j=0}^{q_n^r-1} (D^r f)^{(q_n)}(T^{j q_n} y) = q_n^r (s - \{q_n \beta\}).$$

Therefore, for every $x \in A_n$ there exists $s = s(x) \in \{0, 1\}$ such that $D^r(f_0^{(q_n^{r+1})}) = q_n^r (s - \{q_n \beta\})$ on $[x, x + r q_n^r \|q_n \alpha\|]$, so $f_0^{(q_n^{r+1})}$ restricted to $[x, x + r h]$, with $h := q_n^r \|q_n \alpha\|$, is a polynomial of degree r with leading coefficient $q_n^r (s - \{q_n \beta\})/r!$. In view of Remark 5.2, it follows that

$$\begin{aligned} \sum_{k=0}^r (-1)^{r-k} \binom{r}{k} f_0^{(q_n^{r+1})}(T^k q_n^{r+1} x) &= \sum_{k=0}^r (-1)^{r-k} \binom{r}{k} f_0^{(q_n^{r+1})}(x + k h) \\ &= \Delta_h^r f_0^{(q_n^{r+1})}(x) = q_n^r (s(x) - \{q_n \beta\}) h^r = (s(x) - \{q_n \beta\}) (q_n^{r+1} \|q_n \alpha\|)^r. \end{aligned}$$

Moreover,

$$\begin{aligned} &\sum_{k=1}^{r+1} (-1)^{r+1-k} \binom{r+1}{k} f_0^{(k q_n^{r+1})}(x) \\ &= \sum_{k=1}^{r+1} (-1)^{r+1-k} \binom{r+1}{k} \sum_{l=0}^{k-1} f_0^{(q_n^{r+1})}(T^l q_n^{r+1} x) \\ &= \sum_{l=0}^r f_0^{(q_n^{r+1})}(T^l q_n^{r+1} x) \sum_{k=l+1}^{r+1} (-1)^{r+1-k} \binom{r+1}{k} \\ &= \sum_{l=0}^r f_0^{(q_n^{r+1})}(T^l q_n^{r+1} x) \sum_{k=l+1}^{r+1} (-1)^{r+1-k} \left(\binom{r}{k} + \binom{r}{k-1} \right) \\ &= \sum_{l=0}^r (-1)^{r-l} \binom{r}{l} f_0^{(q_n^{r+1})}(T^l q_n^{r+1} x) = (s(x) - \{q_n \beta\}) (q_n^{r+1} \|q_n \alpha\|)^r. \end{aligned}$$

¹⁵ $D_{q_n}^*$ is the discrepancy of the sequence $\{0, \alpha, \dots, (q_n - 1)\alpha\}$.

For $s = 0, 1$ set $c_s^n := (s - \{q_n\beta\})(q_n^{r+1}\|q_n\alpha\|)^r$ and let

$$A_n^s = \left\{ x \in A_n : \sum_{k=1}^{r+1} (-1)^{r+1-k} \binom{r+1}{k} f_0^{(kq_n^{r+1})}(x) = c_s^n \right\}.$$

By passing to a further subsequence, if necessary, we can assume that

$$(5.4) \quad \mu(A_n^0) \rightarrow \nu_0, \quad \mu(A_n^1) \rightarrow \nu_1,$$

$$(5.5) \quad (f_0^{(rq_n^{r+1})}, \dots, f_0^{(q_n^{r+1})})_*(\mu_{A_n^0}) \rightarrow P_0, \quad (f_0^{(rq_n^{r+1})}, \dots, f_0^{(q_n^{r+1})})_*(\mu_{A_n^1}) \rightarrow P_1$$

in $\mathcal{P}(\mathbb{R}^r)$ and

$$(5.6) \quad (f_0^{((r+1)q_n^{r+1})}, \dots, f_0^{(q_n^{r+1})})_*(\mu_{A_n^c}) \rightarrow P_2 \text{ in } \mathcal{P}(\mathbb{R}^{r+1}).$$

Since $A_n = A_n^0 \cup A_n^1$, by (5.3), we have

$$\nu_0 + \nu_1 \geq 1 - 2(r+1)\kappa > 1/2.$$

Set $\nu_2 := 1 - \nu_0 - \nu_1$. Let us consider the following maps:

$$\theta : \mathbb{R}^{r+1} \rightarrow \mathbb{R}^{r+1}, \quad \theta(x_0, x_1, \dots, x_r) = (x_0, x_0 - x_r, \dots, x_0 - x_1)$$

$$A : \mathbb{R}^r \rightarrow \mathbb{R}^{r+1}, \quad A(x_1, x_2, \dots, x_r) = \left(\sum_{k=1}^r (-1)^{k+1} \binom{r+1}{k} x_k, x_1, \dots, x_r \right),$$

$$R_c : \mathbb{R}^{r+1} \rightarrow \mathbb{R}^{r+1}, \quad R_c(x_0, x_1, \dots, x_r) = (x_0 + c, x_1, \dots, x_r),$$

$$B_c : \mathbb{R}^r \rightarrow \mathbb{R}^r, \quad B_c(x_1, \dots, x_r) = \left(\sum_{k=1}^r (-1)^{k+1} \binom{r+1}{k} x_k - x_{r+1-l} + c \right)_{l=1}^r.$$

Then

$$(5.7) \quad \theta \circ R_c \circ A = R_{-c} \circ A \circ B_c.$$

Indeed, equation (5.7) is valid directly for last r coordinates. The zero coordinate of the LHS of (5.7) is

$$\text{LHS}_0 := \sum_{k=1}^r (-1)^{k+1} \binom{r+1}{k} x_k + c.$$

The zero coordinate of the RHS of (5.7) is

$$\text{RHS}_0 := -c + \sum_{l=1}^r (-1)^{l+1} \binom{r+1}{l} \left(\sum_{k=1}^r (-1)^{k+1} \binom{r+1}{k} x_k - x_{r+1-l} + c \right).$$

Since r is odd,

$$\sum_{l=1}^r (-1)^{l+1} \binom{r+1}{l} = \sum_{l=0}^{r+1} (-1)^{l+1} \binom{r+1}{l} + 2 = -(1-1)^{r+1} + 2 = 2,$$

thus

$$\begin{aligned} \text{RHS}_0 &= -c + 2c + 2 \sum_{k=1}^r (-1)^{k+1} \binom{r+1}{k} x_k - \sum_{l=1}^r (-1)^{l+1} \binom{r+1}{l} x_{r+1-l} \\ &= c + \sum_{k=1}^r (-1)^{k+1} \binom{r+1}{k} x_k = \text{LHS}_0, \end{aligned}$$

which completes the proof of (5.7).

Since

$$\begin{aligned} \left(f_0^{((r+1)q_n^{r+1})}, f_0^{(rq_n^{r+1})}, \dots, f_0^{(q_n^{r+1})} \right) &= \left(\sum_{k=1}^{r+1} (-1)^k \binom{r+1}{k} f_0^{(kq_n^{r+1})}, 0, \dots, 0 \right) \\ &+ \left(\sum_{k=1}^r (-1)^{k+1} \binom{r+1}{k} f_0^{(kq_n^{r+1})}, f_0^{(rq_n^{r+1})}, \dots, f_0^{(q_n^{r+1})} \right), \end{aligned}$$

by the definitions of maps A , R_c and the set A_n^s , it follows that for any $x \in A_n^s$ and $s = 0, 1$ we have

$$\left(f_0^{((r+1)q_n^{r+1})}(x), \dots, f_0^{(q_n^{r+1})}(x) \right) = R_{c_n^s} \circ A \left(f_0^{(rq_n^{r+1})}(x), \dots, f_0^{(q_n^{r+1})}(x) \right).$$

Since additionally $c_n^s \rightarrow (s - \gamma)\kappa^r$ as $n \rightarrow \infty$ for $s = 0, 1$, by Lemmas 4.1, 4.2, 4.3 and combined with (5.4), (5.5), (5.6), we have

$$\begin{aligned} \left(f_0^{((r+1)q_n^{r+1})}(x), \dots, f_0^{(q_n^{r+1})}(x) \right)_* (\mu) \\ \rightarrow \nu_0 \cdot (R_{-\gamma\kappa^r})_* A_*(P_0) + \nu_1 \cdot (R_{(1-\gamma)\kappa^r})_* A_*(P_1) + \nu_2 \cdot P_2. \end{aligned}$$

Therefore,

$$(5.8) \quad P = \nu_0 \cdot (R_{-\gamma\kappa^r})_* A_*(P_0) + \nu_1 \cdot (R_{(1-\gamma)\kappa^r})_* A_*(P_1) + \nu_2 \cdot P_2.$$

As T^f is isomorphic to its inverse, by Corollary 3.12, $\theta_*(P) = P$. In view of (5.7), it follows that the measure P is equal to

$$(5.9) \quad \nu_0 \cdot (R_{\gamma\kappa^r})_* A_*(B_{-\gamma\kappa^r})_*(P_0) + \nu_1 \cdot (R_{-(1-\gamma)\kappa^r})_* A_*(B_{(1-\gamma)\kappa^r})_*(P_1) + \nu_2 \cdot \theta_*(P_2).$$

Since $\nu_0 + \nu_1 > 1/2$ and (5.8) and (5.9) hold, we have

$$(5.10) \quad \begin{aligned} P(R_{-\gamma\kappa^r} \circ A(\mathbb{R}^r) \cup R_{(1-\gamma)\kappa^r} \circ A(\mathbb{R}^r)) &> 1/2, \\ P(R_{\gamma\kappa^r} \circ A(\mathbb{R}^r) \cup R_{-(1-\gamma)\kappa^r} \circ A(\mathbb{R}^r)) &> 1/2. \end{aligned}$$

As $\gamma \neq 0, 1/2$, the sets $\{-\gamma\kappa^r, (1-\gamma)\kappa^r\}$ and $\{\gamma\kappa^r, -(1-\gamma)\kappa^r\}$ are disjoint. Hence the sets $R_{-\gamma\kappa^r} \circ A(\mathbb{R}^r) \cup R_{(1-\gamma)\kappa^r} \circ A(\mathbb{R}^r)$ and $R_{\gamma\kappa^r} \circ A(\mathbb{R}^r) \cup R_{-(1-\gamma)\kappa^r} \circ A(\mathbb{R}^r)$ are disjoint, contrary to (5.10). This completes the proof of non-isomorphism of T^f and its inverse. \square

6. ANALYTIC FLOWS ON \mathbb{T}^2

6.1. Non-isomorphism with the inverse. Let us recall that that analytic special flows over irrational rotations are precisely analytic reparametrizations of two-dimensional rotations and that (in case of ergodicity) they have simple spectra [4], so if they are isomorphic to their inverses, they are automatically reversible.

The aim of this section is to provide analytic examples that are not reversible. For this aim we briefly recall the AACCP¹⁶ constructions [23] (see also [29] for some modifications).

An AACCP is given by a collection of the following parameters: a sequence $(M_k)_{k \geq 1} \subset \mathbb{N}$ together with an infinite real matrix $((d_{ki}, \dots, d_{kM_k}))_{k \geq 1}$ satisfying for each $k \geq 1$

$$\sum_{i=1}^{M_k} d_{ki} = 0.$$

Set $D_k = \max\{|d_{ki}| : i = 1, \dots, M_k\}$. Then we select a sequence $(\varepsilon_k)_{k \geq 1}$ of positive real numbers so that

$$\sum_{k=1}^{\infty} \sqrt{\varepsilon_k} M_k < +\infty, \quad \sum_{k=1}^{\infty} \varepsilon_k < 1, \quad \varepsilon_k < \frac{1}{D_k^2}, \quad k = 1, 2, \dots$$

¹⁶The acronym comes from ‘‘almost analytic cocycle construction procedure’’.

Finally, the completing parameter of the AACCP is a real number $A > 1$.

The above AACCP is said to be *realized over an irrational number* $\alpha \in [0, 1)$ having the continued fraction expansion

$$\alpha = [0; a_1, a_2, \dots]$$

if there exists a strictly increasing sequence $(n_k)_{k \geq 1} \subset \mathbb{N}$ such that for each $k \geq 1$ we have

$$a_{2n_k+1} > 2, \quad A^{N_k} \frac{D_k M_k}{a_{2n_k+1} q_{2n_k}} < \frac{1}{2^k},$$

where N_k is the degree of a real non-negative trigonometric polynomial P_k satisfying

$$\int_0^1 P_k(t) dt = 1 \quad \text{and} \quad P_k(t) < \varepsilon_k \quad \text{for } t \in (\eta_k/2, 1),$$

where we require the numbers $\eta_k > 0$ to satisfy

$$4M_k \eta_k < \frac{\varepsilon_k}{q_{2n_k}} \quad \text{and} \quad \frac{1}{a_{2n_k+1} q_{2n_k}} < \frac{1}{2} \eta_k.$$

Recall now that

$$\zeta_{2n_k} = \left\{ [0, \{q_{2n_k} \alpha\}), T[0, \{q_{2n_k} \alpha\}), \dots, T^{q_{2n_k+1}-1}[0, \{q_{2n_k} \alpha\}) \right\}$$

and

$$\bar{\zeta}_{2n_k} = \left\{ [\{q_{2n_k+1} \alpha\}, 1), T[\{q_{2n_k+1} \alpha\}, 1), \dots, T^{q_{2n_k}-1}[\{q_{2n_k+1} \alpha\}, 1) \right\}$$

are two disjoint Rokhlin towers fulfilling the whole interval $[0, 1)$. It follows that if we set

$$I_k := [0, \{a_{2n_k+1} q_{2n_k} \alpha\}), \quad J_t^k := T^{(t-1)q_{2n_k}} [0, \{q_{2n_k} \alpha\}), \quad \text{for } t = 1, \dots, a_{2n_k+1},$$

then

$$I_k = \bigcup_{t=1}^{a_{2n_k+1}} J_t^k$$

and I_k is the base of a Rokhlin tower of height q_{2n_k} occupying at least $1 - \frac{2}{a_{2n_k+1}}$ of the space.

