ON SPECIAL FLOWS OVER IETS THAT ARE NOT ISOMORPHIC TO THEIR INVERSES

PRZEMYSŁAW BERK AND KRZYSZTOF FRĄCZEK

Abstract. In this paper we give a criterion for special flow to be not isomorphic to its inverse which is a refine of a result of [5]. We apply this criterion to special flows T^f built over ergodic interval exchange transformations $T : [0, 1) \to [0, 1)$ (IET) and under piecewise absolutely continuous roof functions $f : [0, 1) \to \mathbb{R}_+$. We show that for almost every IET T if f is absolutely continuous over exchanged intervals and has non-zero sum of jumps the the special flow T^f is not isomorphic to its inverse. The same conclusion is valid for a typical piecewise constant roof function.

1. Introduction

The problem of isomorphism of probability measure-preserving systems to their own inverse was already stated by Halmos-von Neumann in their seminal paper [10]. In [10] the authors found a complete invariant for ergodic systems with discrete spectrum and then they applied it to prove that any ergodic measure preserving transformation with pure point spectrum is isomorphic to its own inverse. Moreover, Halmos-von Neumann conjectured that the same result is valid for an arbitrary measure preserving transformation. The first counter-example to this conjecture was given by Anzai in [1], it was so called Anzai skew product. Anzai counter-example gave the impetus for further research on the problem isomorphism of measure-preserving systems to their inverse. As shown in [4] (for automorphisms) and in [3], the property of being isomorphic to its inverse is not a typical property. For a fairly detailed introduction to the problem we refer also to [5].

Recall that a measurable flow $\mathcal{T} = \{T_t\}_{t\in\mathbb{R}}$ on a standard probability Borel space $(X, \mathcal{B}, \mu)$ is isomorphic to its inverse if there exists measure-preserving automorphism $S : X \to X$, such that

$$T_t \circ S = S \circ T_{-t} \text{ for all } t \in \mathbb{R}.$$

For any ergodic measure-preserving automorphism $T : X \to X$ and a positive measurable roof function $f : X \to \mathbb{R}_+$ we consider a space $X^f := \{(x, r) \in X \times \mathbb{R}, 0 \le r < f(x)\}$. On X^f we deal with the *special flow* T^f (see e.g. [2], Ch.11), that is the flow which moves points vertically upwards with unit speed and we identify the point $(x, f(x))$ with $(Tx, 0)$. If T is an IET then the flows T^f arise naturally as special representation of flows on compact surfaces.

In [5] the authors developed techniques to prove non-isomorphism of a flow T^f to its inverse that based on studying the weak closure of off-diagonal 3-self-joinings. The idea of detecting non-isomorphism of a dynamical system and its inverse by studying the weak closure of off-diagonal 3-self-joinings was already used by Ryzhikov in [15]. The tools developed in [5] were applied to the special flow built over irrational rotations and under piecewise absolutely continuous roof functions.

Date: October 3, 2013.
2000 *Mathematics Subject Classification.* 37A10, 37E35.
Research partially supported by the Narodowe Centrum Nauki Grant DEC-2011/03/B/ST1/00407.

The main aim of this paper is to extend the techniques of [5] to special flows over ergodic IETs. The paper is motivated by the desire to understand the problem of isomorphism of translation flows on translation surfaces to their own inverse. For the background material concerning translation surfaces, see [20]. Recall that for every translation surface from any hyperelliptic component in the moduli space the vertical flow is isomorphic to its inverse. We conjecture that for a typical translation surface from any non-hyperelliptic component the vertical flow is not isomorphic to its inverse. The result of Section 7 can be regarded as a step toward this conjecture.

In Section 2 we give general background on special flows and joinings. We also recall so called off-diagonal joinings of higher rank, which serve as a main tool in latter calculations and constructions.

In Section 3 we state the conditions under which a sequence of 3-off-diagonal joinings converges weakly in the space $J_3(T^f)$ of all 3-self-joinings and how does the limit looks like, see Theorem 3.10. In [5] the authors give an explicit formula for the whole limit measure. Now under weaker assumptions the limit measure is controlled only partially. Nevertheless, it is enough for our purpose. The proofs are based on ideas drawn from [5] and [6].

In Section 4 (using results of Section 3) we give a sufficient condition for special flow built over partially rigid automorphisms to be not isomorphic to its inverse, see Theorem 4.4. This result reduces the problem of non-isomorphism of T^f and its inverse to establishing that a probability measure ξ_*P on $\mathbb{R}$ does not satisfies a symmetry condition.

In the remainder of the paper we apply newly obtained criterion to special flows over IETs. In Section 5 we state some general background concerning IETs and Rauzy-Veech induction. In Section 6 for a almost every IET T we construct a sequence of Rokhlin towers, based on Rauzy-Veech induction. This combined with Theorem 4.4 leads to proving that special flow over a. e. IET under piecewise linear roof function (over exchanged intervals) with constant non-zero slope is not isomorphic to its inverse, see Theorem 6.4.

In Section 7 we modify the construction of Rokhlin tower and again using Theorem 4.4 we prove that special flows over a. e. IET T and under piecewise constant roof function f with discontinuity points $\beta_1, \ldots, \beta_r$, where $r \geq 3$, are not isomrphic to their inverses, for almost every choice of $\beta_1, \ldots, \beta_r$ and if f has no jumps with opposite value, see Theorem 7.3.

Finally, in Section 8 we prove that the results from Sections 6 and 7 can be extended to piecewise absolutely roof function $f : [0, 1) \to \mathbb{R}$, see Theorem 8.2.

2. Preliminaries

2.1. **Special flows.** Let $(X, \mathcal{B}, \mu)$ be a standard probability Borel space. Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be an ergodic automorphism. We will denote by $\mathcal{B}(\mathbb{R})$ the standard Borel σ-algebra on $\mathbb{R}$, while by *Leb* we will denote the Lebesgue measure on $\mathbb{R}$ or $[0, 1)$ according to the context. Let $\{V_n\}_{n\in\mathbb{N}}$ be a sequence of measurable subsets of X. We say that T is *rigid along a sequence* $\{V_n\}_{n\in\mathbb{N}}$ if there exists an increasing sequence of natural numbers $\{q_n\}_{n\in\mathbb{N}}$, such that $\mu((T^{-q_n}A\triangle A)\cap V_n) \to 0$ for every measurable $A \subset X$. Then $\{q_n\}_{n\in\mathbb{N}}$ is called a *rigidity sequence* for T along $\{V_n\}_{n\in\mathbb{N}}$. Assume that $f \in L^1(X, \mathcal{B}, \mu)$ is a strictly positive function. By $\mathcal{T}^f = (T^f_t)_{t\in\mathbb{R}}$ we will denote the corresponding special flow under f acting on $(X^f, \mathcal{B}^f, \nu^f)$, where $X^f = \{(x, r) \in X \times \mathbb{R};\quad 0 \leq r < f(x)\}$ and $\mathcal{B}^f$ and μ^f are restrictions of $\mathcal{B} \otimes \mathcal{B}(\mathbb{R})$ and $\mu \otimes Leb$ to X^f.

Remark 2.1. If T is ergodic, then T^f is ergodic and aperiodic.

Moreover, if X is endowed with a metric d generating $\mathcal{B}$, then we may consider X^f as a metric space, where as a metric d^f we may take the restriction of product metric on $X \times \mathbb{R}$ to X^f. Then $\mathcal{B}^f$ is generated by d^f.

2.2. **Self-joinings.** Let $\mathcal{T} = (T_t)_{t\in\mathbb{R}}$ be an ergodic flow on $(X, \mathcal{B}, \mu)$. Let $k \geq 2$. By a *k-self-joining of $\mathcal{T}$* we call any probability $(T_t \times \ldots \times T_t)_{t\in\mathbb{R}}$-invariant measure λ on $(X^k, \mathcal{B}^{\otimes k})$ which projects as μ on each coordinate. We will denote by $J_k(\mathcal{T})$ the set of all k-self-joinings of $\mathcal{T}$. We say that k-joining λ is ergodic, if the flow $(T_t \times \ldots \times T_t)_{t\in\mathbb{R}}$ is ergodic on (X^k, λ). For any $(T_t \times \ldots \times T_t)_{t\in\mathbb{R}}$-invariant measure σ we may consider its unique *ergodic decomposition* $\sigma = \int_{\mathcal{M}_e} \nu d\kappa(\nu)$, where $\mathcal{M}_e$ stands for the set of $(T_t \times \ldots \times T_t)_{t\in\mathbb{R}}$-invariant ergodic measures on X^k and κ is some probability measure on $\mathcal{M}_e$.

Remark 2.2. If $\lambda = \int_{\mathcal{M}_e} \nu d\kappa(\nu)$ is the ergodic decomposition of k-joining λ, then the set of measures $\nu \in \mathcal{M}_e$ which are k-joinings is of full measure in measure κ.

Let $\{B_n; n \in \mathbb{N}\}$ be a countable family in $\mathcal{B}$ which i s dense in $\mathcal{B}$ for pseudo-metric $d_\mu(A, B) = \mu(A \triangle B)$. Then on $J_k(\mathcal{T})$ we may consider a metric ρ such that

$$\rho(\lambda, \lambda') = \sum_{n_1,\ldots,n_k\in\mathbb{N}} \frac{1}{2^{n_1+\ldots+n_k}} |\lambda(B_{n_1} \times \ldots \times B_{n_k}) - \lambda'(B_{n_1} \times \ldots \times B_{n_k})|.$$

The set $J_k(\mathcal{T})$ endowed with the weak topology derived from this metric is compact. Moreover, the sequence of joinings $(\lambda_n)_{n\in\mathbb{N}}$ is convergent to λ in this metric, whenever $\lambda_n(A_1 \times \ldots \times A_k) \to \lambda(A_1 \times \ldots \times A_k)$ for all $A_1, \ldots, A_k \in \mathcal{B}$.

Let $t_1, \ldots, t_{k-1} \in \mathbb{R}$. Then we may consider the k-joining determined in following way

$$\mu_{t_1,\ldots,t_{k-1}}(A_1 \times \ldots \times A_{k-1} \times A_k) = \mu(T_{-t_1} A_1 \cap \ldots \cap T_{-t_{k-1}} A_{k-1} \cap A_k),$$

for $A_1, \ldots, A_k \in \mathcal{B}$. Such joining is called *off-diagonal.* As the image of the measure μ via the map $x \mapsto (T_{t_1}x, \ldots, T_{t_{k-1}}x, x)$, the joining $\mu_{t_1,\ldots,t_{k-1}}$ is ergodic.

Lemma 2.3. *Suppose that $(T_t)_{t\in\mathbb{R}}$ is ergodic and aperiodic. Then for any natural $k \geq 2$, the set $\mathcal{A} \subset J_k(\mathcal{T})$ of all k-off-diagonal joinings is Borel in $J_k(\mathcal{T})$. Moreover $h : \mathbb{R}^{k-1} \to \mathcal{A}$ given by $h(t_1, \ldots, t_{k-1}) = \mu_{t_1,\ldots,t_{k-1}}$ is a measurable isomorphism.*

Proof. The map h is continuous and by aperiodicity of $(T_t)_{t\in\mathbb{R}}$ it is an bijection between $\mathbb{R}^{k-1}$ and $\mathcal{A}$. By Souslin's theorem (see [9]), for any measurable set $A \in \mathbb{R}^{k-1}$, $h(A)$ is a Borel set in $J_k(\mathcal{T})$. Hence h is a measurable isomorphism. In particular $\mathcal{A} = h(\mathbb{R}^{k-1})$ is a Borel set. □

For any probability measure $P \in \mathcal{P}(\mathbb{R}^{k-1})$ we also consider integral k-self joining $\int_{\mathbb{R}^{k-1}} \mu_{t_1,\ldots,t_{k-1}} dP(t_1, \ldots, t_{k-1})$ such that

$$\Big(\int_{\mathbb{R}^{k-1}} \mu_{t_1,\ldots,t_{k-1}} dP(t_1, \ldots, t_{k-1})\Big)(A) := \int_{\mathbb{R}^{k-1}} \mu_{t_1,\ldots,t_{k-1}}(A) dP(t_1, \ldots, t_{k-1}),$$

for any $A \in \mathcal{B}^{\otimes k}$. In this paper we will deal with such joinings as partial limits of some sequences of off-diagonal joinings. For additional information about joinings see [5].

3. Limit theorem for off-diagonal joinings

Let $(T_t^f)_{t\in\mathbb{R}}$ be an ergodic special flow on the space X^f, where $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is an ergodic automorphism and $f \in L^2(X, \mathcal{B}, \mu)$ is a roof function such that $f \geq c$ for some $c > 0$. For any measurable subset $W \subset X$ with $\mu(W) > 0$ we will denote by μ_W the conditional measure given by $\mu_W(A) = \mu(A|W)$ for $A \in \mathcal{B}$. Suppose that there exist a sequence $\{W_n\}_{n\in\mathbb{N}}$ of measurable subsets of

X, increasing sequences $\{q_n\}_{n\in\mathbb{N}}$, $\{q'_n\}_{n\in\mathbb{N}}$ of natural numbers and real sequences $\{a_n\}_{n\in\mathbb{N}}$, $\{a'_n\}_{n\in\mathbb{N}}$ such that following conditions are satisfied:

$$\mu(W_n) \to \alpha \text{ with } 0 < \alpha \leq 1, \tag{1}$$

$$\mu(W_n \triangle T^{-1}W_n) \to 0, \tag{2}$$

$$\{q_n\}_{n\in\mathbb{N}} \text{ is a rigidity sequence for } T \text{ along } \{W_n\}_{n\in\mathbb{N}}, \tag{3}$$

$$\{q'_n\}_{n\in\mathbb{N}} \text{ is a rigidity sequence for } T \text{ along } \{W_n\}_{n\in\mathbb{N}}, \tag{4}$$

$$\Big\{\int_{W_n} |f_n(x)|^2 d\mu(x)\Big\}_{n\in\mathbb{N}} \text{ is bounded for } f_n = f^{(q_n)} - a_n, \tag{5}$$

$$\Big\{\int_{W_n} |f'_n(x)|^2 d\mu(x)\Big\}_{n\in\mathbb{N}} \text{ is bounded for } f'_n = f^{(q'_n)} - a'_n, \tag{6}$$

$$(f'_n, f_n)_*(\mu_{W_n}) \to P \text{ weakly in } \mathcal{P}(\mathbb{R}^2). \tag{7}$$

Similar conditions were stated as assumptions of Theorem 6 in [8]. By the definition of weak convergence, we know that for any bounded uniformly continuous function $\phi : \mathbb{R}^2 \to \mathbb{R}$, we have

$$\int_{W_n} \phi(f'_n(x), f_n(x)) d\mu(x) \to \alpha \int_{\mathbb{R}^2} \phi(t,u) dP(t,u). \tag{8}$$

We will now need the following series of auxiliary lemmas.

