Self-similarity for ergodic flows

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Self-similarity

Let us consider a measure-preserving flow $(T_t)_{t\in\mathbb{R}}$ on a probability standard Borel space $(X, \mathcal{B}, \mu)$. The flow $\mathcal{T} = (T_t)_{t\in\mathbb{R}}$ is called **self-similar** if there exists $s \neq \pm 1$ such that the rescaled flow $\mathcal{T}_s = (T_t)_{st\in\mathbb{R}}$ is isomorphic to the original flow $\mathcal{T} = (T_t)_{t\in\mathbb{R}}$, this is there exists a measure-preserving automorphism $S : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ such that

$$S \circ T_t = T_{st} \circ S \text{ for all } t \in \mathbb{R}.$$

If $s = -1$ then $\mathcal{T}$ is usually called **reversible**.

$$\mathcal{I}(\mathcal{T}) := \{s \neq 0 : \mathcal{T} \text{ and } \mathcal{T}_s \text{ are isomorphic}\}$$

$\mathcal{I}(\mathcal{T})$ is a multiplicative subgroup of $\mathbb{R}^*$.

$$\mathcal{I}_{aut}(\mathcal{T}) := \{(t, t') \in \mathbb{R}^2 : T_t \text{ and } T_{t'} \text{ are isomorphic}\}$$

Joining method

By a **joining** between flow $\mathcal{T} = (T_t)_{t\in\mathbb{R}}$ on $(X, \mathcal{B}, \mu)$ and $\mathcal{S} = (S_t)_{t\in\mathbb{R}}$ on $(Y, \mathcal{C}, \nu)$ we mean any probability measure ρ on $(X \times Y, \mathcal{B} \otimes \mathcal{C})$ such that

- ρ is $(T_t \times S_t)_{t\in\mathbb{R}}$–invariant;
- the projections of ρ on X and Y are equal to μ and ν respectively.

$$\mu \times \nu \in \mathcal{J}(\mathcal{T}, \mathcal{S}) := \text{ the set of all joinings.}$$

The flows $\mathcal{T}$, $\mathcal{S}$ are called **disjoint** in the Furstenberg sense if $\mathcal{J}(\mathcal{T}, \mathcal{S}) = \{\mu \times \nu\}$.

$$\mathcal{T} \text{ and } \mathcal{S} \text{ disjoint } \Longrightarrow \mathcal{T},\ \mathcal{S} \text{ are not isomorphic}$$

If $R : (X, \mu) \to (Y, \nu)$ is an isomorphism of $\mathcal{T}$ and $\mathcal{S}$, i.e. $R \circ T_t = S_t \circ R$ then the graph measure μ_R (the image of μ via $X \ni x \mapsto (x, Rx) \in X \times Y$) is a joining.
$\{\mu_{T_t} : t \in \mathbb{R}\}$ an important family of self-joinings of $\mathcal{T}$.

Operator approach - Vershik

Every joining $\rho \in \mathcal{J}(\mathcal{T}, \mathcal{S})$ defines an operator $V_\rho : L^2(X, \mu) \to L^2(Y, \nu)$ by

$$\begin{array}{ccc} L^2(X,\mu) & \hookrightarrow & L^2(X \times Y, \rho) \\ & V_\rho \searrow & \downarrow pr \\ & & L^2(Y,\nu) \end{array}$$

$V_\rho : L^2(X, \mu) \to L^2(Y, \nu)$ is an **intertwining** Makov operator

- $f \geqslant 0 \Longrightarrow V_\rho f \geqslant 0$;
- $V_\rho 1 = 1$, $V_\rho^* 1 = 1$;
- $V\rho \circ T_t = S_t \circ V_\rho$.