Using the above parameters for α over which the AACCP can be realized, one defines a real valued cocycle

$$\varphi = \sum_{k=1}^{\infty} \varphi_k$$

as follows. In I_k we choose consecutively intervals $w_{k,1}, \dots, w_{k,M_k}$ of the same length $\lambda_k \in (\eta_k, 2\eta_k)$, each of which consists of (the same) odd number $e_k \geq 3$ of (consecutive) intervals J_t^k . In general, the intervals $w_{k,i}$ and $w_{k,i+1}$ can be separated by a certain number of intervals of the form J_s^k . Denote by $J_{s_{k,i}}^k$ the middle interval J_t^k in $w_{k,i}$. Then we set

$$\varphi_k(x) = \begin{cases} d_{ki} & \text{if } x \in J_{s_{k,i}}^k \\ 0 & \text{otherwise.} \end{cases}$$

Note that $I_{k+1} \subset J_1^k$, so the supports of φ_k , $k \geq 1$ are pairwise disjoint.

The following two results have been proved in [23].

Proposition 6.1. *The set of $\alpha \in [0, 1)$ over which an AACCP can be realized is residual.*

Proposition 6.2. *The cocycle φ defined above is cohomologous to an analytic cocycle.*

Moreover, we will also make use of the following observation from [23].

Lemma 6.3. *For an arbitrary AACCP and α over which it is realized, the cocycle φ is constant on each interval $T^i I_k$, $i = 1, \dots, q_{2n_k} - 1$, $k \geq 1$. Moreover, for each $k \geq 1$*

$$(6.1) \quad \sum_{i=1}^{q_{2n_k}-1} \varphi|_{T^i I_k} = 0.$$

We now proceed to our special construction. We assume that

$$M_k = 4M'_k \rightarrow \infty.$$

Moreover, we assume that

$$(6.2) \quad a_{2n_k+1} = e_k M_k$$

with $e_k \geq 3$ odd, $k \geq 1$ and

$$(6.3) \quad \frac{1}{\sqrt{M_k}} \sum_{i=1}^{k-1} M_i \leq C_1$$

for a constant $C_1 > 0$. The intervals $w_{k,i}$ are then defined as consecutive unions of e_k (consecutive) subintervals of the form J_t^k . Fix $t_0, u_0 \in \mathbb{R}$. For each $k \geq 1$ we then set

$$(6.4) \quad (d_{ki}) = \left(\underbrace{(t_0, u_0, -u_0, -t_0), \dots, (t_0, u_0, -u_0, -t_0)}_{M'_k \text{ times}} \right).$$

Proceeding as in [29] and using (6.1), by construction, we have

$$(6.5) \quad \varphi^{(e_k q_{2n_k})} - \varphi_k^{(e_k q_{2n_k})} \rightarrow 0 \quad \text{in measure,}$$

$$(6.6) \quad \left(\varphi_k^{(2e_k q_{2n_k})}, \varphi_k^{(e_k q_{2n_k})} \right)_* \rightarrow \frac{1}{4} (\delta_{(t_0+u_0, t_0)} + \delta_{(0, u_0)} + \delta_{(-u_0-t_0, -u_0)} + \delta_{(0, -t_0)}),$$

so

$$(6.7) \quad \left(\varphi^{(2e_k q_{2n_k})}, \varphi^{(e_k q_{2n_k})} \right)_* \rightarrow \frac{1}{4} (\delta_{(t_0+u_0, t_0)} + \delta_{(0, u_0)} + \delta_{(-u_0-t_0, -u_0)} + \delta_{(0, -t_0)}).$$

Moreover,

$$(6.8) \quad \{e_k q_{2n_k} \alpha\} \rightarrow 0 \quad \text{when } k \rightarrow \infty.$$

We now proceed similarly as in [29] and notice that if we set $C := |t_0| + |u_0|$ then for each $k \geq 1$ we have

$$(6.9) \quad \left| \varphi_k^{(e_k q_{2n_k})} \right| \leq C.$$

Since the support of $\sum_{i \geq k+1} \varphi_i$ is included in I_{k+1} and for each $x \in [0, 1)$,

$$\#(\{x, Tx, \dots, T^{e_k q_{2n_k}} x\} \cap I_{k+1}) \leq 1,$$

we also have

$$(6.10) \quad \left| \sum_{i \geq k+1} \varphi_i^{(e_k q_{2n_k})} \right| \leq C.$$

Finally, for $i = 1, \dots, k-1$ we have $\varphi_i = 0$ on I_k , so in view of (6.1), $\varphi_i^{(e_k q_{2n_k})} = 0$ except for the set

$$[0, 1) \setminus \bigcup_{j=0}^{q_{2n_k}-1} T^j I_k \quad \text{and} \quad \bigcup_{j=0}^{e_k q_{2n_k}-1} T^j J_{a_{2n_k+1}-e_k}^k.$$

Moreover, by (6.1), for every $x \in \mathbb{T}$ and $m \geq 0$, $i \geq 1$ we have

$$|\varphi_i^{(m)}(x)| \leq \sum_{j=1}^{M_i} |d_{ij}| \leq CM_i.$$

In view of (6.2) and (6.3), it follows that

$$\begin{aligned} \left\| \left(\sum_{i=1}^{k-1} \varphi_i \right)^{(e_k q_{2n_k})} \right\|_2 &\leq C \sum_{i=1}^{k-1} M_i \cdot \mu \left(\left\{ x \in [0, 1) : \left(\sum_{i=1}^{k-1} \varphi_i \right)^{(e_k q_{2n_k})} (x) \neq 0 \right\} \right)^{1/2} \\ &\leq C \sum_{i=1}^{k-1} M_i \left(\frac{2}{a_{2n_k+1}} + \frac{e_k}{a_{2n_k+1}} \right)^{1/2} \leq 2C \frac{1}{\sqrt{M_k}} \sum_{i=1}^{k-1} M_i \leq \text{const}. \end{aligned}$$

This together with (6.9) and (6.10) implies

$$(6.11) \quad \left\| \varphi^{(e_k q_{2n_k})} \right\|_2 \leq \text{const}.$$

The cocycle φ is clearly bounded (and of zero mean) and by Proposition 6.2, it is cohomologous to an analytic function $f_0 : \mathbb{T} \rightarrow \mathbb{R}$ (of zero mean). Then for each sufficiently large constant $d > 0$ we have

$$\varphi + d > 0, \quad f := f_0 + d > 0$$

and moreover the special flows $T^{\varphi+d}$ and T^f are isomorphic. In view of (6.8), (6.11), Proposition 3.7 and (6.7) we obtain that for some constant $c > 0$ ($c = \int_X f d\mu$)

$$(6.12) \quad \mu_{T_{2ce_k q_{2n_k}}^{\varphi+d}, T_{ce_k q_{2n_k}}^{\varphi+d}} \rightarrow \frac{1}{4} \left(\mu_{T_{t_0+u_0}^{\varphi+d}, T_{t_0}^{\varphi+d}} + \mu_{T_{u_0}^{\varphi+d}, T_{u_0}^{\varphi+d}} + \mu_{T_{-u_0-t_0}^{\varphi+d}, T_{-u_0}^{\varphi+d}} + \mu_{T_{-t_0}^{\varphi+d}, T_{-t_0}^{\varphi+d}} \right).$$

Since $T^{\varphi+d}$ and T^f are isomorphic and (6.12) holds,

$$(6.13) \quad \mu_{T_{2ce_k q_{2n_k}}^f, T_{ce_k q_{2n_k}}^f} \rightarrow \frac{1}{4} \left(\mu_{T_{t_0+u_0}^f, T_{t_0}^f} + \mu_{T_{u_0}^f, T_{u_0}^f} + \mu_{T_{-u_0-t_0}^f, T_{-u_0}^f} + \mu_{T_{-t_0}^f, T_{-t_0}^f} \right).$$

Now, the limit measure

$$\begin{aligned} P &:= \frac{1}{4} \left(\delta_{(t_0+u_0, t_0)} + \delta_{(0, u_0)} + \delta_{(-u_0-t_0, -u_0)} + \delta_{(0, -t_0)} \right) \\ &= \lim_{k \rightarrow \infty} \left(f_0^{(2e_k q_{2n_k})}, f_0^{(e_k q_{2n_k})} \right)_* (\mu) \end{aligned}$$

is not ‘‘symmetric’’ in the sense that $(0, u_0)$ is its atom, while $(0, -u_0)$ is not provided that $u_0 \neq 0$ and $t_0 \neq \pm u_0$ and therefore under these additional assumptions, by Corollary 3.13, T^f is not reversible. In this way we have proved the following result.

Corollary 6.4. *There is an analytic weakly mixing flow on \mathbb{T}^2 (preserving a smooth measure) that is not reversible.*

Remark 6.5. If instead of f (constructed above) we consider $f_\varepsilon := 1 + \varepsilon f$ for small enough $\varepsilon > 0$ then the corresponding special flow T^{f_ε} can be interpreted as arbitrarily small analytic change of time in the linear flow by $(\alpha, 1)$ on \mathbb{T}^2 .

Now, note that $(f_\varepsilon)_0 = \varepsilon f_0$. Hence

$$\left((f_\varepsilon)_0^{(2e_k q_{2n_k})}, (f_\varepsilon)_0^{(e_k q_{2n_k})} \right)_* (\mu) \rightarrow (M_\varepsilon)_* P =: P_\varepsilon,$$

where $M_\varepsilon(x, y) = (\varepsilon x, \varepsilon y)$. It follows that P_ε has the same asymmetries as P and therefore T^{f_ε} is not reversible. Therefore, for some α irrational there are arbitrarily small analytic changes of time in the linear flow by $(\alpha, 1)$ on \mathbb{T}^2 which yield weakly mixing and non-reversible flows.

Remark 6.6. If in (6.4), for each $k \geq 1$, we consider the following pattern

$$(6.14) \quad (d_{ki}) = \underbrace{\left((a, b, c, b, b), \dots, (a, b, c, b, b) \right)}_{M'_k/2 \text{ times}}, \underbrace{\left((-a, -b, -c, -b, -b), \dots, (-a, -b, -c, -b, -b) \right)}_{M'_k/2 \text{ times}}$$

then

$$\left(f_0^{(2e_k q_{2n_k})}, f_0^{(e_k q_{2n_k})} \right)_* (\mu) \rightarrow P$$

and

$$\left(f_0^{(3e_k q_{2n_k})}, f_0^{(2e_k q_{2n_k})}, f_0^{(e_k q_{2n_k})} \right)_* (\mu) \rightarrow Q.$$

We have

$$P = \frac{1}{10} \left(\delta_{(a+b,a)} + \delta_{(b+c,b)} + \delta_{(b+c,c)} + \delta_{(2b,b)} + \delta_{(a+b,b)} \right. \\ \left. + \delta_{(-(a+b),-b)} + \delta_{(-(b+c),-b)} + \delta_{(-(b+c),-c)} + \delta_{(-2b,b)} + \delta_{(-(a+b),-b)} \right)$$

and

$$Q = \frac{1}{10} \left(\delta_{(a+b+c,a+b,a)} + \delta_{(2b+c,b+c,b)} + \delta_{(2b+c,b+c,c)} + \delta_{(a+2b,2b,b)} + \delta_{(a+2b,a+b,b)} \right. \\ \left. + \delta_{(-(a+b+c),-(a+b),-b)} + \delta_{(-(2b+c),-(b+c),-b)} + \delta_{(-(2b+c),-(b+c),-c)} \right. \\ \left. + \delta_{(-(a+2b,-2b,b)} + \delta_{(-(a+2b),-(a+b),-b)} \right).$$

It follows that P is invariant under the map $(x, y) \mapsto (x, x - y)$ and therefore we cannot apply Corollary 3.13 but Q is **not** invariant under the map $(x, y, z) \mapsto (x, x - z, x - y)$, so by Corollary 3.12, the resulting flow is not reversible.

Note however that Corollary 3.13 is sufficient for non-reversibility of T^f if instead of $(2e_k q_{2n_k}, e_k q_{2n_k})$ we consider $(4e_k q_{2n_k}, 2e_k q_{2n_k})$.

Problem. When $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ is weakly mixing then the method of showing non-isomorphism of \mathcal{T} and its inverse passes to non-trivial factors (see Remark 3.10). Since all flows non-isomorphic to their inverses considered in the paper are weakly mixing, in fact, their non-trivial factors are also non-isomorphic to their inverses. A natural question arises if whenever the weak closure joining method applies, \mathcal{T} is disjoint with its inverse.

6.2. Absence of rational self-similarities. In this section we will show that the construction presented in Section 6.1 can be easily modified so that we obtain an analytic weakly mixing flow on \mathbb{T}^2 such that no rational number is its scale of self-similarity (in particular, it is not reversible). It remains an open question whether irrational numbers can be scales of self-similarity for analytic flows (preserving smooth measure)¹⁷ on \mathbb{T}^2 .

We will now recall a result which will ensure that a rational number is not a scale of self-similarity of an ergodic flow.

Proposition 6.7 ([29]). *Let $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ and $\mathcal{S} = (S_t)_{t \in \mathbb{R}}$ be flows on (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) respectively. Assume additionally that \mathcal{T} is weakly mixing and \mathcal{S} is ergodic. Moreover, suppose that for a sequence $(t_k) \subset \mathbb{R}$ with $t_k \rightarrow \infty$*

$$\mu_{T_{t_k}} \rightarrow \int_{\mathbb{R}} \mu_{T_{-t}} dP(t) \quad \text{and} \quad \nu_{S_{t_k}} \rightarrow \int_{\mathbb{R}} \nu_{S_{-t}} dQ(t).$$

¹⁷In [25] it is shown that on each compact orientable surface of genus at least 2 there is a smooth (non-singular) non-self-similar flow. It is unknown whether these constructions are non-reversible. It is also unknown whether a smooth non-self-similar flow can be constructed on \mathbb{T}^2 .

If $P \neq Q$ then the flows \mathcal{T} and \mathcal{S} are disjoint in the sense of Furstenberg.

Notice that when \mathcal{T} and \mathcal{S} are disjoint, then for each $r \in \mathbb{R}^*$ the flows $\mathcal{T} \circ r$ and $\mathcal{S} \circ r$ are also disjoint (indeed, $J(\mathcal{T}, \mathcal{S}) = J(\mathcal{T} \circ r, \mathcal{S} \circ r)$). Therefore, the following result holds.