Lemma 3.1. *If $h_n \to 0$ in measure and $\{h_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(X,\mathcal{B},\mu)$ then $h_n \to 0$ in $L^1(X,\mathcal{B},\mu)$.* □

Lemma 3.2. *A sequence $\{q_n\}_{n\in\mathbb{N}}$ is rigid for T along $\{W_n\}_{n\in\mathbb{N}}$ if and only if for every $f \in L^1(X,\mathcal{B},\mu)$ we have $\chi_{W_n}(f\circ T^{q_n} - f) \to 0$ in measure.*

Proof. Note that $\mu((T^{-q_n}A \triangle A)\cap W_n) \to 0$ is equivalent to

$$\int_{W_n} |\chi_A \circ T^{q_n} - \chi_A| d\mu \to 0.$$

By passing to simple functions and by density of simple functions we have that $\chi_{W_n}(f\circ T^{q_n} - f) \to 0$ in L^1 for all $f \in L^1(X,\mathcal{B},\mu)$. By Markov's inequality, $\chi_{W_n}(f\circ T^{q_n} - f) \to 0$ in measure.

Conversely, suppose that $\chi_{W_n}(f\circ T^{q_n} - f) \to 0$ in measure for any $f \in L^1(X,\mathcal{B},\mu)$. If f is additionally bounded then, by Lemma 3.1, $\chi_{W_n}(f\circ T^{q_n} - f) \to 0$ in L^1. Taking $f = \chi_A$ we obtain $\chi_{W_n}|\chi_A\circ T^{q_n} - \chi_A| \to 0$ in L^1 for every $A \in \mathcal{B}$. This gives the rigidity of the sequence $\{q_n\}_{n\in\mathbb{N}}$ along $\{W_n\}_{n\in\mathbb{N}}$. □

Lemma 3.3. *Suppose that $(X,\mathcal{B},\mu)$ is endowed with a metric d generating the σ-algebra $\mathcal{B}$. If $\sup_{x\in W_n} d(T^{q_n}x, x) \to 0$, then $\{q_n\}_{n\in\mathbb{N}}$ is a rigidity sequence for T along $\{W_n\}_{n\in\mathbb{N}}$.*

Proof. Let $h \in L^1(X,\mathcal{B},\mu)$ and let $\varepsilon > 0$ and $a > 0$ be arbitrary. Then, by Lusin's theorem, there exists a compact set $B_\varepsilon \subset X$ such that $\mu(B_\varepsilon^c) < \frac{\varepsilon}{2}$ and $h : B_\varepsilon \to \mathbb{R}$ is uniformly continuous. Therefore, there exists $\delta > 0$ such that $d(x,y) < \delta$ implies $|h(x) - h(y)| < a$ for all $x, y \in B_\varepsilon$. By assumption, there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \text{ and } x \in W_n \Rightarrow d(x, T^{q_n}x) < \delta.$$

Hence, $x \in W_n \cap B_\varepsilon \cap T^{-q_n}B_\varepsilon$ implies $|h(x) - h(T^{q_n}x)| < a$ for $n > n_0$. Therefore

$$\mu(\{x \in W_n; |h(x) - h(T^{q_n}x)| \geq a)\}) \leq \mu(W_n \cap B_\varepsilon^c \cap T^{-q_n}B_\varepsilon^c) \leq 2\mu(B_\varepsilon^c) < \varepsilon.$$

Since $\varepsilon > 0$ and $a > 0$ were arbitrary, then using Lemma 3.2 we complete the proof. □

Lemma 3.4. *If $\{h_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ are bounded sequences in $L^\infty(X,\mathcal{B},\mu)$ such that $h_n \to 0$ in measure, then $h_n \cdot g_n \to 0$ in L^1.*

Proof. Let $M \geq 0$ be such that $\|g_n\|_\infty \leq M$ for every $n \in \mathbb{N}$. Then

$$\|h_n \cdot g_n\|_{L^1} \leq \|g_n\|_\infty \|h_n\|_{L^1} \leq M\|h_n\|_{L^1}.$$

Since, by Lemma 3.1, $h_n \to 0$ in L^1, it follows that $h_n \cdot g_n \to 0$ in L^1. □

Theorem 3.5. *Suppose that* (1)-(7) *hold. Let h, $h' : X \to \mathbb{R}$ be measurable functions. Let $g \in L^\infty(X,\mathcal{B},\mu)$ and $\phi : \mathbb{R}^2 \to \mathbb{R}$ be bounded and uniformly continuous. Then*

$$\begin{aligned}(9)\qquad &\int_{W_n} \phi(f_n'(x) + h'(x), f_n(x) + h(x))g(x)\, d\mu(x)\\ &\qquad \to \alpha \int_X \int_{\mathbb{R}^2} \phi(t + h'(x), u + h(x))g(x)\, dP(t,u)\, d\mu(x).\end{aligned}$$

Proof. We will divide the proof into steps according to the complexity of the functions h and h'.

Step 1. Assume that $h = h' = 0$. If g is constant then (9) follows from (8) directly. Hence we can assume that $g \in L^1_0(X,\mathcal{B},\mu)$, i.e. has zero mean. By the proof of von Neumann ergodic theorem, coboundaries, i.e. functions of the form $g = \xi - \xi \circ T$ with $\xi \in L^2(X,\mathcal{B},\mu)$ are dense in $L^1_0(X,\mathcal{B},\mu)$. Therefore, it suffices to consider $g = \xi - \xi\circ T$ for $\xi \in L^\infty(X,\mathcal{B},\mu)$, as they are also dense in $L^1_0(X,\mathcal{B},\mu)$. Note that the RHS of (9) is equal to

$$\alpha \int_{\mathbb{R}^2} \phi(t,u) dP(t,u) \int_X g(x)\, d\mu(x) = 0,$$

whenever $g \in L^1_0(X,\mathcal{B},\mu)$. As $g = \xi - \xi \circ T$, we need to prove that

$$(10)\qquad \Big|\int_{W_n} \phi(f_n'(x), f_n(x))\xi(x)\ d\mu(x) - \int_{W_n} \phi(f_n'(x), f_n(x))\xi(Tx)\ d\mu(x)\Big| \to 0.$$

However, by the T-invariance of μ, we have

$$\begin{aligned}&\Big|\int_{W_n} \phi(f_n'(x), f_n(x))\xi(x)\, d\mu(x) - \int_{W_n} \phi(f_n'(x), f_n(x))\xi(Tx)\, d\mu(x)\Big|\\ &= \Big|\int_{T^{-1}W_n} \phi(f_n'(Tx), f_n(Tx))\xi(Tx)\, d\mu(x) - \int_{W_n} \phi(f_n'(x), f_n(x))\xi(Tx)\, d\mu(x)\Big|\\ &\leq \int_{W_n} |\phi(f_n'(Tx), f_n(Tx)) - \phi(f_n'(x), f_n(x))||\xi(Tx)|\, d\mu(x)\\ &\quad + \|\phi\|_\infty \int_{T^{-1}W_n \triangle W_n} |\xi(Tx)|\, d\mu(x).\end{aligned}$$

By (2), we have $\mu(T^{-1}W_n \triangle W_n) \to 0$. Thus, $\int_{T^{-1}W_n\triangle W_n} |\xi(Tx)|\, d\mu(x) \to 0$ as $n \to \infty$.

Now we use the uniform continuity of ϕ. By the definition of f_n and f_n', we have

$$(f_n'(Tx), f_n(Tx)) - (f_n'(x), f_n(x)) = (f(T^{q_n'}x) - f(x), f(T^{q_n}x) - f(x)).$$

By Lemma 3.2, we have that

$$\chi_{W_n}(x)(f(T^{q_n'}x) - f(x)) \to 0 \text{ and } \chi_{W_n}(x)(f(T^{q_n}x) - f(x)) \to 0$$

in measure and thus

$$\chi_{W_n}(x)\Big(\big(f_n'(Tx), f_n(Tx)\big) - \big(f_n'(x), f_n(x)\big)\Big) \to 0$$

in measure. Since ϕ is uniformly continuous, we also have

$$\chi_{W_n}\Big(\phi(f_n' \circ T, f_n \circ T) - \phi(f_n', f_n)\Big) \to 0$$

in measure. By Lemma 3.4, we get that

$$\int_{W_n} \|\phi(f_n'(Tx), f_n(Tx)) - \phi(f_n'(x), f_n(x))\| |\xi(Tx)| \, d\mu(x) \to 0.$$

This concludes the proof of (10), which also completes the proof of (9) for $h = h' = 0$.

Step 2. Now let $h' = \sum_{i=1}^k h_i \chi_{A_i}$ and $h = \sum_{j=1}^l h_j' \chi_{B_j}$ be simple functions, where A_i and B_j for $i = 1, \ldots, k$ and $j = 1, \ldots, l$ make two measurable disjoint partitions of X. Then

$$\begin{aligned}
&\int_{W_n} \phi(f_n'(x) + h'(x), f_n(x) + h(x)) g(x) \, d\mu(x) \\
&\quad = \sum_{i=1}^{k} \sum_{j=1}^{l} \int_{W_n} \phi(f_n'(x) + h_i', f_n(x) + h_j) g(x) \chi_{A_i}(x) \chi_{B_j}(x) \, d\mu(x) \\
&\quad \to \sum_{i=1}^{k} \sum_{j=1}^{l} \alpha \int_X \int_{\mathbb{R}^2} \phi(t + h_i', u + h_j) g(x) \chi_{A_i}(x) \chi_{B_j}(x) dP(t,u) \, d\mu(x) \\
&\quad = \alpha \int_X \int_{\mathbb{R}^2} \phi(t + h'(x), u + h(x)) g(x) dP(t,u) \, d\mu(x),
\end{aligned}$$

where the convergence follows from the first step of the proof applied to functions $(t, u) \mapsto \phi(t + h_i', u + h_j)$. This gives (9) whenever h and h' are simple functions.

Step 3. All we need to show now is that for arbitrary measurable functions h and h', we can find sequences $\{h_m\}_{m\in\mathbb{N}}$, $\{h_m'\}_{m\in\mathbb{N}}$ of simple functions such that

$$\begin{aligned}
&\int_{W_n} \phi(f_n'(x) + h_m'(x), f_n(x) + h_m(x)) g(x) \, d\mu(x) \\
&\qquad \to \int_{W_n} \phi(f_n'(x) + h'(x), f_n(x) + h(x)) g(x) \, d\mu(x)
\end{aligned}$$

and

$$\begin{aligned}
&\int_X \int_{\mathbb{R}^2} \phi(t + h_m'(x), u + h_m(x)) g(x) dP(t,u) \, d\mu(x) \\
&\qquad \to \int_X \int_{\mathbb{R}^2} \phi(t + h'(x), u + h(x)) g(x) dP(t,u) \, d\mu(x),
\end{aligned}$$

as $m \to +\infty$. Take h_m and h_m' simple, such that $h_m \to h$ and $h_m' \to h'$ in measure. Then by the uniform continuity of ϕ we obtain that

$$\phi(f_n'(x) + h_m'(x), f_n(x) + h_m(x)) - \phi(f_n'(x) + h'(x), f_n(x) + h(x)) \to 0,$$

in measure on (X, μ) and

$$\phi(t + h_m'(x), u + h_m(x)) - \phi(t + h'(x), u + h(x)) \to 0$$

in measure on $(\mathbb{R}^2 \times X, P \otimes \mu)$. By Lemma 3.4 and Step 2, this completes the proof of the whole theorem. $\square$

Lemma 3.6. *Let the assumptions* (1)-(7) *hold. Furthermore, assume that* $g, \xi, \xi' \in L^\infty(X, \mathcal{B}, \mu)$. *Then*

$$\int_{W_n} \phi(f_n'(x) + h'(x), f_n(x) + h(x))\, g(x)\, \xi(T^{q_n}x)\, \xi'(T^{q_n'}x)\ d\mu(x)$$
$$\to \alpha \int_X \int_{\mathbb{R}^2} \phi(t + h'(x), u + h(x))\, g(x)\, \xi(x)\, \xi'(x)\, dP(t,u)\ d\mu(x).$$

Proof. By Lemma 3.2,

$$\chi_{W_n}(x)(\xi(x) - \xi(T^{q_n}x)) \to 0 \text{ and } \chi_{W_n}(x)(\xi'(x) - \xi'(T^{q_n'}x)) \to 0,$$

in measure. Then, by Lemma 3.4, it follows that

$$|\int_{W_n} \phi(f_n'(x) + h'(x), f_n(x) + h(x))g(x)\xi(T^{q_n}x)\xi'(T^{q_n'}x)\, d\mu(x)$$
$$- \int_{W_n} \phi(f_n'(x) + h'(x), f_n(x) + h(x))g(x)\xi(x)\xi'(T^{q_n'}x)\, d\mu(x)|$$
$$\leq \int_X |\phi(f_n'(x) + h'(x), f_n(x) + h(x))||g(x)||\xi'(T^{q_n'}x)|$$
$$|\chi_{W_n}(x)(\xi(T^{q_n}x) - \xi(x))|\, d\mu(x) \to 0,$$

and

$$|\int_{W_n} \phi(f_n'(x) + h'(x), f_n(x) + h(x))g(x)\xi(x)\xi'(T^{q_n'}x)\, d\mu(x)$$
$$- \int_{W_n} \phi(f_n'(x) + h'(x), f_n(x) + h(x))g(x)\xi(x)\xi'(x)\, d\mu(x)|$$
$$\leq \int_X |\phi(f_n'(x) + h'(x), f_n(x) + h(x))||g(x)||\xi(x)|$$
$$|\chi_{W_n}(x)(\xi'(T^{q_n'}x) - \xi'(x))|\, d\mu(x) \to 0.$$

Hence to conclude the proof of the lemma, we need to show that

$$\int_{W_n} \phi(f_n'(x) + h'(x), f_n(x) + h(x))g(x)\xi(x)\xi'(x)\, d\mu(x)$$
$$\to \alpha \int_X \int_{\mathbb{R}^2} \phi(t + h'(x), u + h(x))g(x)\xi(x)\xi'(x)dP(t,u)\, d\mu(x).$$

However, that is a direct consequence of Theorem 3.5, taking $g(x)\xi(x)\xi'(x)$ in place of $g(x)$. □

The following auxiliary lemma is well-known and we state it without any proof.