$T_t : L^2(X, \mu) \to L^2(X, \mu)$ standard unitary Koopman operator

$$T_t(f) = f \circ T_t.$$

Operator approach

$\rho \mapsto V_\rho$ gives a one-to-one correspondence between joinings and intertwining Markov operators (Vershik, Ryzhikov).

$$\mu \times \nu \quad \longleftrightarrow \quad \int : L^2(X,\mu) \to L^2(Y,\nu),$$

$$(\int f)(y) = \int_X f\,\mu$$

$$\mu_{T_t} \quad \longleftrightarrow \quad T_t$$

ergodic joining ρ $\longleftrightarrow$ indecomposable operator V_ρ

ρ is an ergodic measure for the flow $(T_t \times S_t)_{t \in \mathbb{R}}$

Obvious examples

- **Positive entropy:** Let $\mathcal{T} = (T_t)_{t\in\mathbb{R}}$ be a measure-preserving flow such that $0 < h_\mu(\mathcal{T}) < +\infty$. Then $h_\mu(\mathcal{T}_s) = |s| h_\mu(\mathcal{T})$. Since entropy is an invariant for isomorphism of flows $\mathcal{I}(\mathcal{T}) \subset \{-1, 1\}$.
- **Zero entropy:** Let $(h_t)_{t\in\mathbb{R}}$ be the **horocycle** flow on a compact surface of constant negative curvature M. $(h_t)_{t\in\mathbb{R}}$ acts on the unit tangent bundle $UT(M)$ and preserves a unique probability measure μ_0. If $(g_s)_{s\in\mathbb{R}}$ stands for the **geodesic** flow then

  $$g_s \circ h_t \circ g_s^{-1} = h_{te^{-2s}},$$

  hence each $s > 0$ is a scale of self-similarity for $(h_t)_{t\in\mathbb{R}}$.
- **Infinite entropy:** Such flows can have also plenty of self-similarities.

Abstract result

Proposition (abstract)

Let (U_t) be a bounded C^0-semigroup on a separable Banach space B ($\|U_t\| \leqslant C$). Suppose that

$$B^0 \subset B^{\odot}(= \{x^* \in B^* : t \mapsto U_t^* x^* \text{ is strongly continuous}\}$$

is a closed (U_t^*)-invariant separable subspace such that $0 \in B^0$ is the only fixed point for (U_t^*) on B^0. Suppose that

$$U_{t_n}^* \to Q : B^0 \to B^* \ *\text{-weakly.}$$

Then there exists $E \subset \mathbb{R}$ of full Lebesgue measure such that if

$$A \circ U_s^* = U_{rs}^* \circ A$$

for some $r \in E$, $s \in \mathbb{R}$, $A : B^0 \to B^0$, then

$$A \circ Q = 0 \text{ on } B^0.$$

Theorem

Let $\mathcal{T} = (T_t)_{t\in\mathbb{R}}$ be a weakly mixing flow such that

$$(*)\quad T_{t_n} \to Q = \alpha \int_{\mathbb{R}} T_s \, dP(s) + (1-\alpha)J \text{ weakly in } L^2(X,\mu),$$

where $0 < \alpha \leqslant 1$, $P \in \mathcal{P}(\mathbb{R})$ and $J \in J(\mathcal{T})$. Then $\mathcal{T}$ and $\mathcal{T}_s$ are disjoint for a.e. $s \in \mathbb{R}$, moreover, $\mathcal{T}_s$ and $\mathcal{T}_t$ are disjoint for a.e. $(s,t) \in \mathbb{R}^2$.

Proof. We apply Abstract Proposition to $B = B^* = B^{\odot} = L^2(X,\mu)$, $U_t = T_{-t}$ and $U_t^* = T_t$. Let $B^0 := L^2_0(X,\mu)$. By ergodicity, zero is the only fixed point of (U_t^*) on B^0. Suppose that $r \in E$. We will show that $\mathcal{T}$ and $\mathcal{T}_r$ are disjoint. Let $A : L^2(X,\mu) \to L^2(X,\mu)$ be a joining between $\mathcal{T}$ and $\mathcal{T}_r$, then

$$A \circ T_s = T_{rs} \circ A \text{ for each } s \in \mathbb{R}.$$

It follows that

$$
\begin{aligned}
0 = A \circ Q = A \circ (\alpha \int_{\mathbb{R}} T_s \, dP(s) + (1-\alpha)J) \\
= \alpha \int_{\mathbb{R}} A \circ T_s \, dP(s) + (1-\alpha) A \circ J \text{ on } L^2_0(X, \mu),
\end{aligned}
$$

hence

$$
\alpha \int_{\mathbb{R}} A \circ T_s \, dP(s) + (1-\alpha) A \circ J = \int \text{ on } L^2(X, \mu).
$$