Corollary 6.8. *Let $a, b \in \mathbb{N}$. Assume that $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ is a weakly mixing flow on (X, \mathcal{B}, μ) . Assume also that for some $t_k \rightarrow \infty$*

$$\mu_{T_{at_k}} \rightarrow \int_{\mathbb{R}} \mu_{T_{-at}} dP(t) \quad \text{and} \quad \mu_{T_{bt_k}} \rightarrow \int_{\mathbb{R}} \mu_{T_{-bt}} dQ(t).$$

for some probability measures P and Q on \mathbb{R} . If $P \neq Q$ then $\mathcal{T} \perp \mathcal{T} \circ (b/a)$.

We will now show what sequence of numbers (d_{ki}) to use in the construction of a weakly mixing non-reversible analytic flow (preserving a smooth measure) on \mathbb{T}^2 instead of the one in (6.4) to fulfill, for each natural $a < b$, the assumptions of the above corollary. To this end we partition $\mathbb{N} = \mathbb{N}_{-1,1} \cup_{a < b; a, b \in \mathbb{N}} \mathbb{N}_{a,b}$ so that $\mathbb{N}_{-1,1}$ and each set $\mathbb{N}_{a,b}$ is infinite. For $k \in \mathbb{N}_{-1,1}$ we repeat the construction described in Section 6.1, so that the resulting flow will not be reversible.

Take $(a, b) \in \mathbb{N}^2$. By reversing the roles of a and b , we may assume that $a < b$. We will consider only $k \in \mathbb{N}_{a,b}$. Assume that $M_k = bM'_k$ and set

$$(d_{ki}) := \underbrace{\left(\left(t_0, \underbrace{-\frac{t_0}{b-1}, \dots, -\frac{t_0}{b-1}}_{b-1 \text{ times}} \right), \dots, \left(t_0, -\frac{t_0}{b-1}, \dots, -\frac{t_0}{b-1} \right) \right)}_{M'_k \text{ times}}.$$

It follows that for the analytic flow $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ on \mathbb{T}^2 constructed in such a way as in Section 6.1 we have (with $c = \int_X f d\mu$)

$$\begin{aligned} \lim_{k \rightarrow \infty, k \in \mathbb{N}_{a,b}} \mu_{T_{acc_k q_2 n_k}} &= \int_{\mathbb{R}} \mu_{T_{-t}} d \left(\frac{a}{b} \delta_{(1-\frac{a}{b-1})t_0} + \frac{b-a}{b} \delta_{-\frac{a}{b-1}t_0} \right) (t) \\ &= \int_{\mathbb{R}} \mu_{T_{-at}} d \left(\frac{a}{b} \delta_{(\frac{1}{a}-\frac{1}{b-1})t_0} + \frac{b-a}{b} \delta_{-\frac{1}{b-1}t_0} \right) (t) \end{aligned}$$

and

$$\lim_{k \rightarrow \infty, k \in \mathbb{N}_{a,b}} \mu_{T_{bcc_k q_2 n_k}} = \mu_{Id} = \int_{\mathbb{R}} \mu_{T_{-t}} d\delta_0(t).$$

In view of Corollary 6.8, this yields $\mathcal{T} \perp \mathcal{T} \circ (a/b)$ for arbitrary natural $a < b$. Since $-1 \notin I(\mathcal{T})$ and $I(\mathcal{T})$ is a multiplicative subgroup of \mathbb{R}^* , we have $\mathbb{Q} \cap I(\mathcal{T}) = \{1\}$. Hence, we have proved the following result.

Corollary 6.9. *There is an analytic weakly mixing flow (preserving a smooth measure) on \mathbb{T}^2 that is not reversible and such that no rational number is its scale of self-similarity.*

7. CHACON'S TYPE AUTOMORPHISMS NON-ISOMORPHIC TO THEIR INVERSES

The results of this section were essentially proved in [40] and [41]¹⁸. We include a detailed proof of Theorem 7.2 below to see a relationship between our methods and earlier works by Ryzhikov.

¹⁸Formally, in [40] and [41] different sequences are considered and two limit joinings (not of the above form) are considered, but the essence of the argument is the same. Note that Proposition 7.1 is a natural automorphism counterpart of Corollary 3.13.

Proposition 7.1. *Assume that T is an ergodic automorphism on (X, \mathcal{B}, μ) . Assume also that*

$$\mu_{T^{2q_n}, T^{q_n}} \rightarrow \int_{\mathbb{Z}^2} \mu_{T^{-a}, T^{-b}} dP(a, b)$$

for some probability measure P on \mathbb{Z}^2 . If the measure P is not invariant under $\theta(a, b) = (a, a - b)$ then T is not isomorphic to its inverse. In particular, T is not reversible.

In this section we consider some rank one automorphisms in construction of which along a subsequence we repeat a Chacon's type construction [17]; we will obtain non-reversible rank one automorphisms.

Recall briefly a rank one construction (see e.g. [31]). For a sequence of positive integers $(r_n)_{n \in \mathbb{N}}$ with all $r_n \geq 2$ and $(s_1^{(n)}, \dots, s_{r_n}^{(n)})_{n \in \mathbb{N}}$ with all $s_i^{(n)}$ non-negative integers we define a rank-one transformation by giving an increasing sequence of Rokhlin towers $(C_n)_{n \in \mathbb{N}}$ such that each C_n consists of q_n pairwise disjoint intervals of the same length (each such interval is called a level of C_n). More precisely, $C_n = \{C_{n,1}, C_{n,2}, \dots, C_{n,q_n}\}$, the dynamics T is defined on $\bigcup_{i=1}^{q_n-1} C_{n,i}$ so that T sends linearly $C_{n,j}$ to $C_{n,j+1}$ for $j = 1, \dots, q_n - 1$. The tower C_{n+1} is obtained first by cutting C_n into r_n subcolumns, say $C_n(i)$, $1 \leq i \leq r_n$, of equal width, placing $s_i^{(n)}$ spacers over each subcolumn $C_n(i)$ and finally stacking each subcolumn $C_n(i)$ on the top of $C_n(i+1)$ for $1 \leq i < r_n$ in order to complete the definition of C_{n+1} . The tower C_{n+1} has the height $q_{n+1} = r_n q_n + \sum_{i=1}^{r_n} s_i^{(n)}$. The ordering of levels in C_{n+1} is lexicographical from the left to the right. The dynamics T on $\bigcup_{i=1}^{q_{n+1}-1} C_{n+1,i}$ is completed by sending linearly C_{n+1,q_n} to the first spacer over the first subcolumn $C_n(1)$, sending this spacer to the one above it, etc., and when reaching the top spacer we send it to C_{n+1,q_n+1} . We keep going the same procedure for the remaining columns and stop at the top spacer over $C_n(r_n)$. In this way we obtain a measure-preserving transformation T defined on a standard Borel space (X, \mathcal{B}, μ) although, in general, μ is only σ -finite. Provided that the number of spacers is not too large [31], μ can be assumed (and this is our tacit standing assumption) to be a probability measure.

We will now describe the details of our particular rank-one construction. Fix an even positive integer $r \geq 4$ and an increasing sequence $(n_k)_{k \in \mathbb{N}}$. Suppose that $r_{n_k} = r$ and $r_{n_k+1} \rightarrow \infty$ ¹⁹ and in the construction we place one spacer over $C_{n_k}(i)$ for $r/2 + 1 \leq i \leq r$ and over $C_{n_k+1}(i)$ for $[r_{n_k+1}/2] + 1 \leq i \leq r_{n_k+1}$.

Theorem 7.2. *Under the above assumptions the constructed rank-one automorphism T is not reversible.*

Proof. We claim that

$$\mu_{T^{2q_{n_k}}, T^{q_{n_k}}} \rightarrow \int_{\mathbb{Z}^2} \mu_{T^{-a}, T^{-b}} dP(a, b),$$

where

$$(7.1) \quad P(1, 0) = \frac{1}{r}, \quad P(\theta(1, 0)) = P(1, 1) = \frac{1}{2r}$$

Once we have shown this, the claim will follow by Proposition 7.1. Since every

¹⁹The assumption on the spacers over the subcolumns $C_{n_k+1}(i)$ for $[r_{n_k+1}/2] + 1 \leq i \leq r_{n_k+1}$ at step $n_k + 1$ of the construction yields some form of "rigidity". This condition can be modified. What is important, is that we prevent the image of a single level of tower C_{n_k} under $T^{r_{n_k}}$ from being "too scattered".

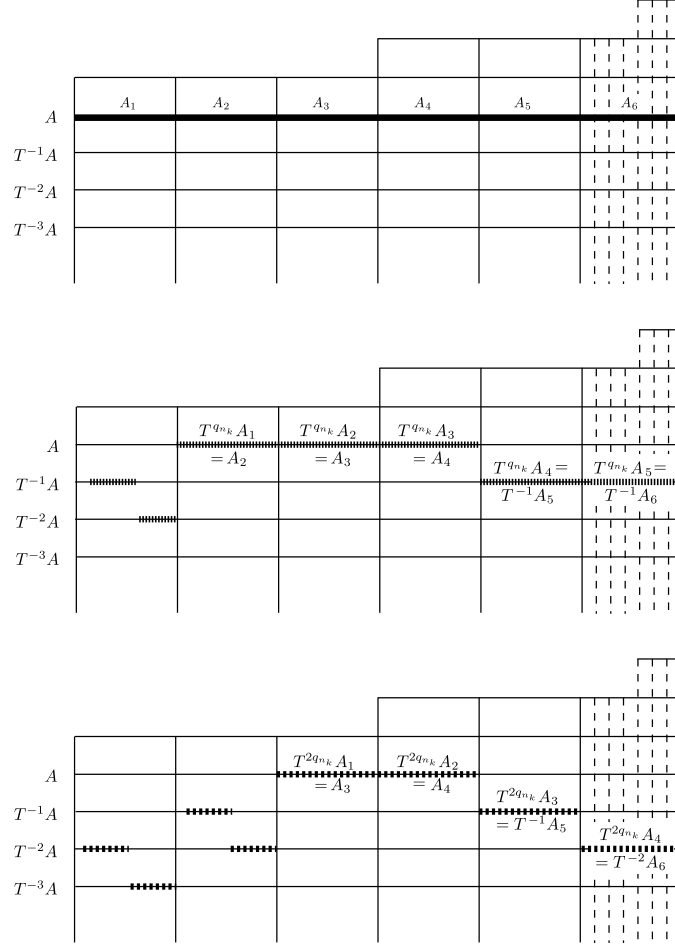


FIGURE 1. Illustration of the proof in the case $r = 6$.

measurable set can be approximated by unions of levels of sufficiently high towers, it suffices to show that

$$\mu_{T^{2q_{n_k}r}, T^{q_{n_k}r}}(A[1] \times A[2] \times A[3]) \rightarrow \int_{\mathbb{Z}^2} \mu_{T^{-a}, T^{-b}}(A[1] \times A[2] \times A[3]) dP(a, b),$$

for $A[1], A[2], A[3]$ being single levels of the tower C_{k_0} for arbitrarily large $k_0 \in \mathbb{N}$ and the measure P satisfies (7.1). Since the towers are arbitrarily high, the levels $A[1], A[2], A[3]$ can be assumed not to be any of the first 3 bottom levels. Fix $k_0 \in \mathbb{N}$ and let $A[1], A[2], A[3]$ be single levels of the tower $C_{n_{k_0}}$. Without loss of generality we may assume that $C_{n_{k_0}}$ is at least of height 4. For each $k \geq k_0$ the sets $A[1], A[2], A[3]$ become finite disjoint unions of levels of tower C_{n_k} :

$$(7.2) \quad A[1] = \bigcup_{l=1}^{l_k} A^{(k)}[1, l], \quad A[2] = \bigcup_{l=1}^{l_k} A^{(k)}[2, l], \quad A[3] = \bigcup_{l=1}^{l_k} A^{(k)}[3, l]$$

for some $l_k \geq 1$. Moreover,

$$(7.3) \quad \text{for any } 1 \leq l < l' \leq l_k \text{ there are at least 3 levels of tower } C_{n_k} \text{ between } A^{(n_k)}[t, l] \text{ and } A^{(n_k)}[t, l'] \text{ for } t = 1, 2, 3.$$

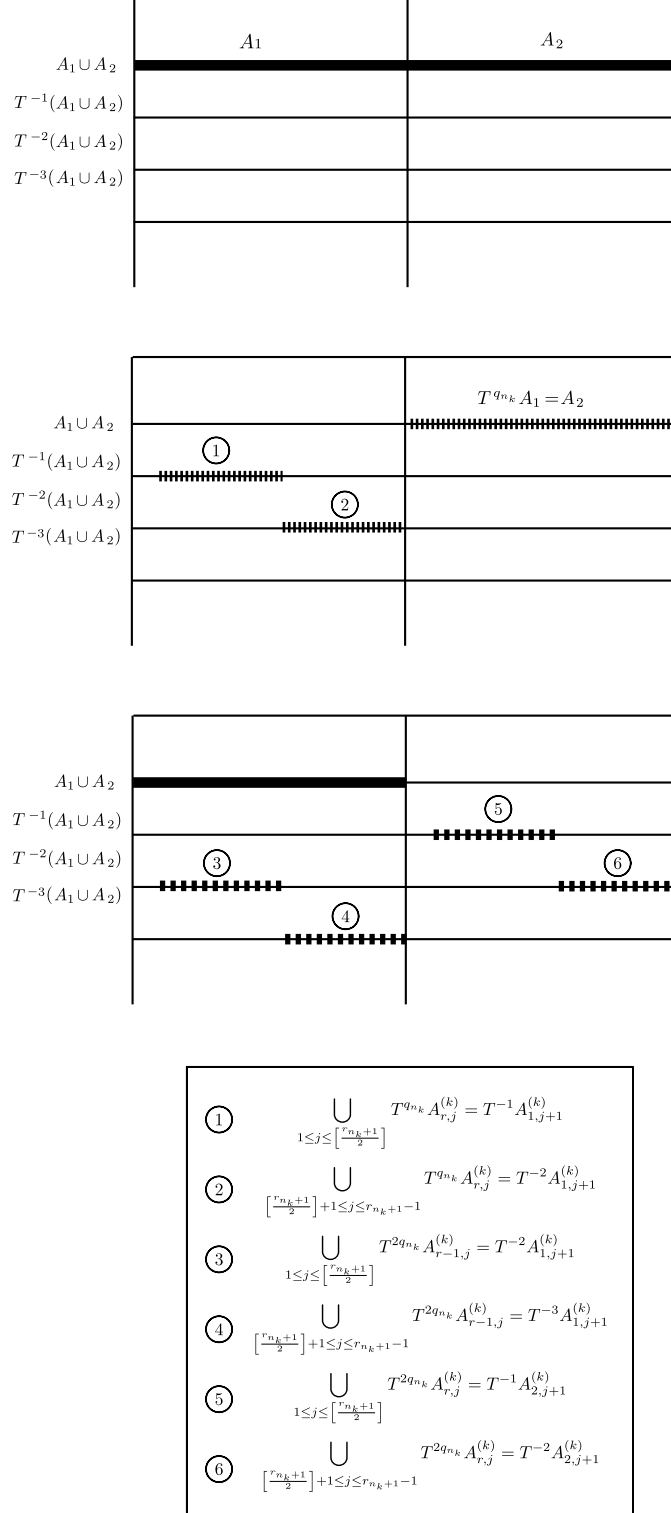


FIGURE 2. A part of Figure 1 magnified.