Lemma 3.7. *Let* $g_n : X \to \mathbb{R}^m$, $n \in \mathbb{N}$ *be measurable maps such that* $(g_n)_*\mu \to P$ *weakly in* $\mathcal{P}(\mathbb{R}^m)$. *Let* $h_n : X \to \mathbb{R}^n$, $n \in \mathbb{N}$ *be measurable such that* $h_n \to 0$ *in measure. Then* $(g_n + h_n)_*\mu \to P$ *weakly in* $\mathcal{P}(\mathbb{R}^m)$. □

Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be an ergodic automorphism, $f : X \to \mathbb{R}$ be an L^2 function such that $f \geq c > 0$. Denote by $T_{-f} : X \times \mathbb{R} \to X \times \mathbb{R}$ the skew product $T_{-f}(x, r) = (Tx, r - f(x))$. Then for every $n \in \mathbb{Z}$ we have $T_{-f}^n(x, r) = (T^n x, r - f^{(n)}(x))$, where

$$f^{(n)}(x) = \begin{cases} f(x) + f(Tx) + \ldots + f(T^{n-1}x) & \text{for } n \geq 0 \\ -(f(T^{-1}x) + \ldots + f(T^n x)) & \text{for } n < 0. \end{cases}$$

Denote by $(\sigma_t)_{t \in \mathbb{R}}$ the flow on $X \times \mathbb{R}$ defined by $\sigma_t(x, r) = (x, r + t)$.

The following lemma follows directly from Lemma 3.2 in [5].

Lemma 3.8. *For all $t,s\in\mathbb{R}$ and all measurable sets $A,B,C\subset X^f$ we have*

$$\mu^f((T^f)_tA\cap(T^f)_sB\cap C)=\sum_{k,l\in\mathbb{Z}}\mu\otimes\lambda((T_{-f})^k\sigma_tA\cap(T_{-f})^l\sigma_sB\cap C).$$

Moreover, the sets that appear on the RHS of the above equation are pairwise disjoint.

Lemma 3.9 (see Lemma 3.4 in [5]). *Suppose that $A=A_1\times A_2$, $B=B_1\times B_2$, $C=C_1\times C_2$ are measurable rectangles in $X\times\mathbb{R}$. Then*

$$\begin{aligned}&\mu\otimes Leb(((T_{-f})^{k_1}A)\cap((T_{-f})^{k_2}B)\cap C)\\&=\int_{(T^{k_1}A_1)\cap(T^{k_2}B_1)\cap C_1}Leb((A_2+f^{(-k_1)}(x))\cap(B_2+f^{(-k_2)}(X))\cap C_2)\,d\mu(x).\end{aligned}$$

The following theorem was inspired by Theorem 6. in [8] and by Proposition 3.7 in [5].

Theorem 3.10. *Suppose that* (1)-(7) *hold. Then*

$$\mu^f_{a_n',a_n}\to\rho=\alpha\int_{\mathbb{R}}\mu^f_{-t,-u}dP(t,u)+(1-\alpha)\nu\quad\text{in } J_3(\mathcal{T}^f),\tag{11}$$

where $\nu\in J_3(\mathcal{T}^f)$.

Proof. By the compactness of $J_3(\mathcal{T}^f)$, and by passing to a subsequence, if necessary, we have $\mu^f_{a_n',a_n}\to\rho$ in $J_3(\mathcal{T}^f)$. First we will prove that for measurable rectangles in X^f

$$A=A_1\times A_2,\quad B=B_1\times B_2,\quad C=C_1\times C_2,$$

with $A_2,B_2,C_2\subset\mathbb{R}$ bounded, we have

$$\begin{aligned}&\mu^f\big((T^f)_{-a_n'}(A\cap(T^{q_n'}W_n\times\mathbb{R}))\cap(T^f)_{-a_n}(B\cap(T^{q_n}W_n\times\mathbb{R}))\cap C\big)\\&\qquad\to\alpha\int_{\mathbb{R}^2}\mu^f\big((T^f)_tA\cap(T^f)_uB\cap C\big)\,dP(t,u).\end{aligned}\tag{12}$$

By Lemma 3.8,

$$\begin{aligned}&\mu^f((T^f)_{-a_n'}((A_1\cap T^{q_n'}W_n)\times A_2)\cap(T^f)_{-a_n}((B_1\cap T^{q_n}W_n)\times B_2)\cap C_1\times C_2)\\&=\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}\mu\otimes Leb((T_{-f})^{-k}(T_{-f})^{-q_n'}\sigma_{-a_n'}((A_1\cap T^{q_n'}W_n)\times A_2)\\&\qquad\cap(T_{-f})^{-l}(T_{-f})^{-q_n}\sigma_{-a_n}((B_1\cap T^{q_n}W_n)\times B_2)\cap C_1\times C_2).\end{aligned}$$

Moreover, in view of Lemma 3.9,

$$\begin{aligned}a^n_{k,l}:=&\mu\otimes Leb\Big((T_{-f})^{-k}(T_{-f})^{-q_n'}\sigma_{-a_n'}((A_1\cap T^{q_n'}W_n)\times A_2)\\&\qquad\cap(T_{-f})^{-l}(T_{-f})^{-q_n}\sigma_{-a_n}((B_1\cap T^{q_n}W_n)\times B_2)\cap C_1\times C_2\Big)\\=&\int_{U_n}Leb\Big((A_2-a_n'+f^{(q_n'+k)}(x))\cap(B_2-a_n+f^{(q_n+l)}(x))\cap C_2\Big)\,d\mu(x),\end{aligned}$$

where

$$\begin{aligned}U_n&:=T^{-q_n'-k}(A_1\cap T^{q_n'}W_n)\cap T^{-q_n-l}(B_1\cap T^{q_n}W_n)\cap C_1\\&=T^{-q_n'-k}(A_1)\cap T^{-k}W_n\cap T^{-q_n-l}(B_1)\cap T^{-l}W_n\cap C_1.\end{aligned}$$

Fix $l \in \mathbb{Z}$. Using Lemma 3.8 and then Lemma 3.9 for $A := X \times \mathbb{R}$ we obtain

$$\begin{aligned}\sum_{k\in\mathbb{Z}} a^n_{k,l} &\leq \mu \otimes Leb((T_{-f})^{-l}(T_{-f})^{-q_n}\sigma_{-a_n}((B_1 \cap T^{q_n}W_n) \times B_2) \cap C_1 \times C_2)\\ &\leq \int_{T^{-l}W_n} Leb((B_2 - a_n + f^{(l+q_n)}(x)) \cap C_2)\,d\mu(x)\\ &= \int_{T^{-l}W_n} Leb((B_2 + f_n(T^l x) + f^{(k)}(x)) \cap C_2)\,d\mu(x),\end{aligned}$$

where in the last equality we used the fact that

$$f^{(l+q_n)}(x) - a_n = f^{(l)}(x) + f^{(q_n)}(T^l x) - a_n = f^{(l)}(x) + f_n(T^l x).$$

Let now $s = diam(B_2 \cup C_2)$ and $V_n = \{x \in T^{-l}W_n; |f_n(T^l x) + f^{(l)}(x)| \leq s\}$. Then

$$\begin{aligned}&\int_{T^{-l}W_n} Leb((B_2 + f_n(T^l x) + f^{(l)}(x)) \cap C_2)\,d\mu(x)\\ &= \int_{V_n} Leb((B_2 + f_n(T^l x) + f^{(l)}(x)) \cap C_2)\,d\mu(x) \qquad (13)\\ &\leq s\mu(V_n) \leq s\mu(\{x \in W_n; |f_n(x)| \geq c|l| - s\}) \leq s\frac{D}{(c|l|-s)^2},\end{aligned}$$

where $D = \sup_{n\in\mathbb{N}} \int_{W_n} |f_n(x)|^2\,d\mu(x)$ and the last inequality follows from Chebyshev's inequality (similar calculation was used in proof of the Lemma [7]). Let $D_l = s\frac{D}{(c|k|-s)^2}$ for integer l with $|l| > \frac{s}{c}$ and let $D_k = 1$ otherwise. Then $\sum_{k\in\mathbb{Z}} a^n_{k,l} \leq D_k$ and $\sum_{l\in\mathbb{Z}} D_l < \infty$. Similarly, we can find a sequence $\{D'_k\}_{k\in\mathbb{Z}}$ such that $\sum_{k\in\mathbb{Z}} D'_k < \infty$ and $\sum_{l\in\mathbb{Z}} a^n_{k,l} \leq D'_k$. Hence, for every $\varepsilon > 0$ there exists $M > 0$ such that

$$\sum_{|k|\geq M}\sum_{l\in\mathbb{Z}} a^n_{k,l} < \sum_{|k|\geq M} D'_k < \frac{\varepsilon}{8} \quad \text{and} \quad \sum_{|l|\geq M}\sum_{k\in\mathbb{Z}} a^n_{k,l} < \sum_{|l|\geq M} D_l < \frac{\varepsilon}{8}.$$

Thus

$$\sum_{\max(|k|,|l|)\geq M} a^n_{k,l} < \sum_{|k|\geq M}\sum_{l\in\mathbb{Z}} a^n_{k,l} + \sum_{|l|\geq M}\sum_{k\in\mathbb{Z}} a^n_{k,l} < \frac{\varepsilon}{4}. \qquad (14)$$

We are now going to prove that for every pair $(k,l) \in \mathbb{Z}^2$ sequence

$$\begin{aligned}a^n_{k,l} = &\int_{T^{-k}W_n \cap T^{-l}W_n} \chi_{C_1}(x)\chi_{T^{-k}A_1}(T^{q'_n}x)\chi_{T^{-l}B_1}(T^{q_n}x)\\ &Leb\big((A_2 + f'_n(x) + f^{(k)}(T^{q'_n}x)) \cap (B_2 + f_n(x) + f^{(l)}(T^{q_n}x)) \cap C_2\big)\,d\mu(x)\end{aligned}$$

converges. Since, by assumption (2), $\mu((T^{-k}W_n \cap T^{-l}W_n) \triangle W_n) \to 0$, it is enough to check the convergence of the sequence

$$\begin{aligned}b^n_{k,l} := \int_{W_n} &Leb\big((A_2 + f'_n(x) + f^{(k)}(T^{q'_n}x)) \cap (B_2 + f_n(x) + f^{(l)}(T^{q_n}x)) \cap C_2\big)\\ &\chi_{C_1}(x)\chi_{T^{-k}A_1}(T^{q'_n}x)\chi_{T^{-l}B_1}(T^{q_n}x)\,d\mu(x).\end{aligned}$$

Let $F'_n, F_n : X \to \mathbb{R}$ be given by

$$F'_n(x) = f'_n(x) + f^{(k)}(T^{q'_n}x) - f^{(k)}(x) \text{ and } F_n(x) = f_n(x) + f^{(l)}(T^{q_n}x) - f^{(l)}(x).$$

Then

$$\begin{aligned}b^n_{k,l} = \int_{W_n} &Leb\big((A_2 + F'_n(x) + f^{(k)}(x)) \cap (B_2 + F_n(x) + f^{(l)}(x)) \cap C_2\big)\\ &\chi_{C_1}(x)\chi_{T^{-k}A_1}(T^{q'_n}x)\chi_{T^{-l}B_1}(T^{q_n}x)\,d\mu(x).\end{aligned}$$

By Lemma 3.2,

$$\chi_{W_n}(x)(f^{(k)}(T^{q_n}x)-f^{(k)}(x))\to 0 \text{ and } \chi_{W_n}(x)(f^{(l)}(T^{q'_n}x)-f^{(l)}(x))\to 0$$

in measure. Since $(f'_n,f_n)_*(\mu_{W_n})\to P$, by Lemma 3.7, it implies

$$(F'_n,F_n)_*(\mu_{W_n})\to P \text{ weakly in } \mathcal{P}(\mathbb{R}^2).$$

Let us apply Lemma 3.6 to

$$\begin{gathered}\phi(t,u):=Leb((A_2+t)\cap(B_2+u)\cap C_2),\\ (h',h):=(f^{(k)},f^{(l)}),\quad (f'_n,f_n):=(F'_n,F_n),\\ g:=\chi_{C_1},\quad \xi:=\chi_{T^{-k}A_1},\quad \xi':=\chi_{T^{-l}B_1}.\end{gathered}$$

We obtain that

$$\begin{aligned}b^n_{k,l}\to c_{k,l}:=\alpha\int_X\int_{\mathbb{R}^2} & Leb\big((A_2+t+f^{(k)}(x))\cap(B_2+u+f^{(l)}(x))\cap C_2\big)\\ & \chi_{C_1}(x)\chi_{T^{-k}A_1}(x)\chi_{T^{-l}B_1}(x)\,dP(t,u)\,d\mu(x).\end{aligned}$$