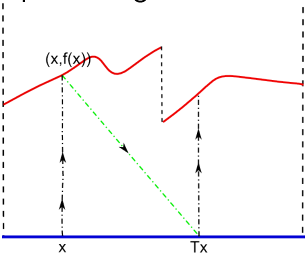
By the weak mixing of $\mathcal{T}$, $\mu \times \mu \in J(\mathcal{T})$ is ergodic, so $\int$ is indecomposable. Consequently, $A \circ T_s = \int$ for P-a.e. s, and hence $A = \int \circ T_{-t} = \int$. □

Abstract Proposition can be applied to the horocycle flow to prove

$$
(h_t)_* \mu \to \mu_0 \text{ weakly in } \mathcal{P}(UTM).
$$

This gives some new equidistribution results for horocyle flows.

How to verify the property $(*)$?
Special flow T^f built over $T:(X,\mu)\to(X,\mu)$ and a positive square integrable $f:X\to\mathbb{R}^+$.



Suppose that T is rigid, i.e. $T^{q_n}\to Id$. Suppose that $(f_0^{(n)})_{n\geqslant 1}$ is bounded in $L^2(X,\mu)$, where

$$f_0(x)=f(x)-\int f\,d\mu,\quad f_0^{(n)}(x)=f_0(x)+f_0(Tx)+\ldots+f_0(T^{n-1}x)$$

By Prokhorov's theorem $(f_0^{(q_n)})_*(\mu) \to P$ weakly in $\mathcal{P}(\mathbb{R})$.

Theorem (Fr.-Lem. 04)

$$T^f_{mq_n} \to \int_{\mathbb{R}} T^f_s \, dP(s).$$

- If T is an irrational rotation by α on S^1 and (q_n) is the sequence of denominators of the continued fraction expansion of α then $T^{q_n} \to Id$ and by Denjoy-Koksma inequality $\|f_0^{(q_n)}\|_{\sup} \leqslant 2 \operatorname{Var} f$, whenever $f \in BV$. Hence $(*)$ holds.
- Similar result holds for so called interval exchange transformations (which need not to be rigid).

Theorem (Fr.-Lem. 06)

If T is an ergodic IET and $f \in BV$ then there exists $a_n \to +\infty$ such that

$$T^f_{a_n} \to \alpha \int_{\mathbb{R}} T^f_s \, dP(s) + (1-\alpha)J$$

for some $0 < \alpha \leqslant 1$ and $P \in \mathcal{P}(\mathbb{R})$.

Abelian differentials

By a **translation surface** we mean any (M, ω), where M is a compact Riemann surface and ω is a holomorphic 1-form (called also Abelian differential). For every direction θ ($\theta \in \mathbb{C}$ and $|\theta| = 1$) the Abelian differential determines the direction vector field $V_\theta : M \to TM$ so that $\omega(V_\theta) = \theta$ (except zeros of ω). The flow $\mathcal{F}_\theta$ associated to V_θ is called a **translation flow** in the direction θ. Each ergodic translation flow has a special representation over an ergodic IET and under a piecewise constant function.

Corollary

If $\mathcal{F}$ is weakly mixing translation flow then $\mathcal{F}_s$ and $\mathcal{F}$ are disjoint for a.e. $s \in \mathbb{R}$, moreover, diffeomorphisms F_s and F_t are also disjoint for a.e. $(s, t) \in \mathbb{R}^2$.