Let $\varepsilon > 0$. We claim that for $k \geq k_0$ sufficiently large and for $A^{(k)}$ being a level of tower C_{n_k} which is not one of the first 3 levels we have

$$\left| \mu_{T^{2rq_{n_k}}, T^{rq_{n_k}}} (A^{(k)} \times A^{(k)} \times A^{(k)}) - \frac{r-2}{2r} \mu(A^{(k)}) \right| < \varepsilon \mu(A^{(k)}),$$

$$\begin{aligned}
(7.4) \quad & \left| \mu_{T^{2r}q_{n_k}, T^{r}q_{n_k}} (A^{(k)} \times A^{(k)} \times TA^{(k)}) - \frac{1}{2r} \mu(A^{(k)}) \right| < \varepsilon \mu(A^{(k)}), \\
& \left| \mu_{T^{2r}q_{n_k}, T^{r}q_{n_k}} (A^{(k)} \times A^{(k)} \times T^2A^{(k)}) - \frac{1}{2r} \mu(A^{(k)}) \right| < \varepsilon \mu(A^{(k)}), \\
& \left| \mu_{T^{2r}q_{n_k}, T^{r}q_{n_k}} (A^{(k)} \times TA^{(k)} \times TA^{(k)}) - \frac{1}{r} \mu(A^{(k)}) \right| < \varepsilon \mu(A^{(k)}), \\
& \left| \mu_{T^{2r}q_{n_k}, T^{r}q_{n_k}} (A^{(k)} \times TA^{(k)} \times T^2A^{(k)}) - \frac{r-3}{2r} \mu(A^{(k)}) \right| < \varepsilon \mu(A^{(k)}), \\
& \left| \mu_{T^{2r}q_{n_k}, T^{r}q_{n_k}} (A^{(k)} \times T^2A^{(k)} \times T^3A^{(k)}) - \frac{1}{2r} \mu(A^{(k)}) \right| < \varepsilon \mu(A^{(k)}).
\end{aligned}$$

The proof of all of these inequalities goes along the same lines, we will prove only the fifth of them, i.e.

$$(7.5) \quad \left| \mu_{T^{2r}q_{n_k}, T^{r}q_{n_k}} (A^{(k)} \times TA^{(k)} \times T^2A^{(k)}) - \frac{r-3}{2r} \mu(A^{(k)}) \right| < \varepsilon \mu(A^{(k)})$$

(the proof of (7.5) contains all elements of the proofs of the other inequalities in (7.4)). To make the notation simpler we will write C for the tower C_{n_k} and we will also change the notation for the subcolumns. Now, tower C_{n_k} is cut into r subcolumns of equal width, denoted from left to right by C^i , $1 \leq i \leq r$, with one spacer placed over subcolumns C^i for $r/2+1 \leq i \leq r$. Then each of C^i is cut into s subcolumns of equal width, denoted from left to right by $C^{i,j}$, $1 \leq j \leq r_{n_k+1}$, with one spacer placed over subcolumns $C^{r,j}$ for $\lceil \frac{r_{n_k+1}}{2} \rceil + 1 \leq j \leq r_{n_k+1}$. Let

$$A_i^{(k)} := A^{(k)} \cap C^i, \quad A_{i,j}^{(k)} := A^{(k)} \cap C^{i,j}$$

for $1 \leq i \leq r$ and $1 \leq j \leq r_{n_k+1}$.

Notice that we have

$$\begin{aligned}
(7.6) \quad & T^{q_{n_k}} A_i^{(k)} = A_{i+1}^{(k)} \quad \text{for } 1 \leq i \leq r/2, \\
& T^{q_{n_k}} A_i^{(k)} = T^{-1} A_{i+1}^{(k)} \quad \text{for } r/2+1 \leq i \leq r-1
\end{aligned}$$

and

$$\begin{aligned}
(7.7) \quad & T^{q_{n_k}} A_{r,j}^{(k)} = T^{-1} A_{1,j+1}^{(k)} \quad \text{for } 1 \leq j \leq \left\lceil \frac{r_{n_k+1}}{2} \right\rceil, \\
& T^{q_{n_k}} A_{r,j}^{(k)} = T^{-2} A_{1,j+1}^{(k)} \quad \text{for } \left\lceil \frac{r_{n_k+1}}{2} \right\rceil + 1 \leq j \leq r_{n_k+1} - 1.
\end{aligned}$$

Moreover

$$(7.8) \quad \mu \left(A^{(k)} \setminus \left(\bigcup_{1 \leq i \leq r-1} A_i^{(k)} \cup \bigcup_{1 \leq j \leq r_{n_k+1}-1} A_{r,j}^{(k)} \right) \right) = \mu(A_{r,r_{n_k+1}}^{(k)}) = \frac{1}{rr_{n_k+2}} \mu(A^{(k)}).$$

We also have

$$\begin{aligned}
(7.9) \quad & T^{2q_{n_k}} A_i^{(k)} = A_{1,i+2}^{(k)} \quad \text{for } 1 \leq i \leq r/2-1, \\
& T^{2q_{n_k}} A_{1,r/2}^{(k)} = T^{-1} A_{1,r/2+2}^{(k)}, \\
& T^{2q_{n_k}} A_{1,j}^{(k)} = T^{-2} A_{1,j+2}^{(k)} \quad \text{for } r/2+1 \leq j \leq r-2
\end{aligned}$$

and

$$\begin{aligned}
(7.10) \quad & T^{2q_{n_k}} A_{r-1,j}^{(k)} = T^{-2} A_{1,j+1}^{(k)} \quad \text{for } 1 \leq j \leq \left\lceil \frac{r_{n_k+1}}{2} \right\rceil, \\
& T^{2q_{n_k}} A_{r-1,j}^{(k)} = T^{-3} A_{1,j+1}^{(k)} \quad \text{for } \left\lceil \frac{r_{n_k+1}}{2} \right\rceil + 1 \leq j \leq r_{n_k+1} - 1
\end{aligned}$$

and

$$\begin{aligned}
(7.11) \quad & T^{2q_{n_k}} A_{r,j}^{(k)} = T^{-1} A_{2,j+1}^{(k)} \quad \text{for } 1 \leq j \leq \left\lceil \frac{r_{n_k+1}}{2} \right\rceil, \\
& T^{2q_{n_k}} A_{r,j}^{(k)} = T^{-2} A_{2,j+1}^{(k)} \quad \text{for } \left\lceil \frac{r_{n_k+1}}{2} \right\rceil + 1 \leq j \leq r_{n_k+1} - 1.
\end{aligned}$$

Moreover

$$(7.12) \quad \mu \left(A^{(k)} \setminus \left(\bigcup_{1 \leq i \leq r-2} A_i^{(k)} \cup \bigcup_{1 \leq j \leq r_{n_k+1}-1} A_{r-1,j}^{(k)} \cup \bigcup_{1 \leq j \leq r_{n_k+1}-1} A_{r,j}^{(k)} \right) \right) \\ = \mu \left(A_{r-1,r_{n_k+1}}^{(k)} \right) + \mu \left(A_{r,r_{n_k+1}}^{(k)} \right) = \frac{2}{rr_{n_k+1}} \mu \left(A^{(k)} \right).$$

Using all of the ?? 7.6 and eqs. (7.7) and (7.8) we obtain

$$(7.13) \quad \mu \left(T^{q_{n_k}} A^{(k)} \setminus \bigcup_{p_2 \in \{0,1,2\}} T^{-p_2} A^{(k)} \right) < \frac{1}{rr_{n_k+2}} \mu \left(A^{(k)} \right)$$

and using eqs. (7.9)–(7.12)

$$(7.14) \quad \mu \left(T^{2q_{n_k}} A^{(k)} \setminus \bigcup_{p_3 \in \{0,1,2,3\}} T^{-p_3} A^{(k)} \right) < \frac{2}{rr_{n_k+2}} \mu \left(A^{(k)} \right).$$

We will show now that (7.5) is true. We have

$$\begin{aligned} & A^{(k)} \cap TT^{q_{n_k}} A^{(k)} \cap T^2 T^{2r_{n_k}} A^{(k)} \\ & \stackrel{(*)}{\simeq} \left(A^{(k)} \cap TT^{q_{n_k}} \left(\bigcup_{1 \leq i \leq r-1} A_i^{(k)} \cup \bigcup_{1 \leq j \leq r_{n_k+1}-1} A_{r,j}^{(k)} \right) \right) \cap T^2 T^{2q_{n_k}} A^{(k)} \\ & \stackrel{(7.6),(7.7),(7.8)}{\simeq} \left(\bigcup_{r/2+1 \leq i \leq r-1} A_{i+1}^{(k)} \cup \bigcup_{1 \leq j \leq \lceil \frac{r_{n_k+1}}{2} \rceil} A_{1,j+1}^{(k)} \right) \cap T^2 T^{2q_{n_k}} A^{(k)} \\ & \stackrel{(**)}{\simeq} \left(\bigcup_{r/2+1 \leq i \leq r-1} A_{i+1}^{(k)} \cup \bigcup_{1 \leq j \leq \lceil \frac{r_{n_k+1}}{2} \rceil} A_{1,j+1}^{(k)} \right) \\ & \quad \cap T^2 T^{2q_{n_k}} \left(\bigcup_{1 \leq i \leq r-2} A_i^{(k)} \cup \bigcup_{1 \leq j \leq r_{n_k+1}-1} A_{r-1,j}^{(k)} \cup \bigcup_{1 \leq j \leq r_{n_k+1}-1} A_{r,j}^{(k)} \right) \\ & \stackrel{(7.9),(7.10),(7.11)}{\simeq} \left(\bigcup_{r/2+1 \leq i \leq r-1} A_{i+1}^{(k)} \cup \bigcup_{1 \leq j \leq \lceil \frac{r_{n_k+1}}{2} \rceil} A_{1,j+1}^{(k)} \right) \\ & \quad \cap \left(\bigcup_{r/2+1 \leq i \leq r-2} A_{i+2}^{(k)} \cup \bigcup_{1 \leq j \leq \lceil \frac{r_{n_k+1}}{2} \rceil} A_{1,j+1}^{(k)} \cup \bigcup_{\lceil \frac{r_{n_k+1}}{2} \rceil + 1 \leq j \leq r_{n_k+1}-1} A_{2,j+1}^{(k)} \right) \\ & = \bigcup_{r/2+3 \leq i \leq r} A_i^{(k)} \cup \bigcup_{2 \leq j \leq \lceil \frac{r_{n_k+1}}{2} \rceil + 1} A_{1,j}^{(k)}, \end{aligned}$$

where (*) and (**) hold up to a set of measure $\frac{1}{rr_{n_k+2}} \mu(A^{(k)})$ and $\frac{2}{rr_{n_k+2}} \mu(A^{(k)})$, respectively. Therefore up to an error of absolute value at most $\frac{3}{rr_{n_k+2}} \mu(A^{(k)})$

$$\begin{aligned} \mu_{T^{2q_{n_k}}, T^{q_{n_k}}} \left(A^{(k)} \times TA^{(k)} \times T^2 A^{(k)} \right) &= \mu \left(T^{-2q_{n_k}} A^{(k)} \cap T^{-q_{n_k}+1} A^{(k)} \cap T^2 A^{(k)} \right) \\ &= \mu \left(A^{(k)} \cap TT^{q_{n_k}} A^{(k)} \cap T^2 T^{2q_{n_k}} A^{(k)} \right) \\ &\simeq \mu \left(\bigcup_{r/2+3 \leq i \leq r} A_i^{(k)} \cup \bigcup_{2 \leq j \leq \lceil \frac{r_{n_k+1}}{2} \rceil + 1} A_{1,j}^{(k)} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{r-r/2-3+1}{r} \cdot \mu(A^{(k)}) + \frac{\lfloor \frac{r_{n_k+1}}{2} \rfloor + 1 - 2 + 1}{rr_{n_k+2}} \cdot \mu(A^{(k)}) \\
&= \frac{r-4}{2r} \cdot \mu(A^{(k)}) + \frac{\lfloor \frac{r_{n_k+1}}{2} \rfloor}{rr_{n_k+2}} \mu(A^{(k)}) = \begin{cases} \frac{r-3}{2r} \mu(A^{(k)}), & 2|r_{n_k+1} \\ \frac{r-3}{2r} \mu(A^{(k)}) - \frac{1}{2rr_{n_k+1}} \mu(A^{(k)}), & 2 \nmid r_{n_k+1}, \end{cases}
\end{aligned}$$

i.e. (7.5) indeed holds. In a similar way, all of the inequalities (7.4) hold. We obtain