By Fubini's theorem and Lemma 3.9, $c_{k,l}$ is equal to

$$\begin{aligned}&\alpha\int_{\mathbb{R}^2}\int_{T^{-k}A_1\cap T^{-l}B_1\cap C_1} Leb\big((A_2+t+f^{(k)}(x))\cap(B_2+u+f^{(l)}(x))\cap C_2\big)\\ &\qquad\qquad d\mu(x)dP(t,u)\\ &=\alpha\int_{\mathbb{R}^2}(\mu\otimes Leb)\big((T_{-f})^{-k}\sigma_t(A_1\times A_2)\cap(T_{-f})^{-l}\sigma_u(B_1\times B_2)\cap(C_1\times C_2)\big)\,dP(t,u).\end{aligned}$$

By Lemma 3.8,

$$\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}c_{k,l}=\alpha\int_{\mathbb{R}^2}\mu^f((T^f)_tA\cap(T^f)_uB\cap C)dP(t,u)<\infty. \tag{15}$$

Hence, by enlarging M if necessary, we have

$$\sum_{\max\{|k|,|l|\}>M}c_{k,l}<\frac{\varepsilon}{4}. \tag{16}$$

We know that $a^n_{k,l}\to c_{k,l}$ for all $k,l\in\mathbb{Z}$. Choose $N>0$ so that for all $n\geq N$ and $k,l\in\mathbb{Z}$ with $\max\{|k|,|l|\}\leq M$, we have

$$|a^n_{k,l}-c_{k,l}|<\frac{\varepsilon}{2(2M+1)^2}.$$

By (14) and (16), it follows that

$$\begin{aligned}&\Big|\sum_{k,l\in\mathbb{Z}}a^n_{k,l}-\sum_{k,l\in\mathbb{Z}}c_{k,l}\Big|\\ &\quad\leq\sum_{\max\{|k|,|l|\}>M}a^n_{k,l}+\sum_{\max\{|k|,|l|\}>M}c_{k,l}+\sum_{\max\{|k|,|l|\}\leq M}|a^n_{k,l}-c_{k,l}|\leq\varepsilon.\end{aligned}$$

Therefore, $\sum_{k,l\in\mathbb{Z}}a^n_{k,l}\to\sum_{k,l\in\mathbb{Z}}c_{k,l}$ and in view of (15) this proves the convergence (12).

Since ρ is the weak limit of $\mu^f_{a'_n,a_n}$, by (12), we get

$$\rho(A\times B\times C)\geq\alpha\int_{\mathbb{R}^2}\mu^f_{-t,-u}(A\times B\times C)dP(t,u),$$

for measurable $A,B,C\subset X^f$. If $\alpha=1$ this implies (11). If $0<\alpha<1$ let us consider

$$\nu:=\frac{1}{1-\alpha}\Big(\rho-\alpha\int_{\mathbb{R}^2}\mu^f_{-t,-u}dP(t,u)\Big).$$

Then ν is a signed measure on $(X^f)^3$ taking non-negative values on product sets and $\nu((X^f)^3) = 1$. All we have to prove is that ν is a 3-joining. First we need to check that ν is positive. Let $K \subset (X^f)^3$ be measurable and $\varepsilon > 0$. Consider the measure $|\nu|$, i.e. the variation of the signed measure ν. Since $|\nu|$ is a measure defined on metric space, it is regular and hence there exists an open set U and a compact set D such that $D \subset K \subset U$ and $|\nu|(U) - |\nu|(D) < \varepsilon$. Let us choose a finite open covering $V_1', \dots, V_m'$ of D by product sets. We can also assume that their union is contained in U. By a refining of this covering, one can obtain finite covering (usually not open) $V_1, \dots, V_r$ of D by pairwise disjoint product measurable sets. Take $V := \bigcup_{i=1\dots r} V_i$. We have $\nu(V) \geq 0$ as V is the disjoint union of product sets. Note that

$$|\nu(V) - \nu(K)| \leq |\nu|(U \setminus D) < \varepsilon.$$

Thus

$$\nu(K) > \nu(V) - \varepsilon \geq -\varepsilon \quad \text{ for every } \quad \varepsilon > 0.$$

It follows that $\nu(K) \geq 0$, so ν is a positive probability measure.

To complete the proof we have to show that ν is $(T_t^f \times T_t^f \times T_t^f)_{t\in\mathbb{R}}$-invariant and that it projects on each coordinate as μ^f. That however follows immediately from the definition of ν and the fact that ρ and $\int_{\mathbb{R}^2} \mu^f_{-t,-u} dP(t,u)$ are 3-joinings. □

4. Non-reversibility criteria for special flows

Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be ergodic automorphism. Let $(T_t^f)_{t\in\mathbb{R}}$ be special flow built over T under a measurable roof function $f : X \to \mathbb{R}$. Suppose that conditions (1)-(7) are satisfied for sequence $\{W_n\}_{n\in\mathbb{N}}$ of measurable sets, integer sequences $\{q_n\}_{n\in\mathbb{N}}$, $\{q_n'\}_{n\in\mathbb{N}}$, real sequences $\{a_n\}_{n\in\mathbb{N}}$, $\{a_n'\}_{n\in\mathbb{N}}$, real number $0 < \alpha \leq 1$ and measure $P \in \mathcal{P}(\mathbb{R}^2)$. Recall that

$$(f_n', f_n)_*(\mu_{W_n}) \to P \text{ weakly in } \mathcal{P}(\mathbb{R}^2).$$

Moreover, suppose that $q_n' = 2q_n$ and $a_n' = 2a_n$. Then, by Theorem 3.10, we obtain the following.

Corollary 4.1. *Let P, a_n, α be defined as above. Then*

$$\mu^f_{2a_n,a_n} \to \alpha \int_{\mathbb{R}} \mu^f_{-t,-u} dP(t,u) + (1-\alpha)\nu, \tag{17}$$

where ν is a 3-joining of the special flow T^f.

We are going to show a necessary condition for T^f to be isomorphic to its inverse. Assume then, that there exists a measure preserving isomorphism $S : X^f \to X^f$ such that

$$ST_t^f = T_{-t}^f S \text{ for } t \in \mathbb{R}. \tag{18}$$

Then we have the following lemma.

Lemma 4.2. *Let* (17) *and* (18) *be satisfied. Let us consider the function $\theta : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\theta(t,u) = (t, t-u)$. Then*

$$\alpha \int_{\mathbb{R}^2} \mu^f_{t,u} dP(t,u) + (1-\alpha)\rho_1 = \alpha \int_{\mathbb{R}^2} \mu_{t,u}\, d\theta_* P(t,u) + (1-\alpha)\rho_2, \tag{19}$$

where $\rho_1, \rho_2 \in J_3(T^f)$.

Proof. Denote by S_* the map which pushes forward any measure on $(X^f)^3$ by the map $S\times S\times S:(X^f)^3\to(X^f)^3$. Note that S_* maps 3-joinings into 3-joinings. Indeed let $\rho\in J_3(T^f)$. Let $A,B,C\subseteq X^f$ be measurable sets. We have

$$\begin{aligned}S_*\rho(T^f_{-t}A\times T^f_{-t}B\times T^f_{-t}C)&=\rho(S^{-1}T^f_{-t}C\times S^{-1}T^f_{-t}B\times S^{-1}T^f_tA)\\&=\rho(T^f_tS^{-1}C\times T^f_tS^{-1}B\times T^f_tS^{-1}A)\\&=\rho(S^{-1}C\times S^{-1}B\times S^{-1}A)\\&=S_*\rho(A\times B\times C).\end{aligned}$$

Moreover if we take $B=C=X^f$ we obtain

$$S_*\rho(A\times X^f\times X^f)=\nu(X^f\times X^f\times S^{-1}A)=\mu^f(S^{-1}A)=\mu^f(A)$$

Additionally $S_*:J_3(T^f)\to J_3(T^f)$ is continuous.

Let $A,B,C\subseteq X^f$ be measurable. Then

$$\begin{aligned}S_*\mu^f_{t,u}(A\times B\times C)&=\mu(T^f_{-t}S^{-1}A\cap T^f_{-u}S^{-1}B\cap S^{-1}C)\\&=\mu(S^{-1}T^f_tA\cap S^{-1}T^f_uB\cap S^{-1}C)\\&=\mu(T^f_tA\cap T^f_uB\cap C)=\mu^f_{-t,-u}(A\times B\times C).\end{aligned}$$

By the continuity of S_* and by (17), we have

$$\begin{aligned}S_*\mu^f_{2a_n,a_n}&\to\alpha\int_{\mathbb{R}^2}S_*\mu^f_{-t,-u}dP(t,u)+(1-\alpha)S_*\nu\\&=\alpha\int_{\mathbb{R}^2}\mu^f_{t,u}dP(t,u)+(1-\alpha)S_*\nu.\end{aligned}\tag{20}$$

However by a direct computation, using again (17) we obtain

$$\begin{aligned}&S_*\mu^f_{2a_n,a_n}(A\times B\times C)=\mu^f_{-2a_n,-a_n}(A\times B\times C)=\mu^f(T^f_{2a_n}A\cap T^f_{a_n}B\cap C)\\&=\mu^f(A\cap T^f_{-a_n}B\cap T^f_{-2a_n}C)=\mu^f_{2a_n,a_n}(C\times B\times A)\\&\to\alpha\int_{\mathbb{R}^2}\mu^f_{-t,-u}(C\times B\times A)dP(t,u)+(1-\alpha)\nu(C\times B\times A)\\&=\alpha\int_{\mathbb{R}}\mu(A\cap T^f_uB\cap T^f_tC)dP(t,u)+(1-\alpha)\nu(C\times B\times A)\\&=\alpha\int_{\mathbb{R}^2}\mu(T^f_{-t}A\cap T^f_{u-t}B\cap C)dP(t,u)+(1-\alpha)\nu(C\times B\times A)\\&=\alpha\int_{\mathbb{R}^2}\mu_{t,t-u}(A\times B\times C)dP(t,u)+(1-\alpha)\nu(C\times B\times A)\\&=\alpha\int_{\mathbb{R}^2}\mu_{t,u}(A\times B\times C)\,d\theta_*P(t,u)+(1-\alpha)\nu(C\times B\times A).\end{aligned}$$

Let $\rho_1:=S_*\nu$ and let $\rho_2=Q_*\nu$, where $Q:(X^f)^3\to(X^f)^3$ is given by $Q(x,y,z):=(z,y,x)$. Since Q permutes coordinates, Q_* maps 3-joinings into 3-joinings. Recall that $S_*\nu$ is also a joining. Thus the proof is concluded. □

Let $\mathcal{A}\subset J_3(T^f)$ be the set of all 3-off-diagonal joinings and let

$$h:\mathbb{R}^2\to\mathcal{A};\quad h(t,u)=\mu_{t,u}.$$

By Lemma 2.3 $\mathcal{A}$ is measurable. Hence we can write

$$\int_{\mathbb{R}^2}\mu^f_{t,u}dP(t,u)=\int_{\mathcal{A}}\eta d(h_*P)(\eta),$$

and analogously

$$\int_{\mathbb{R}^2}\mu^f_{t,u}\,d\theta_*P(t,u)=\int_{\mathcal{A}}\eta d(h_*\theta_*P)(\eta).$$

Thus we can prove the following theorem.

Theorem 4.3. *Let $(T_f)_{t\in\mathbb{R}}$ be a special flow on X^f isomorphic to its inverse. Assume that* (17) *is satisfied. Then*

$$\alpha P + \beta(1-\alpha)\eta_1 = \alpha\theta_* P + \beta(1-\alpha)\eta_2, \tag{21}$$

for some measures $\eta_1, \eta_2 \in \mathcal{P}(\mathbb{R}^2)$ and $0 \le \beta \le 1$.