Almost every translation flow is weakly mixing in the product of the moduli space of Abelian differentials and S^1. [Avila, Forni 2007]

Absence of self-similarity

Theorem (Fr.-Lem. 09)

Let $\mathcal{T} = (T_t)_{t\in\mathbb{R}}$ be an ergodic flow such that

- $T_{t_n} \to \int_{\mathbb{R}} T_s\, dP(s)$ *and* $\mathcal{T}$ *is not rigid, or*
- $T_{t_n} \to \alpha \int_{\mathbb{R}} T_s\, dP(s) + (1-\alpha)J$ $(0 < \alpha \leqslant 1)$ *and* $\mathcal{T}$ *is not partially rigid.*

Then for each $s \neq \pm 1$ *the flows* $\mathcal{T}$ *and* $\mathcal{T}_s$ *are not isomorphic.*

$\mathcal{T}$ is **partially rigid** if $T_{s_n} \to J \geqslant a\, Id$ with $0 < a \leqslant 1$.

Proof. Suppose that $\mathcal{T}$ and $\mathcal{T}_s$ are isomorphic for some $0 < |s| < 1$. Then

$$T_{st} = S \circ T_t \circ S^{-1} \text{ hence } T_{s^m t} = S^m \circ T_t \circ S^{-m}.$$

It follows that

$$\begin{aligned} T_{s^m t_n} &= S^m \circ T_{t_n} \circ S^{-m} \stackrel{n\to\infty}{\to} S^m \circ \int_{\mathbb{R}} T_u \, dP(u) \circ S^{-m} \\ &= \int_{\mathbb{R}} S^m \circ T_u \circ S^{-m} \, dP(u) = \int_{\mathbb{R}} T_{s^m u} \, dP(u) \\ &\stackrel{m\to\infty}{\to} \int_{\mathbb{R}} T_0 \, dP(u) = Id. \end{aligned}$$

Consequently, $\mathcal{T}$ is rigid. □

von Neumann flows are special flows built over irrational rotations on the circle and under piecewise C^1-functions with non-zero sum of jumps. von Neumann proved that such flows are weakly mixing.

Theorem (Fr.-Lem. 06)

von Neumann flows are not partially rigid.

Corollary

von Neumann flows have no self-similarities.

Theorem (Fr.-Lem. 09)

von Neumann flows built over ergodic IETs have no self-similarities.

This approach works also roof functions with zero sum of jumps (piecewise constant). Such flows are partially rigid.
For some Diophantine rotations and for a careful choice of discontinuities of the roof function the special flow is mild mixing, which implies the absence of rigidity [Lemańczyk, Lesigne, Frączek 2007].

Theorem (Fr. 2009)

If the genus of M is greater than 1 then for every stratum $\mathcal{H}_g(m_1, \ldots, m_\kappa)$ of the moduli space of Abelian differentials there exists a dense subset $\mathcal{H}$ such that the vertical flow of each $\omega \in \mathcal{H}$ has no self-similarities.

Theorem (Kułaga 2009)

For every compact surface M with genus greater than 1 there exists a smooth flow with no self-similarities (zero entropy).

Problem: Give a classification of multiplicative subgroups of $\mathbb{R}^*$ that can be obtained as $\mathcal{I}(\mathcal{T})$.
Danilenko proved that $\mathcal{I}(\mathcal{T})$ is always a Borel subgroup. Recall $\mathcal{I}(\mathcal{T}) = \mathbb{R}^*$ for some horocycle flows.
For each countable subgroup $G \subset \mathbb{R}^*$ there exists an ergodic flow such that $\mathcal{I}(\mathcal{T}) = G$.

Theorem (Danilenko-Lemańczyk,private communication)

There exist ergodic flows for which $\mathcal{I}(\mathcal{T})$ is uncountable and has zero Lebesgue measure.

Theorem (Danilenko, Ryzhikov independently)

The absence of self-similarity is generic in the set of measure preserving flows $\mathrm{Flow}(X, \mathcal{B}, \mu)$.

The distance $d_{\mathcal{F}}$ on $\mathrm{Flow}(X, \mathcal{B}, \mu)$ is given by

$$d_{\mathcal{F}}((S_t)_{t\in\mathbb{R}}, (T_t)_{t\in\mathbb{R}}) = \sup_{0\leqslant t\leqslant 1} d(S_t, T_t).$$