$$\begin{aligned}
&\left| \mu_{T^{2q_{n_k}}, T^{q_{n_k}}} (A[1] \times TA[1] \times T^2A[1]) \right. \\
&\quad \left. - \frac{r-3}{2r} \mu_{T^{-2}, T^{-1}} (A[1] \times TA[1] \times T^2A[1]) \right| \\
&\stackrel{(7.2)}{=} \left| \mu_{T^{2q_{n_k}}, T^{q_{n_k}}} \left(\left(\bigcup_{l=1}^{l_k} A^{(k)}[1, l] \right) \times \left(\bigcup_{l=1}^{l_k} TA^{(k)}[1, l] \right) \times \left(\bigcup_{l=1}^{l_k} T^2A^{(k)}[1, l] \right) \right) \right. \\
&\quad \left. - \frac{r-3}{2r} \mu_{T^{-2}, T^{-1}} \left(\left(\bigcup_{l=1}^{l_k} A^{(k)}[1, l] \right) \times \left(\bigcup_{l=1}^{l_k} TA^{(k)}[1, l] \right) \times \left(\bigcup_{l=1}^{l_k} T^2A^{(k)}[1, l] \right) \right) \right| \\
&\leq \sum_{l=1}^{l_k} \left| \mu_{T^{2q_{n_k}}, T^{q_{n_k}}} (A^{(k)}[1, l] \times TA^{(k)}[1, l] \times T^2A^{(k)}[1, l]) \right. \\
&\quad \left. - \frac{r-3}{2r} \mu_{T^{-2}, T^{-1}} (A^{(k)}[1, l] \times TA^{(k)}[1, l] \times T^2A^{(k)}[1, l]) \right| \\
&\quad + \sum_{l=1}^{l_k} \sum_{\substack{1 \leq l', l'' \leq l_k \\ \#\{l, l', l''\} > 1}} \mu_{T^{2q_{n_k}}, T^{q_{n_k}}} (A^{(k)}[1, l] \times TA^{(k)}[1, l'] \times T^2A^{(k)}[1, l'']) \\
&\quad + \frac{r-3}{2r} \sum_{l=1}^{l_k} \sum_{\substack{1 \leq l', l'' \leq l_k \\ \#\{l, l', l''\} > 1}} \mu_{T^{-2}, T^{-1}} (A^{(k)}[1, l] \times TA^{(k)}[1, l'] \times T^2A^{(k)}[1, l'']) \\
&= \sum_{l=1}^{l_k} \left| \mu_{T^{2q_{n_k}}, T^{q_{n_k}}} (A^{(k)}[1, l] \times TA^{(k)}[1, l] \times T^2A^{(k)}[1, l]) \right. \\
&\quad \left. - \frac{r-3}{2r} \mu (T^2A^{(k)}[1, l] \cap T^2A^{(k)}[1, l] \cap T^2A^{(k)}[1, l]) \right| \\
&\quad + \sum_{l=1}^{l_k} \sum_{\substack{1 \leq l', l'' \leq l_k \\ \#\{l, l', l''\} > 1}} \mu_{T^{2q_{n_k}}, T^{q_{n_k}}} (A^{(k)}[1, l] \times TA^{(k)}[1, l'] \times T^2A^{(k)}[1, l'']) \\
&\quad + \frac{r-3}{2r} \sum_{l=1}^{l_k} \sum_{\substack{1 \leq l', l'' \leq l_k \\ \#\{l, l', l''\} > 1}} \mu (T^2A^{(k)}[1, l] \cap T^2A^{(k)}[1, l'] \cap T^2A^{(k)}[1, l'']) \\
&= \sum_{l=1}^{l_k} \left| \mu_{T^{2q_{n_k}}, T^{q_{n_k}}} (A^{(k)}[1, l] \times TA^{(k)}[1, l] \times T^2A^{(k)}[1, l]) - \frac{r-3}{2r} \mu(A^{(k)}[1, l]) \right| \\
&\quad + \sum_{l=1}^{l_k} \sum_{\substack{1 \leq l', l'' \leq l_k \\ \#\{l, l', l''\} > 1}} \mu_{T^{2q_{n_k}}, T^{q_{n_k}}} (A^{(k)}[1, l] \times TA^{(k)}[1, l'] \times T^2A^{(k)}[1, l'']) \\
&\stackrel{(7.4)}{<} \sum_{l=1}^{l_k} \varepsilon \mu (A^{(k)}[1, l])
\end{aligned}$$

$$\begin{aligned}
& + \sum_{l=1}^{l_k} \sum_{\substack{1 \leq l'' \leq l_k \\ l'' \neq l}} \mu_{T^{2q_{n_k}}, T^{q_{n_k}}} \left(A^{(k)}[1, l] \times TA^{(k)}[1, l] \times T^2 A^{(k)}[1, l''] \right) \\
& + \sum_{l=1}^{l_k} \sum_{\substack{1 \leq l' \leq l_k \\ l' \neq l}} \mu_{T^{2q_{n_k}}, T^{q_{n_k}}} \left(A^{(k)}[1, l] \times TA^{(k)}[1, l'] \times T^2 A^{(k)}[1, l] \right) \\
& + \sum_{l=1}^{l_k} \sum_{\substack{1 \leq l', l'' \leq l_k \\ l', l'' \neq l}} \mu_{T^{2q_{n_k}}, T^{q_{n_k}}} \left(A^{(k)}[1, l] \times TA^{(k)}[1, l'] \times T^2 A^{(k)}[1, l''] \right) \\
& = \varepsilon \mu(A[1]) \\
& + \sum_{l=1}^{l_k} \sum_{\substack{1 \leq l'' \leq l_k \\ l'' \neq l}} \mu \left(T^{-2q_{n_k}} A^{(k)}[1, l] \cap T^{-q_{n_k}} TA^{(k)}[1, l] \cap T^2 A^{(k)}[1, l''] \right) \\
& + \sum_{l=1}^{l_k} \sum_{\substack{1 \leq l' \leq l_k \\ l' \neq l}} \mu \left(T^{-2q_{n_k}} A^{(k)}[1, l] \cap T^{-q_{n_k}} TA^{(k)}[1, l'] \cap T^2 A^{(k)}[1, l] \right) \\
& + \sum_{l=1}^{l_k} \sum_{\substack{1 \leq l', l'' \leq l_k \\ l', l'' \neq l}} \mu \left(T^{-2q_{n_k}} A^{(k)}[1, l] \cap T^{-q_{n_k}} TA^{(k)}[1, l'] \cap T^2 A^{(k)}[1, l''] \right) \\
& \leq \varepsilon \mu(A[1]) + \sum_{l=1}^{l_k} \sum_{\substack{1 \leq l'' \leq l_k \\ l'' \neq l}} \mu \left(T^{-2q_{n_k}} A^{(k)}[1, l] \cap T^2 A^{(k)}[1, l''] \right) \\
& + 2 \sum_{l=1}^{l_k} \sum_{\substack{1 \leq l' \leq l_k \\ l' \neq l}} \mu \left(T^{-2q_{n_k}} A^{(k)}[1, l] \cap T^{-q_{n_k}} TA^{(k)}[1, l'] \right) \\
& = \varepsilon \mu(A[1]) + \sum_{l=1}^{l_k} \sum_{\substack{1 \leq l'' \leq l_k \\ l'' \neq l}} \mu \left(A^{(k)}[1, l] \cap T^{2q_{n_k}} T^2 A^{(k)}[1, l''] \right) \\
& + 2 \sum_{l=1}^{l_k} \sum_{\substack{1 \leq l' \leq l_k \\ l' \neq l}} \mu \left(A^{(k)}[1, l] \cap T^{q_{n_k}} TA^{(k)}[1, l'] \right) \\
& = \varepsilon \mu(A[1]) + \sum_{\substack{l''=1 \\ l \neq l''}}^{l_k} \sum_{1 \leq l \leq l_k} \mu \left(A^{(k)}[1, l] \cap T^{2q_{n_k}} T^2 A^{(k)}[1, l''] \right) \\
& + 2 \sum_{\substack{l'=1 \\ l \neq l'}}^{l_k} \sum_{1 \leq l \leq l_k} \mu \left(A^{(k)}[1, l] \cap T^{q_{n_k}} TA^{(k)}[1, l'] \right) \\
& = \varepsilon \mu(A[1]) + \sum_{l''=1}^{l_k} \mu \left(\left(\bigcup_{\substack{1 \leq l \leq l_k \\ l \neq l''}} A^{(k)}[1, l] \right) \cap T^{2q_{n_k}} T^2 A^{(k)}[1, l''] \right)
\end{aligned}$$

$$\begin{aligned}
& + 2 \sum_{l'=1}^{l_k} \mu \left(\left(\bigcup_{\substack{1 \leq l \leq l_k \\ l \neq l'}} A^{(k)}[1, l] \right) \cap T^{q_{n_k}} T A^{(k)}[1, l'] \right) \\
& \stackrel{(7.13), (7.14), (7.3)}{\leq} \varepsilon \mu(A[1]) + \sum_{l''=1}^{l_k} \frac{2}{r r_{n_k+2}} \mu(A^{(k)}[1, l'']) + 2 \sum_{l'=1}^{l_k} \frac{1}{r r_{n_k+2}} \mu(A^{(k)}[1, l']) \\
& = \left(\varepsilon + \frac{4}{r r_{n_k+2}} \right) \mu(A[1]).
\end{aligned}$$

Hence

$$\begin{aligned}
(7.15) \quad & \mu_{T^{2q_{n_k}}, T^{q_{n_k}}} (A[1] \times T A[1] \times T^2 A[1]) \\
& \rightarrow \frac{r-3}{2r} \mu_{T^{-2}, T^{-1}} (A[1] \times T A[1] \times T^2 A[1]) = \frac{r-3}{2r} \mu(A[1]).
\end{aligned}$$

In a similar way

$$\begin{aligned}
(7.16) \quad & \mu_{T^{2q_{n_k}}, T^{q_{n_k}}} (A[1] \times A[1] \times A[1]) \\
& \rightarrow \frac{r-2}{2r} \mu_{I, I} (A[1] \times A[1] \times A[1]) = \frac{r-2}{2r} \mu(A[1]),
\end{aligned}$$

$$\begin{aligned}
(7.17) \quad & \mu_{T^{2q_{n_k}}, T^{q_{n_k}}} (A[1] \times A[1] \times T A[1]) \\
& \rightarrow \frac{1}{2r} \mu_{T^{-1}, T^{-1}} (A[1] \times A[1] \times T A[1]) = \frac{1}{2r} \mu(A[1]),
\end{aligned}$$

$$\begin{aligned}
(7.18) \quad & \mu_{T^{2q_{n_k}}, T^{q_{n_k}}} (A[1] \times A[1] \times T^2 A[1]) \\
& \rightarrow \frac{1}{2r} \mu_{T^{-2}, T^{-2}} (A[1] \times A[1] \times T^2 A[1]) = \frac{1}{2r} \mu(A[1]),
\end{aligned}$$

$$\begin{aligned}
(7.19) \quad & \mu_{T^{2q_{n_k}}, T^{q_{n_k}}} (A[1] \times T A[1] \times T A[1]) \\
& \rightarrow \frac{1}{r} \mu_{T^{-1}, I} (A[1] \times T A[1] \times T A[1]) = \frac{1}{r} \mu(A[1]),
\end{aligned}$$

$$\begin{aligned}
(7.20) \quad & \mu_{T^{2q_{n_k}}, T^{q_{n_k}}} (A[1] \times T^2 A[1] \times T^3 A[1]) \\
& \rightarrow \frac{1}{2r} \mu_{T^{-3}, T^{-1}} (A[1] \times T^2 A[1] \times T^3 A[1]) = \frac{1}{2r} \mu(A[1]).
\end{aligned}$$

Let $I = \{(0, 0), (0, 1), (0, 2), (1, 1), (1, 2), (2, 3)\}$. Since

$$\frac{r-3}{2r} + \frac{r-2}{2r} + \frac{1}{2r} + \frac{1}{2r} + \frac{1}{r} + \frac{1}{2r} = 1,$$

we have

$$\mu_{T^{2q_{n_k}}, T^{q_{n_k}}} \left(\bigcup_{(p_2, p_3) \in I} A[1] \times T^{p_2} A[1] \times T^{p_3} A[1] \right) \rightarrow \mu(A[1]).$$

Notice that for $(p'_2, p'_3) \notin I$ such that $T^{p'_2} A[1]$ and $T^{p'_3} A[1]$ are levels of tower $C_{n_{k_0}}$

$$A[1] \times T^{p'_2} A[1] \times T^{p'_3} A[1] \subset A[1] \times \left(\bigcup_{(p_2, p_3) \in I} T^{p_2} A[1] \times T^{p_3} A[1] \right)^c.$$

Therefore

$$\begin{aligned}
& \mu_{T^{2q_{n_k}}, T^{q_{n_k}}} \left(A[1] \times T^{p'_2} A[1] \times T^{p'_3} A[1] \right) \\
& \leq \mu_{T^{2q_{n_k}}, T^{q_{n_k}}} \left(A[1] \times \left(\bigcup_{(p_2, p_3) \in I} T^{p_2} A[1] \times T^{p_3} A[1] \right)^c \right) \\
(7.21) \quad & = \mu(A[1]) - \mu_{T^{2q_{n_k}}, T^{q_{n_k}}} \left(A[1] \times \left(\bigcup_{(p_2, p_3) \in I} T^{p_2} A[1] \times T^{p_3} A[1] \right) \right) \\
& \rightarrow \mu(A[1]) - \mu(A[1]) = 0.
\end{aligned}$$

Let now

$$P = \frac{r-2}{2r} \delta_{(0,0)} + \frac{1}{2r} \delta_{(1,1)} + \frac{1}{2r} \delta_{(2,2)} + \frac{1}{r} \delta_{(1,0)} + \frac{r-3}{2r} \delta_{(2,1)} + \frac{1}{2r} \delta_{(3,1)}.$$

Notice that for $(p'_2, p'_3) \notin I$ such that $T^{p'_2} A[1], T^{p'_3} A[1]$ are levels of tower $C_{n_{k_0}}$ we have

$$(7.22) \quad \int_{\mathbb{Z}^2} \mu_{T^{-a}, T^{-b}} \left(A[1] \times T^{p'_2} A[1] \times T^{p'_3} A[1] \right) dP(a, b) = 0.$$

Using eqs. (7.15)–(7.22) we obtain

$$\mu_{T^{2q_{n_k}}, T^{q_{n_k}}} (A[1] \times A[2] \times A[3]) \rightarrow \int_{\mathbb{Z}^2} \mu_{T^{-a}, T^{-b}} (A[1] \times A[2] \times A[3]) dP(a, b).$$

This implies (7.1) and the claim follows. \square

Remark 7.3. In the same way, one can show that some rank one flows are not reversible. The construction of such flows is similar to the rank one automorphisms considered in the above theorem. They are also determined by a sequence of integers $(r_n)_{n \in \mathbb{N}}$ which denote the number of subcolumns at each step of the construction. Now, the role of spacers is played by rectangles placed above the subcolumns. The additional assumption in the “flow version” of our theorem is that along the subsequence (n_k) the rectangles are of fixed height and at steps $n_k, n_k + 1$ they are placed over the same subcolumns as in Theorem 7.2.