Proof. Let ρ_1 and ρ_2 be measures satisfying (19). Then let

$$\rho_1 = \int_{J_3(T^f)} \eta d\kappa_1(\eta) = \beta \int_{\mathcal{A}} \eta d(\kappa_1)_{\mathcal{A}}(\eta) + (1-\beta)\int_{\mathcal{A}^c} \eta d(\kappa_1)_{\mathcal{A}^c}(\eta)$$
$$\rho_2 = \int_{J_3(T^f)} \eta d\kappa_2(\eta) = \beta' \int_{\mathcal{A}} \eta d(\kappa_2)_{\mathcal{A}} + (1-\beta')\int_{\mathcal{A}^c} \eta d(\kappa_2)_{\mathcal{A}^c}(\eta),$$

be ergodic decompositions, where κ_1, κ_2 are Borel measures on $J_3(T^f)$. Note that ergodic decomposition of 3-joining consists of ergodic 3-joinings. By the uniqueness of ergodic decomposition and by (19) we obtain that

$$\alpha \int_{\mathcal{A}} \eta dh_* P(\eta) + \beta'(1-\alpha)\int_{\mathcal{A}} \eta d\kappa_2'(\eta) = \alpha \int_{\mathcal{A}} \eta dh_*\theta_* P(\eta) + \beta(1-\alpha)\int_{\mathcal{A}} \eta d\kappa_1'(\eta),$$

and hence $\beta = \beta'$. Again, by uniqueness of ergodic decomposition, we obtain that

$$\alpha h_* P + \beta(1-\alpha)\kappa_2' = \alpha h_*\theta_* P + \beta(1-\alpha)\kappa_1'. \tag{22}$$

And thus, by pushing forward with h^{-1} (recall that h was a measure theoretic isomorphism), we conclude the proof of the theorem. □

Let us consider

$$\xi : \mathbb{R}^2 \to \mathbb{R}; \quad \xi(x,y) = x - 2y.$$

Recall that

$$P = \lim_{n\to\infty} (f_n', f_n)_* \mu_{W_n},$$

where $f_n = f^{(q_n)} - a_n$ and $f_n' = f^{(2q_n)} - 2a_n$. Then for any $x \in X$ we have

$$\begin{aligned}\xi \circ (f_n', f_n)(x) &= f_n'(x) - 2f_n(x)\\ &= \sum_{i=0}^{2q_n-1} f(T^i x) - 2a_n - 2(\sum_{i=0}^{q_n-1} f(T^i x) - a_n)\\ &= \sum_{i=q_n}^{2q_n-1} f(T^i x) - \sum_{i=0}^{q_n-1} f(T^i x)\\ &= f^{(q_n)}(T^{q_n}x) - f^{(q_n)}(x).\end{aligned} \tag{23}$$

It follows that

$$\xi_* P = \lim_{n\to\infty} (\xi \circ (f_n', f_n))_* \mu_{W_n} = \lim_{n\to\infty} (f^{(q_n)} \circ T^{q_n} - f^{(q_n)})_* \mu_{W_n}. \tag{24}$$

Theorem 4.4. *Suppose that $\xi_* P = \sum_{i=0}^m c_i \delta_{d_i}$ is a discrete measure with $d_0 = 0$. Assume that $\sum_{i=1}^m c_i > \frac{1-\alpha}{\alpha}$, where $0 < \alpha \le 1$ is given by* (1)*, and $d_i \ne -d_j$ for $i \ne j$. Then the special flow T^f is not isomorphic to its inverse.*

Proof. Note that

$$\xi \circ \theta(t,u) = 2u - t = -\xi(t,u).$$

Hence we also have

$$\xi_*(\theta_* P) = \sum_{i=0}^m c_i \delta_{-d_i}.$$

Combining this with (21) we get

$$\alpha\sum_{i=0}^{m}c_i\delta_{d_i}+\beta(1-\alpha)\xi_*\eta_1=\alpha\sum_{i=0}^{m}c_i\delta_{-d_i}+\beta(1-\alpha)\xi_*\eta_2$$

with $0\leq\beta\leq 1$. After normalization this gives

$$\begin{aligned}&\frac{\alpha}{\alpha+\beta(1-\alpha)}\sum_{i=0}^{m}c_i\delta_{d_i}+\frac{\beta(1-\alpha)}{\alpha+\beta(1-\alpha)}\xi_*\eta_1\\ =&\frac{\alpha}{\alpha+\beta(1-\alpha)}\sum_{i=0}^{m}c_i\delta_{-d_i}+\frac{\beta(1-\alpha)}{\alpha+\beta(1-\alpha)}\xi_*\eta_2.\end{aligned}$$

The LHS probability measure has non-zero atoms at d_i, $i=1,\dots,m$ and the sum of their measures is equal to

$$\frac{\alpha}{\alpha+\beta(1-\alpha)}\sum_{i=1}^{m}c_i>\frac{\alpha}{\alpha+\beta(1-\alpha)}\frac{1-\alpha}{\alpha}=\frac{1-\alpha}{\alpha+\beta(1-\alpha)}\geq\frac{\beta(1-\alpha)}{\alpha+\beta(1-\alpha)}.$$

The RHS probability measure has atoms at $-d_i$, $i=0,\dots,m$ and the sum of their measures is greater or equal to $1-\frac{\beta(1-\alpha)}{\alpha+\beta(1-\alpha)}$. By assumption, the union of these atoms is disjoint from the set of atoms $\{d_i : i=1,\dots,m\}$. It follows that the LHS measure with at least $2m+1$ different atoms with total measure greater than 1. This yields a contradiction, and hence the proof is complete. □

5. Interval Exchange Transformations

On the interval $I=[0,1)$ we consider the standard Lebesgue measure *Leb*. Let S_d ($d\geq 2$) be the set of all permutations of d elements. Let $\Lambda^d=\{\lambda\in\mathbb{R}^d_+;|\lambda|=1\}$ be the standard unit simplex, where $|\lambda|=\sum_{i=1}^{d}\lambda_i$ is the length of a vector λ. Let $I_k=[\sum_{j<k}\lambda_k,\sum_{j\leq k}\lambda_k)$ for $k=1,\dots,d$. The *interval exchange transformation* $T_{\pi,\lambda}$ is a map that rearranges intervals I_k according to permutation π. In other words it is an automorphism of I such that $T_{\pi,\lambda}$ acts as a translation on I_k for $k=1,\dots,d$. Note that $T_{\pi,\lambda}$ preserves *Leb* (see [16], [17] or [20] for detailed information about IETs).

In this section we will construct a sequence of sets W_n for interval exchange transformation (IET) such that the assumptions of Theorem 3.10 are met. The construction is based on the construction made by Veech in [19] (see also [11]). Let $T_{\pi,\lambda}:[0,1)\to[0,1)$ be an ergodic interval exchange transformation of d intervals given by a permutation $\pi\in S_d$, and a length vector $\lambda=(\lambda_1,\dots,\lambda_d)\in\Lambda^d$. Then π is irreducible, i.e.

$$\pi(\{1,\dots,k\})=\{1,\dots,k\}\text{ for some }k\in\{1,\dots,d\}\quad\Rightarrow\quad k=d.$$

Denote by S^0_d the set of all irreducible permutations. Let $(I_j)_{j=1\dots d}$ be the exchanged intervals and by ∂I_j we will denote the left endpoint of the interval I_j. Note that $I_j=[\sum_{i<j}\lambda_i,\sum_{i\leq j}\lambda_i)$ and $\partial I_j=\sum_{i<j}\lambda_i$. We say that $T_{\pi,\lambda}$ satisfies *Keane's condition*, if

$$T^k_{\pi,\lambda}(\partial I_i)=\partial I_j\text{ for some }k\in\mathbb{N}\text{ and }i,j\in\{1\dots d\}\Rightarrow k=1\text{ and }j=1.$$

Theorem 5.1 (see Proposition 3.2. in [20]). *If $\pi\in S^0_d$ then for a.e. $\lambda\in\Lambda^d$ the interval exchange transformation $T_{\pi,\lambda}$ is ergodic and satisfies Keane's condition.*

From now on for any interval J we will denote by $|J|$ the length of this interval. Let $s=min\{|I_d|,|I_{\pi^{-1}(d)}|\}$. If $|I_d|\neq|I_{\pi^{-1}(d)}|$ then we can consider the first return map to the interval $[0,1-s)$. Note that newly obtained automorphism is also an interval exchange of d subintervals of $[0,1-s)$ which is given by a pair $(\pi^1,\lambda^1)\in S^0_d\times(0,+\infty)^d$. The map $(\pi,\lambda)\mapsto(\pi^1,\lambda^1)$ is called *the Rauzy-Veech* *induction* and we will denote it by $\hat{R}$. If possible we can iterate Rauzy-Veech induction.

Proposition 5.2 (see Veech in [18]). *Let $T_{\pi,\lambda}$ be an interval exchange transformation satisfying Keane's condition. Then for each $n \in \mathbb{N}$ the n-th Rauzy-Veech induction is well defined.*

Let $(\pi^n, \lambda^n) = \hat{R}(\pi, \lambda)$. By I^n we will denote the interval obtained after n-th step of Rauzy-Veech inductions. Denote by $A^n(\pi, \lambda) = A^n = [A^n_{ij}]_{i,j=1\ldots d}$ the n-th Rauzy-Veech induction matrix (or shortly Rauzy's matrix), i.e. A^n_{ij} is the number of times that the interval I^n_j visits the interval I_i under iterations of $T_{\pi,\lambda}$ before its first return to I^n.

Remark 5.3. Let $T = T_{\pi,\lambda} : I \to I$ be an interval exchange transformation of d intervals satisfying Keane's condition. Then for every $n \in \mathbb{N}$ (see ...), the interval I is decomposed into d Rokhlin towers of the form $\bigcup_{i=0}^{s^n_j - 1} T^i I^n_j$ (i.e. $T^i I^n_j$ for $i = 0, \ldots, s^n_j - 1$ are pairwise disjoint intervals), where $s^n_j := \sum_{i=1}^d A^n_{ij}(\pi, \lambda)$ for $j = 1, \ldots, d$.

The following theorem states some properties of Rauzy's matrices.

Theorem 5.4 (see [16]). *Let $(\pi, \lambda) \in S^0_d \times \Lambda^d$ be such that $T_{\pi,\lambda}$ satisfies Keane's condition. Then*
1. $A^n(\pi, \lambda)\lambda^n = \lambda$;
2. $A^n(\pi, \lambda) = A^1(\pi, \lambda) \cdot \ldots A^1(\pi^{n-1}, \lambda^{n-1}) \cdot A^1(\pi^n, \lambda^n)$;
3. *There exists $n \in \mathbb{N}$ such that $A^n(\pi, \lambda)$ is strictly positive matrix and $\pi^n = \pi$.*

For any positive $d \times d$ matrix B let

$$\rho(B) = \max_{1 \le i,k,l \le d} \frac{B_{ij}}{B_{ik}}.$$

Set $b_j = \sum_{i=1}^d B_{ij}$ and let A be any nonnegative nonsingular $d \times d$ matrix. The following properties are easy to prove

$$b_j \le \rho(B) b_k \quad \text{for any} \quad 1 \le j, k \le d, \tag{25}$$

$$\rho(AB) \le \rho(B). \tag{26}$$

By *normalized Rauzy-Veech induction* we call the map

$$R : S^0_d \times \Lambda^d \to S^0_d \times \Lambda^d, \qquad R(\pi, \lambda) = \Big(\pi^1, \frac{\lambda^1}{|\lambda^1|}\Big).$$

The set of permutations S^0_d splits into subsets called Rauzy graphs $G \subset S^0_d$ such that the product $G \times \Lambda^d$ is R-invariant (see [20] for details). The following theorem was proven by Veech in [18].

Theorem 5.5. *For every Rauzy graph $G \subset S^0_d$ there exists a σ-finite R-invariant measure ζ_G on $G \times \Lambda^d$ equivalent to the product of the counting measure on G and the Lebesgue measure on Λ^d such that the normalized Rauzy-Veech induction R is ergodic and recurrent on $(G \times \Lambda^d, \zeta_G)$.*

6. Non-reversibility of special flows over interval exchange transformation

6.1. **Piecewise linear roof functions.** Let $I = [0, 1)$ be the unit interval. Let $(I, \mathcal{B}, Leb, T)$ be an interval exchange transformation of d intervals given by a pair (π, λ), where π is a non-reducible permutation of d elements, and λ is a length vector. Let $(T^f_t)_{t \in \mathbb{R}}$ be the special flow built over T and under a roof function

$f : [0,1) \to \mathbb{R}$ which is linear over each exchanged interval with the same slope $r \neq 0$. Suppose that T is ergodic and satisfies Keane condition.

Now we will perform the main construction.

Lemma 6.1. *For every $\pi \in S_d^0$ and a.e. $\lambda \in \Lambda^d$ there exists sequence $\{W_n\}_{n\in\mathbb{N}}$ of measurable sets, increasing sequences $\{q_n\}_{n\in\mathbb{N}}$, $\{q'_n\}_{n\in\mathbb{N}}$ of integer numbers such that $q'_n = 2q_n$ and conditions (1)-(4) are satisfied.*

Proof. Fix $(\pi_0, \lambda_0) \in S_d^0 \times \Lambda^d$ and let $\pi_0 \in G$ be the corresponding Rauzy graph. By Theorem 5.4 there exists $m > 1$ such that $B := A^m_{(\pi_0,\lambda_0)}$ is positive matrix and $\pi_0 = \pi_0^m$. Let $\varepsilon > 0$ and let $\delta > 0$ be such that

$$6\delta < \varepsilon \text{ and } (1-3\delta)(1-\rho(B)\frac{\delta}{1-\delta}) > 1-\varepsilon.$$

Let $Y \subseteq \{\pi_0\} \times \Lambda^d$ be the set of (π_0, λ) such that $\lambda_1 > (1-\delta)|\lambda|$ and $\lambda_j > \frac{\delta}{2d}|\lambda|$ for $2 \le j \le d$. Moreover, let $V = \{(\pi_0, B\lambda); (\pi_0,\lambda) \in Y\}$. Since $\zeta(V) > 0$, by the ergodicity and recurrence of R, for a.e. $(\pi,\lambda) \in G_d \times \Lambda^d$, there are infinitely many times r_n for $n \in \mathbb{N}$ such that $R^{r_n}(\pi,\lambda) \in V$. Since $\pi_0 = \pi^m$, we have that $R^{m+r_n}(\pi,\lambda) \in Y$, as by the definition of V, we have that $R^m V \subseteq Y$. Moreover, by the Theorem 5.4

$$A^{m+r_n}(\pi,\lambda) = A^{r_n}(\pi,\lambda)B.$$

Hence, by (26), we obtain that

$$\rho(A^{m+r_n}(\pi,\lambda)) \le \rho(B). \tag{27}$$

Since $R^{m+r_n}(\pi,\lambda) = (\pi^{m+r_n}, \frac{\lambda^{m+r_n}}{|\lambda^{m+r_n}|}) \in Y$, we have

$$\lambda_1^{m+r_n} > (1-\delta)|\lambda^{m+r_n}| \tag{28}$$

$$\lambda_j^{m+r_n} > \frac{\delta}{2d}|\lambda^{m+r_n}| \text{ for } 2 \le j \le d. \tag{29}$$

Let $T := T_{\pi,\lambda}$ and $s_j^n = \sum_{i=1}^d A_{ij}^{m+r_n}$. Note that s_j^n is the first return time of interval $I_j^{m+r_n}$ to I^{m+r_n} under T. Let

$$J^n := I_1^{m+r_n} \cap T^{-s_1^n} I_1^{m+r_n} \cap T^{-2s_1^n} I_1^{m+r_n}.$$

Since $T^{s_1^n}(I_1^{m+r_n}) \subset I^{m+r_n}$ and Leb is $T_{\pi,\lambda}$-invariant, we have

$$Leb(J^n) = Leb(I^{m+r_n} \cap T^{s_1^n} I_1^{m+r_n} \cap T^{2s_1^n} I_1^{m+r_n}),$$

and $T^{s_1^n} I_1^{m+r_n} \cap T^{2s_1^n} I_1^{m+r_n} \subset I^{m+r_n}$. Moreover, by (28)

$$Leb(T^{s_1^n} I_1^{m+r_n} \cap T^{2s_1^n} I_1^{m+r_n}) = Leb(I_1^{m+r_n} \cap T^{s_1^n} I_1^{m+r_n}) \ge (1-2\delta)|I^{m+r_n}|.$$

Thus

$$\begin{aligned} Leb(J^n) &\ge |I^{m+r_n}| - Leb(I^{m+r_n} \setminus I_1^{m+r_n}) \\ &\quad - Leb(I^{m+r_n} \setminus (T^{s_1^n} I_1^{m+r_n} \cap T^{2s_1^n} I_1^{m+r_n})) \\ &> |I^{m+r_n}| - \delta|I^{m+r_n}| - 2\delta|I^{m+r_n}| = (1-3\delta)|I^{m+r_n}| \end{aligned} \tag{30}$$

Since $\bigcup_{i=0}^{s_1^n-1} T^i I_1^{m+r_n}$ is a Rokhlin tower (see Remark 5.3), we have

$$J^n \cap T^l J^n = \emptyset \text{ for } 1 \le l < s_1^n,$$

and by the fact that $T^{s_1^n} J \subseteq I^{m+r_n}$ we have that

$$\begin{aligned} Leb(J^n \cap T^{s_1^n} J^n) &\ge |I^{m+r_n}| - Leb(I^{m+r_n} \setminus J^n) - Leb(I^{m+r_n} \setminus T^{s_1^n} J^n) \\ &> (1-6\delta)|I^{m+r_n}| > (1-\varepsilon)|I^{m+r_n}|. \end{aligned}$$

By Remark 5.3, we have that $|\lambda| = \sum_{j=1}^{d} \lambda_j^{m+r_n} s_j^n$ and by (25), (27) and (28) we obtain that

$$|\lambda| - s_1^n \lambda_1^{m+r_n} = \sum_{j=2}^{d} s_j^n \lambda_j^{m+r_n} \leq \rho(A^{m+r_n}) s_1^n \delta |\lambda^{m+r_n}| \leq \rho(B) s_1^n \delta |\lambda^{m+r_n}|$$
$$\leq \rho(B) s_1^n \frac{\delta}{1-\delta} \lambda_1^{m+r_n} \leq \rho(B) \frac{\delta}{1-\delta} |\lambda|.$$

Therefore

$$s_1^n \lambda_1^{m+r_n} = Leb(\bigcup_{l=0}^{s_1^n - 1} T^l I_1^{m+r_n}) > \Big(1 - \rho(B)\frac{\delta}{1-\delta}\Big)|\lambda|,$$

which, by (30), implies that

$$Leb(\bigcup_{l=0}^{s_1^n - 1} T^l J^n) = s_1^n Leb(J^n) \geq s_1^n |I^{m+r_n}|$$
$$> (1-3\delta)\Big(1 - \rho(B)\frac{\delta}{1-\delta}\Big)|\lambda| > (1-\varepsilon)|\lambda|.$$

Note that final inequality does not depend on n.

Let

$$W_n = \bigcup_{l=0}^{s_1^n - 1} T^l J^n. \tag{31}$$

By passing to a subsequence if necessary, we get

$$\lim_{n\to\infty} Leb(W_n) = \alpha \text{ for some } 1 \geq \alpha \geq 1-\varepsilon. \tag{32}$$

Set

$$q_n := s_1^n. \tag{33}$$

Since $T^l J^n \cap T^k J^n = \emptyset$ for $0 \leq k < l < q_n$ and $k \neq l$, we get that

$$|W_n| = s_1^n |J^n| \to \alpha. \tag{34}$$

Thus condition (1) is satisfied.

Let $x \in W_n$, and let $0 \leq l < q_n$ be such that $x \in T^l J^n$. By the definition of J^n we have that

$$T^{q_n}(x) \in T^{q_n}(T^l J^n) \subset T^l I_1^{m+r_n}. \tag{35}$$

By the properties of Rauzy-Veech induction $|I^{m+r_n}| \to 0$. Hence

$$\limsup_{n\to\infty} |T^{q_n} x - x| = 0. \tag{36}$$

Analogously we can prove that in (35), we can replace q_n by $2q_n$ and as a result

$$\limsup_{n\to\infty} |T^{2q_n} x - x| = 0. \tag{37}$$

Be Lemma 3.3, (36) and (37) imply (3) and (4). Moreover

$$Leb(W_n \triangle T^{-1} W_n) \leq 2 Leb(J^n) \to 0 \text{ with } n \to \infty, \tag{38}$$

which verifies condition (2). □

Lemma 6.2. *Assume that sequences $\{W_n\}_{n\in\mathbb{N}}$, $\{q_n\}_{n\in\mathbb{N}}$ and $\{q_n'\}_{n\in\mathbb{N}}$ are as in Lemma 6.1 and (29) is satisfied. Then for every $x \in W_n$ and $0 \leq l < q_n$ the points $T^l x$ and $T^{q_n+l} x$ belong to the same exchanged interval and the sequence defined by*

$$\gamma_n := \sum_{l=0}^{q_n - 1} (T^{q_n+l}(x) - T^l(x)),$$

converges, up to taking a subsequence, to some $\gamma > 0$.

Proof. Let $x \in W_n$, and let $0 \leq l < q_n$ be such that $x \in T^l J^n$. By (35) we have that $T^{q_n} x \in T^l I_1^{m+r_n}$. By the non-reducibility of considered interval exchange transformation there is $2 \leq p \leq d$ such that $\pi^{m+r_n}(p) < \pi^{m+r_n}(1)$. Hence by (29) we get

$$T^{q_n}(x) - x > |I_p^{m+r_n}| > \frac{\delta}{2d}|I^{m+r_n}|.$$

Since $|I^{m+r_n}| > Leb(J^n)$, we get

$$\gamma_n = \sum_{l=0}^{q_n-1}(T^{q_n+l}(x) - T^l(x)) \geq q_n \frac{\delta}{d} Leb(J^n) = \frac{\delta}{d} Leb(W_n) \to \frac{\delta\alpha}{d} > 0,$$

for $x \in W_n$. Note that γ_n does not depend on $x \in W_n$, because T acts as a linear function on $T^l J^n$ for each $0 \leq l < q_n$. Hence, by passing to a subsequence if necessary, we obtain that

$$\gamma_n \to \gamma \text{ for some } \gamma > 0. \tag{39}$$

□

Lemma 6.3 (see also [11])**.** *Let* $T : I \to I$ *be an IET and let* $f : I \to \mathbb{R}$ *be a function of bounded variation. Let* $\{T^i J\}_{i=0}^{q-1}$ *be a Rokhlin tower such that* $T^i J$, $i = 0, \ldots, q-1$ *are intervals. Then there exists* $a \in \mathbb{R}$ *such that*

$$|f^{(q)}(x) - a| \leq Var_{[0,1)} f \quad \textit{for} \quad x \in \bigcup_{i=0}^{q-1} T^i(J \cap T^{-q}J).$$

Moreover,

$$|f^{(2q)}(x) - 2a| \leq 2Var_{[0,1)} f \quad \textit{for} \quad x \in \bigcup_{i=0}^{q-1} T^i(J \cap T^{-q}J \cap T^{-2q}J).$$

Proof. Let

$$a := \frac{1}{|J|}\int_{\bigcup_{l=0}^{q-1} T^l J} f(t)dt. \tag{40}$$

Then for $x \in T^k(J \cap T^{-q}J)$ we have

$$\begin{aligned} &|f^{(q)}(x) - a| \\ &\leq \sum_{k \leq i < q} \frac{1}{|J|}\int_{T^i J} |f(T^{i-k}x) - f(t)|dt + \sum_{0 \leq i < k} \frac{1}{|J|}\int_{T^i J} |f(T^{q+i-k}x) - f(t)|dt \\ &\leq \sum_{0 \leq i < q} Var_{T^i J} f \leq Var_{[0,1]} f. \end{aligned}$$

If $x \in T^k(J \cap T^{-q}J \cap T^{-2q}J)$ then $T^q x \in T^k(J \cap T^{-q}J)$. Hence

$$|f^{(2q)}(x) - 2a| \leq |f^{(q)}(x) - a| + |f^{(q)}(T^q x) - a| \leq 2Var_{[0,1]} f,$$

which concludes the proof of the lemma. □

Theorem 6.4. *Let* $\pi \in S_d^0$. *For a.e.* $\lambda \in \Lambda^d$ *if* $f : [0,1) \to \mathbb{R}$ *is a roof function which is linear over each exchanged interval with the same slope* $r \neq 0$ *then the special flow* $T_{\pi,\lambda}^f$ *is not isomorphic to its inverse.*

Proof. Let W_n be the tower obtained in (31) and let $\{q_n\}_{n\in\mathbb{N}}$ be the sequence of integer numbers defined in (33). Take $q_n' = 2q_n$. Then by Lemma 6.1 conditions $(1)-(4)$ are satisfied. Moreover, by Lemma 6.3, we obtain a real sequence $\{a_n\}_{n\in\mathbb{N}}$ such that sequences

$$\Big\{\int_{W_n} |f^{(q_n)}(x) - a_n|^2 dx\Big\}_{n\in\mathbb{N}} \quad \text{and} \quad \Big\{\int_{W_n} |f^{(2q_n)}(x) - 2a_n|^2 dx\Big\}_{n\in\mathbb{N}}$$

are bounded, that is (5) and (6) are satisfied. Also, by Lemma 6.3 and by Prokhorov's theorem, passing to a subsequence if necessary, we can assume that

$$P = \lim_{n\to\infty} (f_n', f_n)_* Leb_{W_n} \quad \text{weakly in} \quad \mathcal{P}(\mathbb{R}^2),$$

where

$$f_n = f^{(q_n)} - a_n \quad \text{and} \quad f_n' = f^{(2q_n)} - 2a_n.$$

By (35) and (39), for $x \in W_n$ we have

$$\begin{aligned} f^{(q_n)} \circ T^{q_n}(x) - f^{(q_n)}(x) &= \sum_{i=0}^{q_n-1} (f(T^{q_n+i}x) - f(T^i x)) \\ &= r \sum_{i=0}^{q_n-1} (T^{q_n+i}x - T^i x) = r\gamma_n \to r\gamma, \end{aligned}$$

for some $\gamma > 0$. Therefore, by (24), $\xi_* P = \delta_{r\gamma}$. Since, by (32), we can take $\lim Leb(W_n) = \alpha > 0$ arbitrary close to 1 (in this case it is enough to take $\alpha > \frac{1}{2}$). Then, by Theorem 4.4, this concludes the proof of the theorem. □

7. Piecewise constant roof functions

In this section we will consider special flows built over interval exchange transformations and under piecewise constant roof functions. For some class of these functions we will prove the non-reversibility of such flows, by using similar arguments as in previous sections.

We will need the following general lemma.

Lemma 7.1 (cf. [12])**.** *Let T be an ergodic automorphism of a standard probability space $(X, \mathcal{B}, \mu)$. Let $\{W_n\}_{n\in\mathbb{N}}$ be the sequence of Rokhlin towers such that $\liminf_{n\to\infty} \mu(W_n) > 0$ and such that $\mu(J_n) \to 0$, where J_n is the basis of tower W_n. Then almost every $x \in X$ belongs to W_n for infinitely many $n \in \mathbb{N}$.*

We will apply a construction and arguments similar to those shown in the proof of Lemma 6.1 as well as some notation. We will also need the following fact considering interval exchange transformations.

Lemma 7.2 (see [14])**.** *Let $T_{\pi,\lambda} : [0,1) \to [0,1)$ be an interval exchange transformation of d intervals satisfying Keane's condition. Let G_π be a Rauzy graph associated with π. Then there exists $\pi_0 \in G$ such that $\pi_0(1) = d$ and $\pi_0(d) = 1$.*

Theorem 7.3. *Let $\pi \in S_d^0$. Suppose that $T : [0,1) \to [0,1)$ is an interval exchange transformation of d intervals defined by (π, λ). Assume that $f : [0,1) \to \mathbb{R}_+$ is a piecewise constant function with $r \geq 3$ discontinuity points $\beta_1, \ldots, \beta_r$ and such that f has no jumps with opposite value. Then for almost every $\lambda \in \Lambda^d$ and almost every choice of discontinuity points of the roof function f the special flow T^f on $[0,1)^f$ is non-reversible.*

Proof. Fix $\pi_0 \in G_\pi$ such that $\pi_0(1) = d$ and $\pi_0(d) = 1$. By Theorem 5.4, there exists $m \in \mathbb{N}$ such that $B := A^m(\pi_0, \lambda_0)$ has positive entries and $\pi_0^m = \pi_0$.