Remark 7.4. A similar method can be used to show non-reversibility of the classical Chacon’s automorphism, i.e. the rank one automorphism which can be constructed as described in the beginning of this section, dividing the column at each step of the construction into three subcolumns and placing a spacer above the middle one. More precisely, to show that this automorphism is not reversible, one can use the “automorphism counterpart” of Corollary 3.12.

8. TOPOLOGICAL SELF-SIMILARITIES OF SPECIAL FLOWS

In this section we will deal with topological self-similarities of continuous flows $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ on a compact metric spaces. For each such flow denote by $I_{top}(\mathcal{T})$ the subgroup of all $s \in \mathbb{R}^*$ such that the flows \mathcal{T} and $\mathcal{T} \circ s$ are topologically conjugate. If $I_{top}(\mathcal{T}) \not\subseteq \{-1, 1\}$ then the flows \mathcal{T} is called topologically self-similar. More precisely we will deal with continuous time changes of minimal linear flows on the two torus. Each such flow is topologically conjugate to the special flow T^f build over an irrational rotation $Tx = x + \alpha$ on the circle and under a continuous roof function $f : \mathbb{T} \rightarrow \mathbb{R}_+$.

We will show that if T^f is topologically self-similar then α must be a quadratic irrational and f is topologically cohomological to a constant function. It follows that if a continuous time change of a minimal linear flow on the two torus is topologically self-similar then it is topologically conjugate to a minimal linear flows on the two torus.

Let (X, d) be a compact connected topological manifold. Then X is locally path connected and locally simply connected, so denote by \tilde{X} the universal covering space of X and let $\pi' : \tilde{X} \rightarrow X$ be the covering map. Denote by Θ the deck transformation group of the covering $\pi' : \tilde{X} \rightarrow X$, i.e. Θ is the group of homeomorphisms $\theta : \tilde{X} \rightarrow \tilde{X}$ such that $\pi' \circ \theta = \pi'$. Then Θ is countable (isomorphic to the fundamental group of X) and it acts in the properly discontinuous way, that is, for each $x \in X$ there exists an open $V \ni x$ such that $\theta(V) \cap V = \emptyset$ whenever $Id \neq \theta \in \Theta$. In what follows we need the following simple observation.

Lemma 8.1. *Assume that Z is a topological space and let G be a countable group (considered with the discrete topology). Assume that G acts on Z as homeomorphisms in the properly discontinuous way. Assume moreover that $\Phi : \mathbb{R} \rightarrow G$ and that*

$$\mathbb{R} \ni t \mapsto \Phi(t)z \in Z$$

is continuous for each $z \in Z$. Then Φ is continuous, hence constant.

Proof. Fix $t_0 \in \mathbb{R}$ and $z \in Z$. Select an open $V \ni z$, so that $gV \cap V = \emptyset$ whenever $1 \neq g \in G$. By the continuity assumption, there is an open interval $W \ni t_0$ such that $\Phi(t)(z) \in \Phi(t_0)(V)$ for $t \in W$. Thus $\Phi(t) = \Phi(t_0)$. It follows directly that the map $W \ni t \mapsto \Phi(t)$ is constant, whence Φ is continuous. \square

Let $T : X \rightarrow X$ be a homeomorphism and $f : X \rightarrow \mathbb{R} \setminus \{0\}$ a continuous function, which is globally either positive or negative. Let us consider the skew product $T_{-f} : X \times \mathbb{R} \rightarrow X \times \mathbb{R}$, $T_{-f}(x, r) = (Tx, r - f(x))$ and the orbit equivalence relation \equiv on $X \times \mathbb{R}$ defined by $(x_2, r_2) \equiv (x_1, r_1)$ if there exists $n \in \mathbb{Z}$ such that $(x_2, r_2) = T_{-f}^n(x_1, r_1)$. Denote by X^f the quotient space $(X \times \mathbb{R}) / \equiv$. Then X^f is a compact topological manifold and the canonical projection $\pi_1 = \pi_1^f : X \times \mathbb{R} \rightarrow X^f$ is a covering map.

Let us consider the (continuous) flow $(\sigma_t)_{t \in \mathbb{R}}$ on $X \times \mathbb{R}$ given by $\sigma_t(x, r) = (x, r + t)$. Since T_{-f}^n commutes with σ_t for every $n \in \mathbb{Z}$ and $t \in \mathbb{R}$, each σ_t transforms the equivalence classes for \equiv into the equivalence classes. Therefore $(\sigma_t)_{t \in \mathbb{R}}$ defines a continuous flow on X^f , this flow is denoted by T^f . If the function f is positive the flow T^f is called the special flow built over the homeomorphism T and under the roof function f . Of course, $T_t^f \circ \pi_1 = \pi_1 \circ \sigma_t$ for every $t \in \mathbb{R}$.

Then, the map $\pi_2 : \tilde{X} \times \mathbb{R} \rightarrow X \times \mathbb{R}$ given by $\pi_2(\tilde{x}, r) = (\pi'(\tilde{x}), r)$ is a covering map and $\tilde{X} \times \mathbb{R}$ is the universal covering of X^f with the covering map $\pi^f = \pi_1^f \circ \pi_2$.

For every $\theta \in \Theta$ denote by $\underline{\theta} : \tilde{X} \times \mathbb{R} \rightarrow \tilde{X} \times \mathbb{R}$ the trivial extension $\underline{\theta}(\tilde{x}, r) = (\theta(\tilde{x}), r)$. Note that $\underline{\theta}$ belongs to the deck group of the universal covering π^f .

Let $\tilde{T} : \tilde{X} \rightarrow \tilde{X}$ be a lift of $T : X \rightarrow X$. Recall that $\tilde{T} : \tilde{X} \rightarrow \tilde{X}$ is a homeomorphism. Let us consider the group automorphism $\gamma : \Theta \rightarrow \Theta$ given by

$$\gamma(\theta) = \tilde{T} \circ \theta \circ \tilde{T}^{-1}$$

and the semidirect product $\Theta \rtimes_{\gamma} \mathbb{Z}$ with multiplication

$$(\theta, m) \cdot (\theta', m') = (\theta \circ \gamma^m(\theta'), m + m').$$

Denote by $\tilde{T}_{-\tilde{f}} : \tilde{X} \times \mathbb{R} \rightarrow \tilde{X} \times \mathbb{R}$ the skew product $\tilde{T}_{-\tilde{f}}(\tilde{x}, r) = (\tilde{T}(\tilde{x}), r - \tilde{f}(\tilde{x}))$, where $\tilde{f} = f \circ \pi'$.

Proposition 8.2 (Proposition 1.1 in [20]). *The deck transformation group Θ^f of the universal covering $\pi^f : \tilde{X} \times \mathbb{R} \rightarrow X^f$ is equal to*

$$\{\underline{\theta} \circ \tilde{T}_{-\tilde{f}}^m : \theta \in \Theta, m \in \mathbb{Z}\}$$

and $(\theta, m) \mapsto \underline{\theta} \circ \tilde{T}_{-\tilde{f}}^m$ establishes the group isomorphism of $\Theta \rtimes_{\gamma} \mathbb{Z}$ and Θ^f .

We will identify the groups $\Theta \rtimes_{\gamma} \mathbb{Z}$ and Θ^f .

For any $s \in \mathbb{R}^* \setminus \{1\}$ let us consider the flow $T^f \circ (s^{-1}) = (T_{s^{-1}t}^f)_{t \in \mathbb{R}}$ on X^f . This flow is topologically isomorphic to the flow $(T_t^{sf})_{t \in \mathbb{R}}$ on X^{sf} . Indeed, the homeomorphism $U : X \times \mathbb{R} \rightarrow X \times \mathbb{R}$ given by $U(x, r) = (x, sr)$ satisfies

$$U \circ T_{-f} = T_{-sf} \circ U \quad \text{and} \quad U \circ \sigma_{s^{-1}t} = \sigma_t \circ U.$$

Therefore, U induces a homeomorphism $U' : X^f \rightarrow X^{sf}$ with $U' \circ T_{s^{-1}t}^f = T_t^{sf} \circ U'$.

Suppose that $s \in I_{top}(T^f) \setminus \{1\}$. As the flows T^f and $T^f \circ (s^{-1})$ on X^f are topologically isomorphic, the flows $(T_t^f)_{t \in \mathbb{R}}$ on X^f and $(T_t^{sf})_{t \in \mathbb{R}}$ on X^{sf} are also topologically isomorphic. Thus there exists a homeomorphism $S : X^f \rightarrow X^{sf}$ such that $S \circ T_t^f = T_t^{sf} \circ S$. Let $\tilde{S} : \tilde{X} \times \mathbb{R} \rightarrow \tilde{X} \times \mathbb{R}$ be a lift of S . Then \tilde{S} is a homeomorphism such that $S \circ \pi^f = \pi^{sf} \circ \tilde{S}$. Since $S^{-1} \circ T_{-t}^{sf} \circ S \circ T_t^f = id_{X^f}$, its lift $\tilde{S}^{-1} \circ \sigma_{-t} \circ \tilde{S} \circ \sigma_t$ is an element of the deck transformation group Θ^f , so there exists a map $\mathbb{R} \ni t \mapsto (\theta(t), m(t)) \in \Theta \rtimes_{\gamma} \mathbb{Z}$ such that

$$\tilde{S}^{-1} \circ \sigma_{-t} \circ \tilde{S} \circ \sigma_t = \underline{\theta}(t) \circ \tilde{T}_{-f}^{m(t)}.$$

Now, for each $(\tilde{x}, r) \in \tilde{X} \times \mathbb{R}$ the map

$$\mathbb{R} \ni t \mapsto \tilde{S}^{-1} \circ \sigma_{-t} \circ \tilde{S} \circ \sigma_t(\tilde{x}, r) \in \tilde{X} \times \mathbb{R}$$

is continuous. By Lemma 8.1 applied to Θ^f we obtain that the map $t \mapsto (\theta(t), m(t))$ is constant. Moreover, $(\theta(0), m(0)) = (id_X, 0)$, so

$$(8.1) \quad \tilde{S} \circ \sigma_t = \sigma_t \circ \tilde{S}.$$

For every $\tilde{\theta} \in \Theta^f$ the homeomorphism $\tilde{S} \circ \tilde{\theta} \circ \tilde{S}^{-1}$ is a deck transformation of π^{sf} , so there exists $A : \Theta^f \rightarrow \Theta^{sf}$ such that

$$(8.2) \quad \tilde{S} \circ \tilde{\theta} \circ \tilde{S}^{-1} = A(\tilde{\theta}).$$

Moreover, $A : \Theta^f \rightarrow \Theta^{sf}$ is a group isomorphism which can be identified with the automorphism $A : \Theta \rtimes_{\gamma} \mathbb{Z} \rightarrow \Theta \rtimes_{\gamma} \mathbb{Z}$.

Let $\tilde{S} = (S_1, S_2)$, where $S_1 : \tilde{X} \times \mathbb{R} \rightarrow \tilde{X}$ and $S_2 : \tilde{X} \times \mathbb{R} \rightarrow \mathbb{R}$. Let $A = (A_1, A_2)$, where $A_1 : \Theta \rtimes_{\gamma} \mathbb{Z} \rightarrow \Theta$ and $A_2 : \Theta \rtimes_{\gamma} \mathbb{Z} \rightarrow \mathbb{Z}$. In view of (8.1),

$$\tilde{S}(\tilde{x}, t) = \tilde{S} \circ \sigma_t(\tilde{x}, 0) = \sigma_t \circ \tilde{S}(\tilde{x}, 0) = (S_1(\tilde{x}, 0), S_2(\tilde{x}, 0) + t).$$

Therefore

$$\tilde{S}(\tilde{x}, t) = (V(\tilde{x}), t + g(\tilde{x})),$$

where $V : \tilde{X} \rightarrow \tilde{X}$ is a homeomorphism and $g : \tilde{X} \rightarrow \mathbb{R}$ is a continuous function. Note that if (θ, m) denotes $\underline{\theta} \circ \tilde{T}_{-f}^m$ then

$$\begin{aligned} \tilde{S} \circ (\theta, m)(\tilde{x}, r) &= \tilde{S}(\theta \circ \tilde{T}^m(\tilde{x}), r - \tilde{f}^{(m)}(\tilde{x})) \\ &= (V \circ \theta \circ \tilde{T}^m(\tilde{x}), r - \tilde{f}^{(m)}(\tilde{x}) + g(\theta \circ \tilde{T}^m(\tilde{x}))), \end{aligned}$$

while if we set $(\theta, m) = \underline{\theta} \circ \tilde{T}_{-sf}^m$ then

$$\begin{aligned} (\theta, m) \circ \tilde{S}(\tilde{x}, r) &= (\theta, m)(V(\tilde{x}), r + g(\tilde{x})) \\ &= (\theta \circ \tilde{T}^m \circ V(\tilde{x}), r + g(\tilde{x}) - s\tilde{f}^{(m)}(V(\tilde{x}))) \end{aligned}$$

Therefore, in view of (8.2), we have

$$\begin{aligned} V \circ \theta \circ \tilde{T}^m(\tilde{x}) &= A_1(\theta, m) \circ \tilde{T}^{A_2(\theta, m)} \circ V(\tilde{x}) \\ g(\theta \circ \tilde{T}^m(\tilde{x}) - \tilde{f}^{(m)}(\tilde{x})) &= g(\tilde{x}) - s\tilde{f}^{(A_2(\theta, m))}(V(\tilde{x})). \end{aligned}$$

Let us consider the action of the group $\Theta \rtimes_{\gamma} \mathbb{Z}$ on \tilde{X} defined by $(\theta, m)(\tilde{x}) = \theta \circ \tilde{T}^m(\tilde{x})$. Then as a conclusion we have the following.

Theorem 8.3. *The number $s \in I_{top}(T^f) \setminus \{1\}$ if and only if there exist a homeomorphism $V : \tilde{X} \rightarrow \tilde{X}$, a group automorphism $A : \Theta \rtimes_{\gamma} \mathbb{Z} \rightarrow \Theta \rtimes_{\gamma} \mathbb{Z}$ and a continuous function $g : \tilde{X} \rightarrow \mathbb{R}$ such that for every $(\theta, m) \in \Theta \rtimes_{\gamma} \mathbb{Z}$*

$$\begin{aligned} V \circ (\theta, m)(\tilde{x}) &= A(\theta, m) \circ V(\tilde{x}) \\ {}_s\tilde{f}^{(A_2(\theta, m))}(V(\tilde{x})) - \tilde{f}^{(m)}(\tilde{x}) &= g(\tilde{x}) - g((\theta, m)(\tilde{x})). \end{aligned}$$

Remark 8.4. If T is uniquely ergodic, then so is T^f and therefore in this case $I_{top}(T^f) \subset I_{T^f}$.

8.1. Special flows over irrational rotations. Suppose that T is the rotation by an irrational number $\alpha \in \mathbb{R}$ on the additive circle $X = \mathbb{T} = \mathbb{R}/\mathbb{Z}$. Then $\tilde{X} = \mathbb{R}$ and the deck transformation group is the group of translations of \mathbb{R} by integer numbers, so $\Theta = \mathbb{Z}$ with $n(x) = x + n$. As each such translation commutes with the lift $\tilde{T} : \mathbb{R} \rightarrow \mathbb{R}$, $\tilde{T}x = x + \alpha$, $\gamma = id_{\mathbb{Z}}$. Thus $\Theta^f = \mathbb{Z} \times \mathbb{Z}$ and the action of this group on $\tilde{X} = \mathbb{R}$ is given by

$$(8.3) \quad (n, m)x = x + n + m\alpha.$$

We will now prove the following result describing topological self-similarities of T^f whose second part is to be compared with Remark 2.3.

Proposition 8.5. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a continuous positive function. Then $s \in I_{top}(T^f) \setminus \{1\}$ if and only if there exist a matrix $[a_{ij}] = A \in GL_2(\mathbb{Z})$, $\delta \in \mathbb{R}$ and a continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that*

$$(8.4) \quad a_{12} + a_{22}\alpha = (a_{11} + a_{21}\alpha)\alpha,$$

$$(8.5) \quad a_{11} + a_{21}\alpha = \sigma s = \sigma(a_{22} - a_{21}\alpha)^{-1}$$

$$(8.6) \quad {}_s\tilde{f}^{(a_{21}n + a_{22}m)}(\sigma s x + \delta) - \tilde{f}^{(m)}(x) = g(x) - g(x + n + m\alpha),$$

for all $m, n \in \mathbb{Z}$, where $\sigma = \det A$.

Moreover, $-1 \in I_{top}(T^f)$ if and only if there exist a continuous map $g : \mathbb{T} \rightarrow \mathbb{R}$ and $\delta \in \mathbb{T}$ such that

$$(8.7) \quad f(\delta - x) - f(x) = g(x) - g(x + \alpha).$$

Proof. By Theorem 8.3 and (8.3), $s \in I_{top}(T^f) \setminus \{1\}$ if and only if there exist a homeomorphism $V : \mathbb{R} \rightarrow \mathbb{R}$, a group automorphism

$$A : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2, \quad A(n, m) = (a_{11}n + a_{12}m, a_{21}n + a_{22}m) \quad \text{with } A := [a_{ij}] \in GL_2(\mathbb{Z})$$

and a continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that for every $(n, m) \in \mathbb{Z}^2$ and $x \in \mathbb{R}$

$$(8.8) \quad V(x + n + m\alpha) = V(x) + (a_{11}n + a_{12}m) + (a_{21}n + a_{22}m)\alpha$$

$$(8.9) \quad {}_s\tilde{f}^{(a_{21}n + a_{22}m)}(V(x)) - \tilde{f}^{(m)}(x) = g(x) - g(x + n + m\alpha).$$

Let us consider $v : \mathbb{R} \rightarrow \mathbb{R}$, $v(x) = V(x) - (a_{11} + a_{21}\alpha)x$. In view of (8.8), we obtain

$$\begin{aligned} v(x + n + m\alpha) &= V(x + n + m\alpha) - (a_{11} + a_{21}\alpha)(x + n + m\alpha) \\ &= v(x) + (a_{12} + a_{22}\alpha - (a_{11} + a_{21}\alpha)\alpha)m. \end{aligned}$$

It follows that v is \mathbb{Z} -periodic, so $v : \mathbb{T} \rightarrow \mathbb{R}$. Moreover (by taking $n = 0$ and $m = 1$ above), $v(x + \alpha) = v(x) + a_{12} + a_{22}\alpha - (a_{11} + a_{21}\alpha)\alpha$, so

$$\int_{\mathbb{T}} v(x) dx = \int_{\mathbb{T}} v(x + \alpha) dx = \int_{\mathbb{T}} v(x) dx + a_{12} + a_{22}\alpha - (a_{11} + a_{21}\alpha)\alpha.$$

Thus $a_{12} + a_{22}\alpha = (a_{11} + a_{21}\alpha)\alpha$, so (8.4) holds. Moreover, v is a constant function and $V(x) = \gamma x + \delta$ with $\gamma := a_{11} + a_{21}\alpha$ and some real δ .

Case 1. Suppose that a_{12} or a_{21} is equal to zero. As α is irrational and by (8.4), $a_{12} + (a_{22} - a_{11})\alpha - a_{21}\alpha^2 = 0$, it follows that $a_{12} = a_{21} = 0$ and $a_{11} = a_{22} = \pm 1$. Hence $V(x) = \pm x + \delta$ and, by (8.9),

$$s\tilde{f}^{(\pm m)}(\pm x + \delta) - \tilde{f}^{(m)}(x) = g(x) - g(x + n + m\alpha).$$

Setting $m = 0$ we have $g(x + n) = g(x)$, so g is \mathbb{Z} -periodic. Therefore, g can be treated as a map on \mathbb{T} and taking $m = 1$ we have

$$sf^{(\pm 1)}(\pm x + \delta) - f(x) = g(x) - g(x + \alpha) \quad \text{for all } x \in \mathbb{T}.$$

Recalling that $f^{(-1)}(y) = -f(y - \alpha)$, it follows that

$$\begin{aligned} (\pm s - 1) \int_{\mathbb{T}} f(x) dx &= s \int_{\mathbb{T}} f^{(\pm 1)}(\pm x + \delta) dx - \int_{\mathbb{T}} f(x) dx \\ &= \int_{\mathbb{T}} g(x) dx - \int_{\mathbb{T}} g(x + \alpha) dx = 0. \end{aligned}$$

Since $s \neq 1$ and f is positive, it follows that $s = -1$ and $a_{11} = a_{22} = -1$. Therefore, (8.5), (8.6) and (8.7) hold.

Case 2. Suppose that both a_{12} and a_{21} are non-zero. Since $a_{12} + (a_{22} - a_{11})\alpha - a_{21}\alpha^2 = 0$, the irrational number α is a quadratic irrational. In view of (8.9) (by substituting m by 0, and by substituting n by $-a_{22}m$ and m by $a_{21}m$, respectively), we obtain

$$(8.10) \quad s\tilde{f}^{(a_{21}n)}(\gamma x + \delta) = g(x) - g(x + n),$$

$$(8.11) \quad -\tilde{f}^{(a_{21}m)}(x) = g(x) - g(x + (a_{21}\alpha - a_{22})m).$$

It follows that for every $x \in \mathbb{R}$

$$(8.12) \quad \lim_{|y| \rightarrow \infty} \frac{g(x) - g(x + y)}{y} = sa_{21} \int_{\mathbb{T}} f dx.$$

Indeed, if $|y|$ is large enough

$$\begin{aligned} \frac{g(x) - g(x + y)}{y} &= \frac{g(x) - g(x + \{y\})}{y} + \frac{g(x + \{y\}) - g(x + \{y\} + [y])}{[y]} \frac{[y]}{y} \\ &\stackrel{(8.10)}{=} \frac{g(x) - g(x + \{y\})}{y} + sa_{21} \frac{f^{(a_{21}[y])}(V(x + \{y\}))}{a_{21}[y]} \frac{[y]}{y}. \end{aligned}$$

Since $|g(x) - g(x + \{y\})| \leq 2\|g\|_{C[x, x+1]}$, $f^{(n)}/n$ tends to $\int_{\mathbb{T}} f dx$ uniformly and $[y]/y \rightarrow 1$ as $|y| \rightarrow \infty$, we get (8.12). Furthermore (by taking $y = (a_{21}\alpha - a_{22})m$ in (8.12)),

$$\begin{aligned} \frac{-\int_{\mathbb{T}} f dx}{a_{21}\alpha - a_{22}} &\leftarrow \frac{-\tilde{f}^{(a_{21}m)}(x)/a_{21}m}{(a_{21}\alpha - a_{22})} \\ &\stackrel{(8.11)}{=} \frac{g(x) - g(x + (a_{21}\alpha - a_{22})m)}{a_{21}(a_{21}\alpha - a_{22})m} \rightarrow s \int_{\mathbb{T}} f dx, \end{aligned}$$

hence $s = (a_{22} - a_{21}\alpha)^{-1}$.

In view of (8.4), $(1, \alpha)A = \gamma(1, \alpha)$, so $(1, \alpha)A^{-1} = \gamma^{-1}(1, \alpha)$. Moreover,

$$A^{-1} = \sigma \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}, \quad \text{where } \sigma := \det A = \pm 1.$$

It follows that $\gamma^{-1} = \sigma(a_{22} - a_{21}\alpha)$, hence $\gamma = \sigma s$, this yields (8.5). Therefore, $V(x) = \sigma s x + \delta$, so (8.9) gives (8.6).

It follows that if $-1 \in I_{top}(T^f)$ then a_{12} or a_{21} is equal to zero. Otherwise, using Case 2 we have $a_{11} + \alpha a_{21} = \gamma = -\sigma$, so $a_{21} = 0$, a contradiction. Moreover, by Case 1, this yields the second part of the proposition. \square

Corollary 8.6. *If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is not a quadratic irrational then $I_{top}(T^f) \subset \{1, -1\}$.*

Corollary 8.7. *There exists a continuous time change $(\varphi_t)_{t \in \mathbb{R}}$ of a minimal linear flow on \mathbb{T}^2 such that $I((\varphi_t)_{t \in \mathbb{R}}) = \mathbb{R}^*$ and $I_{top}((\varphi_t)_{t \in \mathbb{R}}) = \{1\}$.*

Proof of Corollary 8.7. On the modular space $\Gamma \backslash PSL_2(\mathbb{R})$, $\Gamma = PSL_2(\mathbb{Z})$, by Corollary 1.1 and Corollary 1.2, $I((h_t)_{t \in \mathbb{R}}) = \mathbb{R}^*$ and $C((h_t)_{t \in \mathbb{R}}) = \{h_t : t \in \mathbb{R}\}$. As it was shown in [36] that this flow is loosely Bernoulli, so $(h_t)_{t \in \mathbb{R}}$ is isomorphic to a special flow over any irrational rotation $Tx = x + \alpha$ on the circle, see [18], [34]. Moreover, the roof function $f : \mathbb{T} \rightarrow \mathbb{R}_+$ can be chosen continuous, see [22].

If α is not a quadratic irrational, then $I_{top}(T^f) \subset \{-1, 1\}$. On the other hand, since T^f is measure-theoretically isomorphic with $(h_t)_{t \in \mathbb{R}}$, we have $I(T^f) = \mathbb{R}^*$ and $C(T^f) = \{T_t^f : t \in \mathbb{R}\}$. Moreover, T^f is a special representation of a continuous time change of a minimal linear flow on \mathbb{T}^2 .

Now we will see that f can be chosen so that $-1 \notin I_{top}(T^f)$ which will finish the proof. Suppose that there exists a special representation T^f of the horocycle flow $(h_t)_{t \in \mathbb{R}}$ such that $-1 \in I_{top}(T^f)$. Then we can construct another continuous function $f' : \mathbb{T} \rightarrow \mathbb{R}$ such that $T^{f'}$ is isomorphic to T^f and $-1 \notin I_{top}(T^{f'})$.

Since $-1 \in I_{top}(T^f)$, by Proposition 8.5, there exist $\delta \in \mathbb{T}$ and $g : \mathbb{T} \rightarrow \mathbb{R}$ continuous such that

$$(8.13) \quad f(\delta - x) - f(x) = g(x) - g(x + \alpha).$$

Let $j : \mathbb{T} \rightarrow \mathbb{R}$ be a measurable map such that $x \mapsto j(x) - j(x + \alpha)$ is continuous and

$$(8.14) \quad x \mapsto j(x) + j(\delta - x) \text{ is not a.e. equal to any continuous function;}$$

the existence of such a map will be discussed at the end of the proof.

We claim that if $f' = f + j - j \circ T$ then $-1 \notin I_{top}(T^{f'})$. Otherwise, by Proposition 8.5, there exist $\delta' \in \mathbb{T}$ and $g' : \mathbb{T} \rightarrow \mathbb{R}$ continuous such that

$$f'(\delta' - x) - f'(x) = g'(x) - g'(x + \alpha).$$

In view of (8.13), it follows that

$$\begin{aligned} f(\delta - x) - f(\delta' - x) &= (f(\delta - x) - f(x)) - (f'(\delta' - x) - f'(x)) \\ &\quad - (j(x) - j(x + \alpha)) + (j(\delta' - x) - j(\delta' - x + \alpha)) \\ &= ((g - g' - j)(x) - (g - g' - j)(x + \alpha)) + (j(\delta' - x) - j(\delta' - x + \alpha)). \end{aligned}$$

Replacing x by $\delta' - x$ we have

$$f(x + \delta - \delta') - f(x) = h(x) - h(x + \alpha),$$

with

$$h(x) = (j + g' - g)(\delta' - x + \alpha) + j(x).$$

Since $C(T^f) = \{T_t^f : t \in \mathbb{R}\}$, in view of Lemma 2.6, there exist $k \in \mathbb{Z}$ and $t_0 \in \mathbb{R}$ such that $\delta - \delta' = k\alpha$ and $h = t_0 - f^{(k)}$ a.e. Therefore

$$j(\delta' - x + \alpha) + j(x) = (g - g')(\delta' - x + \alpha) + t_0 - f^{(k)}(x) \quad \text{a.e.}$$

Moreover,

$$\begin{aligned} j(\delta - x) - j(\delta' - x + \alpha) &= j(\delta - x) - j(\delta - x - (k - 1)\alpha) \\ &= (f')^{(-k+1)}(\delta - x) - f^{(-k+1)}(\delta - x), \end{aligned}$$

Adding both equations we obtain that the map $x \mapsto j(\delta - x) + j(x)$ is a.e. equal to a continuous map, contrary to (8.14).