Let $0<\varepsilon<\min\left(1/\rho(B)20, 1/8(2r+1)\right)$ and let $\varepsilon/3<\delta'<\delta<\varepsilon/2$ be such that

$$\delta-\delta'<\frac{\varepsilon}{4\rho(B)}. \tag{41}$$

Let $Y\subseteq\{\pi_0\}\times\Lambda^d$ stand for the set of (π_0,λ) such that

$$\frac{1}{2}-\delta<\lambda_1<\frac{1}{2}-\delta+\frac{\delta-\delta'}{4},\qquad \frac{1}{2}+\delta'<\lambda_d<\frac{1}{2}+\delta'+\frac{\delta-\delta'}{4}.$$

Then

$$\lambda_d-\lambda_1>\delta'+\delta-\frac{\delta-\delta'}{4}>\frac{2}{3}\varepsilon-\frac{1}{16}\varepsilon>\frac{\varepsilon}{2}.$$

Note that Y is an open set and hence it is of positive measure. By the arguments used in the proof of Lemma 6.1 we have that for almost every $\lambda\in\Lambda^d$ there exists an increasing sequence of natural numbers $\{r_n\}_{n\in\mathbb{N}}$, such that $R^{m+r_n}(\pi,\lambda)\in Y$. Therefore

$$\lambda_1^{m+r_n}>\Big(\frac{1}{2}-\delta\Big)|\lambda^{m+r_n}|,\qquad \lambda_d^{m+r_n}>\Big(\frac{1}{2}+\delta'\Big)|\lambda^{m+r_n}|\quad\text{and} \tag{42}$$

$$\lambda_d^{m+r_n}-\lambda_1^{m+r_n}>\frac{\varepsilon}{2}|\lambda^{m+r_n}|. \tag{43}$$

Set $s_j^n=\sum_{i=1}^d A_{ij}^{m+r_n}$ and define

$$J^n:=I_1^{m+r_n}\cap T^{-s_1^n-s_d^n}I_1^{m+r_n}\cap T^{2(-s_1^n-s_d^n)}I_1^{m+r_n}.$$

Recall that s_j^n is the first return time of interval $I_j^{m+r_n}$ to I^{m+r_n} under T and, by (25) and (27), we have

$$s_j^n\leq\rho(B)s_k^n\quad\text{for}\quad 1\leq j,k\leq d. \tag{44}$$

Since $\pi_0(1)=d$ and $\pi_0(d)=1$, the interval $I_1^{m+r_n}$ is translated by $T^{s_1^n}$ to the interval ending at the end of I^{m+r_n} and $I_d^{m+r_n}$ is translated by $T^{s_d^n}$ to the interval starting at 0. Moreover, in view of (42), $T^{s_1^n}I_1^{m+r_n}\subset I_d^{m+r_n}$. It follows that $T^{s_1^n+s_d^n}$ acts on $I_1^{m+r_n}$ by the translation by $|I_d^{m+r_n}|-|I_1^{m+r_n}|=\lambda_d^{m+r_n}-\lambda_1^{m+r_n}>0$. Moreover, since $\varepsilon<1/20$, by (42),

$$\lambda_d^{m+r_n}-\lambda_1^{m+r_n}<|\lambda^{m+r_n}|-2\lambda_1^{m+r_n}\leq 2\delta|I^{m+r_n}|\leq\varepsilon|I^{m+r_n}|\leq\frac{1}{4}|I_1^{m+r_n}|. \tag{45}$$

Therefore J^n is an interval starting from 0 and

$$\begin{aligned}|J^n|&=\lambda_1^{m+r_n}-2(\lambda_d^{m+r_n}-\lambda_1^{m+r_n})>\lambda_1^{m+r_n}-4\delta|I^{m+r_n}|\\&>(1-4\delta)|I^{m+r_n}|>\frac{1}{2}|I^{m+r_n}|.\end{aligned} \tag{46}$$

Note also that by the Remark 5.3 and by the fact that $T^{s_1^n}I_1^{m+r_n}\subset I_d^{m+r_n}$, we have that

$$J^n\cap T^iJ^n=\emptyset\text{ for }i=1,\ldots,s_1^n+s_d^n-1. \tag{47}$$

Let now $W_n=\bigcup_{i=0}^{s_1^n+s_d^n-1}T^iJ^n$. Let $x\in T^iJ^n$ for some $0\leq i<s_1^n+s_d^n$. We have $T^{-i}x\in J^n$ and thus

$$\begin{aligned}T^{s_1^n+s_d^n}x&=T^i\circ T^{s_1^n+s_d^n}\circ T^{-i}x=T^i(T^{-i}x+\lambda_d^{m+r_n}-\lambda_1^{m+r_n})\\&=x+\lambda_d^{m+r_n}-\lambda_1^{m+r_n},\end{aligned} \tag{48}$$

where the last equality follows from the fact that

$$T^{-i}x,\ T^{-i}x+\lambda_d^{m+r_n}-\lambda_1^{m+r_n}\in I_1^{m+r_n}$$

and T^i acts on $I_1^{m+r_n}$ as a translation. Hence we obtain

$$\sup_{x\in W_n}d(x,T^{s_1^n+s_d^n}x)=\lambda_d^{m+r_n}-\lambda_1^{m+r_n}<|I^{m+r_n}|\to 0. \tag{49}$$

Similar argument shows also that

$$\sup_{x\in W_n} d(x,T^{2(s_1^n+s_d^n)}x)=2(\lambda_d^{m+r_n}-\lambda_1^{m+r_n})<2|I^{m+r_n}|\to 0. \tag{50}$$

Note that the points that do not belong to W_n come from three sources, namely: the tower of height s_1^n built over $I_1^{m+r_n}\setminus J^n$, the towers built of height s_j^n over the intervals $I_j^{m+r_n}$ for $j=2,\dots,d-1$ and the tower of height s_d^n built over $I_d^{m+r_n}\setminus T^{s_1^n}J^n$. Since, by (46),

$$Leb(I_1^{m+r_n}\setminus J^n)=|I_1^{m+r_n}|-|J^n|=2(\lambda_d^{m+r_n}-\lambda_1^{m+r_n}),$$
$$Leb(I_d^{m+r_n}\setminus T^{s_1^n}J^n)=|I_d^{m+r_n}|-|J^n|=3(\lambda_d^{m+r_n}-\lambda_1^{m+r_n})$$

and by (42) the sum of lengths of intervals $I_j^{m+r_n}$, $j=2,\dots,d-1$ is

$$|I^{m+r_n}|-\lambda_d^{m+r_n}-\lambda_1^{m+r_n}\le(\delta-\delta')|I^{m+r_n}|,$$

it follows that

$$Leb([0,1)\setminus W_n)<(\lambda_d^{m+r_n}-\lambda_1^{m+r_n})(2s_1^n+3s_d^n)+(\delta-\delta')|I^{m+r_n}|\max_{1<j<d}(s_j^n). \tag{51}$$

Let us consider r disjoint segments

$$J_l^n=\Big[\frac{(2l-1)|J^n|}{2r+1},\frac{2l|J^n|}{2r+1}\Big)\subset J^n\quad\text{ for } l=1,\dots,r.$$

In view of (47), $W_n^l=\bigcup_{i=0}^{s_1^n+s_d^n-1}T^iJ_l^n$ are also Rokhlin towers for $1\le l\le r$. Note that, by (44),

$$s_1^n|I^{m+r_n}|\ge\sum_{j=1}^d\frac{s_j^n}{\rho(B)}|I_j^{m+r_n}|=1/\rho(B). \tag{52}$$

Hence, by (46)

$$Leb(W_n^l)=(s_1^n+s_d^n)\frac{1}{2r+1}|J^n|>s_1^n\frac{1}{2(2r+1)}|I^{m+r_n}|\ge\frac{1}{2(2r+1)\rho(B)}>0.$$

By Lemma 7.1, for almost every choice $(\beta_1,\dots\beta_r)\in[0,1)^r$ we have $\beta_l\in W_l^n$ for infinitely many $n\in\mathbb{N}$ for all $1\le l\le r$. Consider now the sets

$$V_l^n=\bigcup_{i=0}^{s_1^n+s_d^n-1}T^{-i}\big[\beta_l-(\lambda_d^{m+r_n}-\lambda_1^{m+r_n}),\beta_l\big)\quad\text{ for }\quad l=1,\dots,r.$$

Note that V_l^n are Rokhlin towers. Indeed, since $\beta_l\in W_n^l$, there exists $0\le k=k(l)<s_1^n+s_d^n$ such that $T^{-k}\beta_l\in J^n$ for $l=1,\dots,r$. Note also that, by (45) and (46),

$$\lambda_d^{m+r_n}-\lambda_1^{m+r_n}<\varepsilon|I^{m+r_n}|<\frac{1}{8(2r+1)}|I^{m+r_n}|<\frac{1}{2(2r+1)}|J^n|. \tag{53}$$

Hence

$$[\beta_l-2(\lambda_d^{m+r_n}-\lambda_1^{m+r_n}),\beta_l)\subset T^k\Big[\frac{2l-2}{2r+1}|J^n|,\frac{2l}{2r+1}|J^n|\Big)\subset W_n.$$

Since $T^{s_1^n+s_d^n}$ acts on W_n as the translation by $\lambda_d^{m+r_n}-\lambda_1^{m+r_n}$, we have

$$T^{s_1^n+s_d^n}[\beta_l-2(\lambda_d^{m+r_n}-\lambda_1^{m+r_n}),\beta_l-(\lambda_d^{m+r_n}-\lambda_1^{m+r_n}))=[\beta_l-(\lambda_d^{m+r_n}-\lambda_1^{m+r_n}),\beta_l).$$

It follows that

$$T^{-s_1^n-s_d^n}[\beta_l-(\lambda_d^{m+r_n}-\lambda_1^{m+r_n}),\beta_l)\subset T^k\Big[\frac{2l-2}{2r+1}|J^n|,\frac{2l}{2r+1}|J^n|\Big).$$

Thus for $0 \le i \le k$ we have

$$T^{-i}[\beta_l - (\lambda_d^{m+r_n} - \lambda_1^{m+r_n}), \beta_l) \subset T^{k-i}\Big[\frac{2l-2}{2r+1}|J^n|, \frac{2l}{2r+1}|J^n|\Big) \subset W_n$$

and for $k < i < s_1^n + s_d^n$ we have

$$T^{-i}[\beta_l - (\lambda_d^{m+r_n} - \lambda_1^{m+r_n}), \beta_l) \subset T^{s_1^n+s_d^n+k-i}\Big[\frac{2l-2}{2r+1}|J^n|, \frac{2l}{2r+1}|J^n|\Big) \subset W_n.$$

By (47), this implies that $T^{-i}[\beta_l - (\lambda_d^{m+r_n} - \lambda_1^{m+r_n}), \beta_l)$ are pairwise disjoint for $i = 0, \ldots, s_1^n + s_d^n - 1$ and $l = 1, \ldots, r$ and all these sets are subsets of W_n. It follows that V_l^n, $l = 1, \ldots, r$ are pairwise disjoint Rokhlin towers all included in W_n. Hence

$$Leb(\bigcup_{l=1}^{r} V_l^n) = r(\lambda_d^{m+r_n} - \lambda_1^{m+r_n})(s_1^n + s_d^n) \ge \frac{r\varepsilon}{2} s_1^n |I^{m+r_n}|. \tag{54}$$

By (52) and by passing to a subsequence, if necessary, we get

$$\lim_{n\to\infty} Leb\Big(\bigcup_{l=1}^{r} V_l^n\Big) = \Gamma \ge \frac{r\varepsilon}{2\rho(B)} > 0, \tag{55}$$

and in particular

$$\lim_{n\to\infty} Leb(W_n) = \alpha > 0. \tag{56}$$

Moreover, by (51), (54), (43), (44), (41) and (52), we have

$$\begin{aligned}
&Leb\Big(\bigcup_{l=1}^{r} V_l^n\Big) - Leb([0,1) \setminus W_n) \\
&> (\lambda_d^{m+r_n} - \lambda_1^{m+r_n})\Big((r-2)s_1^n + (r-3)s_d^n\Big) - (\delta - \delta') \max_{1<j<d}(s_j^n)|I^{m+r_n}| \\
&> \frac{\varepsilon}{2} s_1^n |I^{m+r_n}| - (\delta - \delta') \max_{1<j<d}(s_j^n)|I^{m+r_n}| \\
&> \Big(\frac{\varepsilon}{2} - (\delta - \delta')\rho(B)\Big) s_1^n |I^{m+r_n}| > \frac{\varepsilon}{4} s_1^n |I^{m+r_n}| > \frac{\varepsilon}{4\rho(B)}.
\end{aligned} \tag{57}$$

Therefore

$$\Gamma - (1-\alpha) \ge \frac{\varepsilon}{4\rho(B)} > 0. \tag{58}$$

Let $f : [0,1) \to \mathbb{R}_+$ be a piecewise constant roof function for which $\beta_1, \ldots, \beta_r$ are all discontinuities and with jumps equal to $d_1, \ldots, d_r$ respectively. By assumption, $d_j \ne -d_k$ for all $1 \le j, k \le r$.

Let $q_n = s_1^n + s_d^n$ and $q_n' = 2q_n$. By the definition of $\{W_n\}_{n\in\mathbb{N}}$, (49), (50) and Lemma 6.3, there exist $\{a_n\}_{n\in\mathbb{N}}$, $\{a_n'\}_{n\in\mathbb{N}}$ that meet the conditions (1)-(6). Passing to a subsequence, if necessary, we can assume that $P = \lim(f_n', f_n)_* Leb_{W_n}$ and let us consider the measure

$$\xi_* P = \lim_{n\to\infty} (\xi \circ (f_n', f_n))_* \mu_{W_n} = \lim_{n\to\infty} (f^{(q_n)} \circ T^{q_n} - f^{(q_n)})_* \mu_{W_n}.$$

Since $f^{(q_n)}(T^{q_n}x) - f^{(q_n)}(x) = \sum_{i=0}^{q_n-1}(f(T^{q_n+i}x) - f(T^i x))$, by the definition of sets $V_l^n \subset W_n$, for every $x \in W_n$ we have

$$f^{(q_n)}(T^{q_n}x) - f^{(q_n)}(x) = \begin{cases} d_l & \text{if } x \in V_l^n \text{ for } l = 1, \ldots, r \\ 0 & \text{otherwise.} \end{cases}$$

Thus, by (55) and (58), it follows that the measure $\xi_* P$ has r non-zero atoms at d_l for $l = 1, \dots, r$ with total mass

$$\frac{\Gamma}{\alpha} > \frac{1-\alpha}{\alpha}.$$

In view of Theorem 4.4, this gives that T^f is not isomorphic to its inverse. □

8. Piecewise absolutely continuous roof functions

We will use the results of the previous section to prove the non-reversibility of special flows built over IETs under piecewise absolutely continuous functions (AC functions). Let $T := T_{\pi,\lambda} : [0,1) \to [0,1)$ be an uniquely ergodic IET. Let $f : [0,1) \to \mathbb{R}$ be a positive piecewise absolutely continuous roof function. Then the derivative Df is well defined almost everywhere, $Df \in L^1([0,1), Leb)$ and we can define the sum of jumps of f as

$$S(f) = \int_0^1 Df(x)dx.$$

We can decompose the function f into the sum of functions f_{pl} and f_{ac}, where f_{pl} is a piecewise linear function with the slope $S(f)$ and f_{ac} given by the formula

$$f_{ac}(x) = \int_0^x Df(t)dt - S(f)x, \tag{59}$$

is an absolutely continuous function. Note that $\int_0^1 Df_{ac}(t)dt = f_{ac}(1) - f_{ac}(0) = 0$.

The proof of the following result is partially based on the proof of Lemma 4.8 in [13].

Lemma 8.1. *Let $g : [0,1) \to \mathbb{R}$ be a function of bounded variation. Let $\{T^i\Delta_n\}_{i=0}^{q_n-1}$ be a sequence of Rokhlin towers such that $T^i\Delta_n$ for $i = 0, \dots, q_n - 1$ and $n \in \mathbb{N}$ are intervals and $q_n \to +\infty$. Let $J_n := \Delta_n \cap T^{-q_n}\Delta_n \cap T^{-2q_n}\Delta_n$ and $W_n = \bigcup_{i=0}^{q_n-1} T^i J_n$. Suppose that* $\liminf Leb(W_n) = \alpha > 0$. *Then there exists a sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$, such that for $x \in W_n$ the sequences of functions $\{g^{(q_n)} - a_n\}_{n\in\mathbb{N}}$ and $\{g^{(2q_n)} - 2a_n\}_{n\in\mathbb{N}}$ are uniformly bounded for $n \in \mathbb{N}$. Moreover, there exists measure $Q \in \mathcal{P}(\mathbb{R}^2)$ such that*

$$\left(g^{(2q_n)} - 2a_n, g^{(q_n)} - a_n\right)_* Leb_{W_n} \to Q \text{ weakly in } \mathcal{P}(\mathbb{R}^2).$$

up to taking a subsequence. If additionally g is absolutely continuous function with $\int_0^1 Dg(t)dt = 0$ then

$$(g^{(q_n)} \circ T^{q_n} - g^{(q_n)})\chi_{W_n} \to 0 \quad \text{uniformly.}$$

Proof. By Lemma 6.3, we immediately obtain a real sequence $\{a_n\}_{n\in\mathbb{N}}$ such that

$$|g^{(q_n)}(x) - a_n| \le Var(g) \text{ and } |g^{(2q_n)}(x) - 2a_n| < 2Var(g) \text{ for } x \in W_n \text{ and } n \in \mathbb{N}.$$

Moreover, by Prokhorov's theorem, the weak limit measure of the sequence

$$\{\left(g^{(2q_n)} - 2a_n, g^{(q_n)} - a_n\right)_* Leb_{W_n}\}_{n\in\mathbb{N}}$$

in $\mathcal{P}(\mathbb{R}^2)$ exists up to taking a subsequence. Denote the limit measure by $Q \in \mathcal{P}(\mathbb{R}^2)$.

Assume that g is absolutely continuous function with $\int_0^1 Dg(t)dt = 0$. Since g is absolutely continuous, for every $\varepsilon > 0$ there exists a function $g_\varepsilon \in C^1([0,1))$ such that $Var(g_\varepsilon - g) = \|Dg - Dg_\varepsilon\|_{L^1} < \varepsilon$ and $g_\varepsilon(0) = g(0)$. Then

$$\Big|\int_0^1 Dg_\varepsilon(t)dt\Big| = \Big|\int_0^1 (Dg_\varepsilon(t) - Dg(t))dt\Big| \le \int_0^1 |Dg_\varepsilon(t) - Dg(t)|dt < \varepsilon.$$

By the unique ergodicity of T, we have

$$\frac{1}{q_n}\Big|\sum_{i=0}^{q_n-1} Dg_\varepsilon\circ T^i\Big| \to \Big|\int_0^1 Dg_\varepsilon(t)dt\Big| \quad \text{uniformly.}$$

For sufficiently large n and for $x\in W_n$, we have

$$\begin{aligned}\Big|\sum_{i=0}^{q_n-1}(g_\varepsilon(T^{q_n+i}x)-g_\varepsilon(T^ix))\Big| &= \Big|\sum_{i=0}^{q_n-1}\int_{T^ix}^{T^{q_n+i}x} Dg_\varepsilon(t)dt\Big|\\ &\leq \int_x^{T^{q_n}x}\Big|\sum_{i=0}^{q_n-1} Dg_\varepsilon(T^it)dt\Big|dt < q_n\varepsilon|T^{q_n}x-x|.\end{aligned}$$

Since $|T^{q_n}x-x|\leq Leb(\Delta_n)$ and $q_nLeb(\Delta_n)=Leb(\bigcup_{i=0}^{q_n-1}T^i\Delta_n)\leq 1$, we have $q_n|T^{q_n}x-x|\leq 1$, and thus

$$\Big|\sum_{i=0}^{q_n-1}(g_\varepsilon(T^{q_n+i}x)-g_\varepsilon(T^ix))\Big| < \varepsilon. \tag{60}$$

Since $[T^ix,T^{q_n+i}x)$ for $0\leq i<q_n$ are included in different levels of the tower $\{T^i\Delta_n\}_{i=0}^{q_n-1}$, $[T^ix,T^{q_n+i}x)$ for $0\leq i<q_n$ are pairwise disjoint. Hence for $x\in W_n$ we get

$$\begin{aligned}\Big|\sum_{i=0}^{q_n-1}(g_\varepsilon(T^{q_n+i}x)-g_\varepsilon(T^ix)) &- \sum_{i=0}^{q_n-1}(g(T^{q_n+i}x)-g(T^ix))\Big|\\ &= \sum_{i=0}^{q_n-1}\Big|(g_\varepsilon-g)(T^{q_n+i}x)-(g_\varepsilon-g)(T^ix)\Big| \qquad (61)\\ &\leq \sum_{i=0}^{q_n-1} Var_{[T^ix,T^{q_n+i}x)}(g_\varepsilon-g)\leq Var_{[0,1)}(g_\varepsilon-g)<\varepsilon.\end{aligned}$$

Combining (60) and (61), for sufficiently large $n>0$ and $x\in W_n$ we have

$$|g^{(q_n)}(T^{q_n}x)-g^{(q_n)}(x)| = \Big|\sum_{i=0}^{q_n-1}(g(T^{q_n+i}x)-g(T^ix))\Big| < 2\varepsilon,$$

which completes the proof. □

Now we will state a more general version of Theorems 6.4 and 7.3.

Theorem 8.2. *Let $f:[0,1)\to\mathbb{R}_+$ be a piecewise absolutely continuous function with $\beta_1,\ldots,\beta_r$ its discontinuity points. Then for almost every $(\pi,\lambda)\in S_d^0\times\Lambda^d$ we have:*

1. *if $S(f)\neq 0$ and f is absolutely continuous over exchanged intervals, or*
2. *for almost every choice of $(\beta_1,\ldots,\beta_r)\in[0,1)^r$ with $r\geq 3$ if $S(f)=0$ and f has no jumps with opposite value,*

then $T^f_{\pi,\lambda}$ is not isomorphic to its inverse.

Proof. Let $f=f_{pl}+f_{ac}$ be the decomposition of f into its piecewise linear part with slope $S(f)$ and the absolutely continuous part satisfying $\int_0^1 Df_{ac}(t)dt=0$. If $f_{ac}=0$, then the assertion of the theorem follows straightforwardly from Theorems 7.3 and 6.4. We will show that the result remains unchanged when f_{ac} is non-zero.

Let the sequence of Rokhlin towers $\{W_n\}_{n\in\mathbb{N}}$, the sequence of integer numbers $\{q_n\}_{n\in\mathbb{N}}$ and real sequence $\{a_n\}_{n\in\mathbb{N}}$ be as in the proof of Theorems 6.4 and 7.3, where the roof function is equal to f_{pl}. Then

$|f_{pl}^{(q_n)}(x)-a_n|\leq Var(f_{pl})$ and $|f_{pl}^{(2q_n)}(x)-2a_n|<2Var(f_{pl})$ for $x\in W_n$ and $n\in\mathbb{N}$

and

$$\lim_{n\to\infty}(f_{pl}^{(2q_n)}-2a_n, f_{pl}^{(q_n)}-a_n)_*Leb_{W_n}=P \quad \text{in} \quad \mathcal{P}(\mathbb{R}^2).$$

In both cases we obtain that the measure $\xi_*P\in\mathcal{P}(\mathbb{R})$ satisfies the assumption of Theorem 4.4. By applying Lemma 6.3 to the function f and using Prokhorov's theorem we obtain a sequence $\{b_n\}_{n\in\mathbb{N}}$ such that

$$|f^{(q_n)}(x)-b_n|\le Var(f) \text{ and } |f^{(2q_n)}(x)-2b_n|<2Var(f) \text{ for } x\in W_n \text{ and } n\in\mathbb{N}$$

and there exists the weak limit of measures

$$\lim_{n\to\infty}(f^{(2q_n)}-2b_n, f^{(q_n)}-b_n)_*Leb_{W_n}=Q\in\mathcal{P}(\mathbb{R}^2),$$

up to taking a subsequence. We will show that $\xi_*Q=\xi_*P$, that is the measure Q also satisfies the assumption of Theorem 4.4. Indeed, by (24) we have

$$\begin{aligned}\xi_*Q &\leftarrow (f^{(q_n)}\circ T^{q_n}-f^{(q_n)})_*Leb_{W_n}\\ &=(f_{pl}^{(q_n)}\circ T^{q_n}-f_{pl}^{(q_n)}+f_{ac}^{(q_n)}\circ T^{q_n}-f_{ac}^{(q_n)})_*Leb_{W_n}.\end{aligned}$$

By Lemma 8.1 $(f_{ac}^{(q_n)}\circ T^{q_n}-f_{ac}^{(q_n)})\chi_{W_n}\to 0$ uniformly, as $n\to\infty$. Using Lemma 3.7 we obtain that

$$\xi_*Q=\lim_{n\to\infty}(f_{pl}^{(q_n)}\circ T^{q_n}-f_{pl}^{(q_n)})_*Leb_{W_n}=\xi_*P.$$

This completes the proof. □

References

[1] H. Anzai, *On an example of a measure preserving transformation which is not conjugate to its inverse*, Proc. Japan Acad. 27 (1951), 517-522.

[2] I. P. Cornfeld, S. V. Fomin, Ya. G. Sinai, *Ergodic theory.* Springer-Verlag, New York, 1982.

[3] A. I. Danilenko, Ryzhikov, Valery V. *On self-similarities of ergodic flows.* Proc. Lond. Math. Soc., 104 (2012), no. 3, 431–454.

[4] A. del Junco, *Disjointness of measure-preserving transformations, minimal self-joinings and category.* Ergodic theory and dynamical systems, I (College Park, Md., 1979–80), pp. 81–89, Progr. Math., 10, Birkhäuser, Boston, Mass., 1981.

[5] K. Frączek, J. Kułaga, M. Lemańczyk, *Non-reversibility and self-joinings of higher orders for ergodic flows*, `arXiv:1206.3053`, accepted for publication in Journal d'Analyse Mathematique.

[6] K. Frączek, M. Lemańczyk, *On the self-similarity problem for ergodic flows.* Proc. Lond. Math. Soc. 99 (2009), no. 3, 658–696.

[7] K. Frączek, M. Lemańczyk, *A class of special flows over irrational rotations which is disjoint from mixing flows.* Ergodic Theory Dynam. Systems 24 (2004), no. 4, 1083–1095.

[8] K. Frączek, M. Lemańczyk, *On disjointness properties of some smooth flows.* Fund. Math. 185 (2005), 117–142.

[9] E. Glasner, *Ergodic theory via joinings.* Mathematical Surveys and Monographs, 101. American Mathematical Society, Providence, RI, 2003.

[10] P.R. Halmos, J. von Neumann, *Operator methods in classical mechanics. II*, Ann. of Math. (2) 43 (1942), 332-350.

[11] A. Katok, *Interval exchange transformations and some special flows are not mixing*, Israel J. Math. 35 (1980), 301-310.

[12] J. King, *Joining-rank and the structure of finite rank mixing transformations*, J. Analyse Math. 51 (1988), 182-227.

[13] J. Kułaga, *On the self-similarity problem for smooth flows on orientable surfaces*, Ergodic Theory Dynam. Systems 32 (2012), 1615-1660.

[14] G. Rauzy, *Échanges d'intervalles et transformations induites*, Acta Arith. 34 (1979), 315-328.

[15] V.V. Ryzhikov, *Partial multiple mixing on subsequences can distinguish between automorphisms T and T^{-1}*, Math. Notes 74 (2003), 841–847

[16] W. Veech, *Interval exchange transformations.* J. Analyse Math. 33 (1978), 222-272.

[17] W. Veech, *Projective Swiss cheeses and uniquely ergodic interval exchange transformations.* Ergodic theory and dynamical systems, I (College Park, Md., 1979–80), pp. 113–193, Progr. Math., 10, Birkhäuser, Boston, Mass., 1981.

[18] W. Veech, *Gauss measures for transformations on the space of interval exchange maps.* Ann. of Math. (2) 115 (1982), 201-242.
[19] W. Veech, *The metric theory of interval exchange transformations. I. Generic spectral properties.* Amer. J. Math. 106 (1984), 1331-1359.
[20] M. Viana, *Ergodic theory of interval exchange maps.* Rev. Mat. Complut. 19 (2006), 7–100.

Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, ul. Chopina 12/18, 87-100 Toruń, Poland
E-mail address: zimowy@mat.umk.pl, fraczek@mat.umk.pl