Finally we point out a measurable map $j : \mathbb{T} \rightarrow \mathbb{R}$ such that $j - j \circ T$ is continuous and satisfies (8.14). Let $(q_n)_{n \geq 1}$ be a subsequence of denominators of α such that $q_{n+1} \geq 2q_n$ for $n \geq 1$. Let us consider an L^2 map $j : \mathbb{T} \rightarrow \mathbb{R}$ with the Fourier series

$$j(x) = \sum_{n \geq 1} \frac{1}{n} \cos 2\pi q_n(x - \delta/2).$$

Since

$$j(x) - j(x + \alpha) = \sum_{n \geq 1} \frac{2}{n} \sin 2\pi q_n(x + \alpha/2 - \delta/2) \sin \pi q_n \alpha$$

with $|\sin \pi q_n \alpha| \leq \|q_n \alpha\| < 1/q_{n+1} < 1/2^n$ for $n \geq 1$, we can choose j such that $j - j \circ T$ is continuous. Moreover, $j(\delta - x) = j(x)$ for a.e. $x \in \mathbb{T}$. Therefore, we need to show that j (or equivalently $j_\delta(x) = j(x + \delta/2)$) is not a.e. equal to any continuous function. Some elementary arguments show that the Fourier series of j_δ is not Cesàro summable at 0. Then, by classical Fejer's theorem, j_δ is not a.e. equal to any continuous function, which completes the proof. \square

The following lemma is easily obtained by induction.

Lemma 8.8. *Let $0 < |\rho| < 1$ and let $(x_n)_{n \geq 0}$ be a real sequence such that*

$$|\rho x_{n+1} - x_n| \leq M \quad \text{for } n \geq 0.$$

Then

$$|\rho^n x_n - x_0| \leq \frac{1 - |\rho|^n}{1 - |\rho|} M \leq \frac{M}{1 - |\rho|} \quad \text{for } n \geq 0.$$

Theorem 8.9. *If there exists $s \in I_{\text{top}}(T^f) \setminus \{-1, 1\}$ then f is cohomologous to a constant function via a continuous transfer function.*

Proof. Without loss of generality we can assume that $|s| < 1$. By Proposition 8.5, there exist a matrix $[a_{ij}] = A \in GL_2(\mathbb{Z})$, $\delta \in \mathbb{R}$ and a continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$ satisfying (8.4)-(8.6). Let us consider

$$F(x) := \tilde{f}(x) - \int_{\mathbb{T}} f(t) dt \quad \text{and} \quad G(x) := g(x) + x s a_{21} \int_{\mathbb{T}} f(t) dt.$$

In view of (8.6) and (8.5),

$$(8.15) \quad s F^{(a_{21}n + a_{22}m)}(\sigma s x + \delta) - F^{(m)}(x) = G(x) - G(x + n + m\alpha).$$

with (remembering that F is 1-periodic) $\int_{\mathbb{T}} F(t) dt = 0$.

Choose any $x_0 \in [0, 1]$ and $m_0 \geq \frac{2|s a_{21}|}{1 - |s|}$. Let us define inductively three sequences: $(x_k)_{k \geq 0}$ taking values in $[0, 1]$ and two other integer-valued sequences $(m_k)_{k \geq 0}$, $(n_k)_{k \geq 0}$ so that:

$$n_k := -[x_k + m_k \alpha], \quad m_{k+1} := a_{21} n_k + a_{22} m_k, \quad x_{k+1} := \{\sigma s x_k + \delta\}$$

for all $k \geq 0$. In view of (8.15), it follows that

$$s F^{(m_{k+1})}(x_{k+1}) - F^{(m_k)}(x_k) = G(x_k) - G(x_k + n_k + m_k \alpha)$$

and $x_k + n_k + m_k \alpha = \{x_k + m_k \alpha\} \in [0, 1]$. Therefore,

$$|s F^{(m_{k+1})}(x_{k+1}) - F^{(m_k)}(x_k)| \leq C := 2 \max_{x \in [0, 1]} |G(x)|.$$

By Lemma 8.8,

$$(8.16) \quad |s^k F^{(m_k)}(x_k) - F^{(m_0)}(x_0)| \leq \frac{C}{1 - |s|} \quad \text{for } k \geq 0.$$

Moreover, as $s(a_{22} - a_{21}\alpha) = 1$ (see (8.5)), we have

$$\begin{aligned} sm_{k+1} - m_k &= sa_{21}n_k + (sa_{22} - 1)m_k = -sa_{21}[x_k + m_k\alpha] + (sa_{22} - 1)m_k \\ &= -sa_{21}(x_k + m_k\alpha) + (sa_{22} - 1)m_k + sa_{21}\{x_k + m_k\alpha\} \\ &= (s(a_{22} - a_{21}\alpha) - 1)m_k + sa_{21}(\{x_k + m_k\alpha\} - x_k) \\ &= sa_{21}(\{x_k + m_k\alpha\} - x_k). \end{aligned}$$

Hence $|sm_{k+1} - m_k| \leq |sa_{21}|$ for $k \geq 0$. In view of Lemma 8.8, it follows that

$$|s^k m_k - m_0| \leq \frac{|sa_{21}|}{1 - |s|},$$

so the sequence $(s^k m_k)_{k \geq 0}$ is bounded and (by the lower bound of m_0) bounded away from zero. Thus $|m_k| \rightarrow \infty$ as $k \rightarrow \infty$.

By the unique ergodicity of the rotation T , the sequence $F^{(n)}/n$ tends uniformly to $\int_{\mathbb{T}} F(t) dt = 0$ as $|n| \rightarrow \infty$. It follows that,

$$s^k F^{(m_k)}(x_k) = s^k m_k \frac{F^{(m_k)}(x_k)}{m_k} \rightarrow 0.$$

Therefore, passing to $k \rightarrow \infty$ in (8.16), we have $|F^{(m_0)}(x_0)| \leq C/(1 - |s|)$. Consequently, $\|F^{(m)}\|_{\text{sup}} \leq C/(1 - |s|)$ for every $m \geq 1$. In view of the classical Gottschalk-Hedlund theorem (Theorem 14.11 in [14]), $F = f - \int_{\mathbb{T}} f(t) dt$ is a coboundary with a continuous transfer function. \square

Remark 8.10. Consider the quadratic number $\alpha \in (0, 1)$ satisfying $\frac{1}{\alpha} = \alpha + 1$. Let $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ be the linear flow on \mathbb{T}^2 given by $(\alpha, 1)$, that is $T_t(x, y) = (x + t\alpha, y + t)$. Then $1/\alpha \in I_{\text{top}}(\mathcal{T})$. Indeed, the rescaled flow $\mathcal{S} = (S_t)_{t \in \mathbb{R}}$ is given by the formula $S_t(x, y) = (x + t, y + \frac{1}{\alpha}t)$ and it is easy to see that the homeomorphism $A : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ given by the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ satisfies $A \circ T_t = S_t \circ A$ for each $t \in \mathbb{R}$.

REFERENCES

- [1] J. Aaronson, M. Lemańczyk, C. Mauduit, H. Nakada, *Koksma inequality and group extensions of Kronecker transformations*, in Algorithms, Fractals and Dynamics edited by Y. Takahashi, Plenum Press 1995, 27-50.
- [2] H. Anzai, *Ergodic skew product transformations on the torus*, Osaka Math. J. **3** (1951), 83-99.
- [3] J. Bourgain, P. Sarnak, T. Ziegler, *Disjointness of Mobius from horocycle flows*, "From Fourier and Number Theory to Radon Transforms and Geometry" in memory of Leon Ehrenpreis, Developments in Mathematics, Vol. 28 (2012), 67-83, Springer Verlag.
- [4] I.P. Cornfeld, S.V. Fomin, Y.G. Sinai, *Ergodic Theory*, Springer-Verlag, New York, 1982.
- [5] A. Danilenko, M. Lemańczyk, *Spectral multiplicities for ergodic flows*, to appear in Discrete Continuous Dynam. Systems, [arXiv:1008.4845](https://arxiv.org/abs/1008.4845).
- [6] A. Danilenko, V.V. Ryzhikov, *On self-similarities of ergodic flows*, Proc. London Math. Soc. **104** (2012), 431-454.
- [7] M. Einsiedler, T. Ward, *Ergodic Theory: With a View Towards Number Theory*, Graduate Texts in Mathematics, Springer 2010.
- [8] K. Frączek, M. Lemańczyk, *A class of special flows over irrational rotations which is disjoint from mixing flows*, Ergodic Theory Dynam. Systems **24** (2004), 1083-1095.
- [9] K. Frączek, M. Lemańczyk, *On the self-similarity problem for flows*, Proc. London Math. Soc. **99** (2009), 658-696.
- [10] H. Furstenberg, *Disjointness in ergodic theory, minimal sets and diophantine approximation*, Math. Syst. Th. **1** (1967), 1-49.
- [11] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton University Press, Princeton, New Jersey, 1981.
- [12] E. Glasner, *Ergodic Theory via Joinings*, Mathematical Surveys and Monographs **101**, AMS, Providence, RI, 2003.
- [13] G.R. Goodson, A. del Junco, M. Lemańczyk, D. Rudolph, *Ergodic transformations conjugate to their inverse by involutions*, Ergodic Theory Dynam. Systems **16** (1996), 97-124.

- [14] W. Gottschalk, G. Hedlund, *Topological dynamics*, American Mathematical Society Colloquium Publications, Vol. 36. AMS, Providence, R. I., 1955.
- [15] M. Herman, *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations*, Publ. Math. IHES **49** (1979), 5-234.
- [16] A. del Junco, *Disjointness of measure-preserving transformations, minimal self-joinings and category*, Ergodic theory and dynamical systems, I (College Park, Md., 1979-80), pp. 81-89, Progr. Math., 10, Birkhauser, Boston, Mass., 1981.
- [17] A. del Junco, M. Rahe, L. Swanson, *Chacon's automorphism has minimal self joinings*, Journal d'Analyse Mathématique **37** (1980), 276-284.
- [18] A.B. Katok, *Monotone equivalence in ergodic theory*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **41** (1977), 104-157.
- [19] A. Katok, J.-P. Thouvenot, *Spectral Properties and Combinatorial Constructions in Ergodic Theory*, Handbook of dynamical systems. Vol. 1B, 649-743, Elsevier B. V., Amsterdam, 2006.
- [20] H. Keynes, N. Markley, M. Sears, *The structure of automorphisms of real suspension flows*, Ergodic Theory Dynam. Systems **11** (1991), 349-364.
- [21] A.Ya. Khinchin, *Continued fractions*, The University of Chicago Press, Chicago-London, 1964.
- [22] A.V. Kochergin, *The homology of functions over dynamical systems*, (Russian) Dokl. Akad. Nauk SSSR **231** (1976), 795-798.
- [23] J. Kwiatkowski, M. Lemańczyk, D. Rudolph, *A class of real cocycles having an analytic coboundary modification*, Israel J. Math. **87** (1994), 337-360.
- [24] L. Kuipers, H. Niederreiter, *Uniform distribution of sequences*, Pure and Applied Mathematics, Wiley-Interscience, New York-London-Sydney, 1974.
- [25] J. Kułaga, *On the self-similarity problem for smooth flows on orientable surfaces*, Ergodic Theory Dynam. Systems **32** (2012), 1615-1660.
- [26] J. Lee, K.K. Park, S. Shin, *Reversible topological Markov shifts*, Ergodic Theory Dynam. Systems **26** (2006), 267-280.
- [27] M. Lemańczyk, *Spectral Theory of Dynamical Systems*, Encyclopedia of Complexity and System Science, Springer-Verlag (2009), 8554-8575.
- [28] M. Lemańczyk, F. Parreau, J.-P. Thouvenot, *Gaussian automorphisms whose ergodic self-joinings are Gaussian*, Fund. Math. **164** (2000), 253-293.
- [29] M. Lemańczyk, M. Wysokińska, *On analytic flows on the torus which are disjoint from systems of probabilistic origin*, Fund. Math. **195** (2007), 97-124.
- [30] S.A. Malkin, *An example of two metrichally nonisomorphic ergodic automorphisms with the same simple spectrum*, (Russian) Izv. Vyssh. Uchebn. Zaved. Matematika **6** (1968), 69-74.
- [31] M.G. Nadkarni, *Spectral theory of dynamical systems*, Birkhäuser Advanced Texts, Birkhäuser Verlag, Basel, 1998.
- [32] D. Ornstein, *Ergodic theory, randomness, and "chaos"*, Science **243** (1989), 182-187.
- [33] D. Ornstein, *Imbedding Bernoulli shifts in flows*, 1970 Contributions to Ergodic Theory and Probability (Proc. Conf., Ohio State Univ., Columbus, Ohio, 1970) pp. 178-218 Springer, Berlin.
- [34] D. Ornstein, D. Rudolph, B. Weiss, *Equivalence of measure preserving transformations*, Mem. Amer. Math. Soc. **37** (1982), no. 262.
- [35] V.I. Oseledets, *An example of two nonisomorphic systems with the same simple singular spectrum*, Functional Anal. Appl. **5** (1971), 75-79.
- [36] M. Ratner, *Horocycle flows are loosely Bernoulli*, Israel J. Math. **31** (1978), 122-132.
- [37] M. Ratner, *Rigidity of horocycle flows*, Ann. of Math. (2) **115** (1982), 597-614.
- [38] M. Ratner, *Horocycle flows, joinings and rigidity of products*, Ann. of Math. (2) **118** (1983), 277-313.
- [39] V.V. Ryzhikov, *Joinings, intertwining operators, factors, and mixing properties of dynamical systems*, Russian Acad. Sci. Izv. Math. **42** (1994), 91-114.
- [40] V.V. Ryzhikov, *On the asymmetry of cascades*, Trudy Mat. Inst. Steklova **216** (1997), 154-157; translation in Proc. Steklov Inst. Math. **216** (1997), 147-150.
- [41] V.V. Ryzhikov, *Partial multiple mixing on subsequences may distinguish between automorphisms T and T^{-1}* , Mathematical Notes **74** (2003), 889-895.
- [42] M. Viana, *Dynamics of Interval Exchange Transformations and Teichmüller Flows*, lecture notes available from <http://w3.impa.br/~viana/out/ietf.pdf>