DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **11**, Numbers **2&3**, September & October **2004**
Website: http://AIMsciences.org
pp. **489–516**

POLYNOMIAL GROWTH OF THE DERIVATIVE FOR DIFFEOMORPHISMS ON TORI

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Abstract. We consider area–preserving zero entropy ergodic diffeomorphisms on tori. We classify such diffeomorphisms for which the sequence $\{Df^n\}$ has a polynomial growth on the 3–torus: they are necessary of the form

$$\mathbb{T}^3 \ni (x_1, x_2, x_3) \mapsto (x_1 + \alpha, \varepsilon x_2 + \beta(x_1), x_3 + \gamma(x_1, x_2)) \in \mathbb{T}^3,$$

where $\varepsilon = \pm 1$. We also indicate why there is no 4–dimensional analogue of the above result. Random diffeomorphisms on the 2–torus are studied as well.

1. **Introduction.** Let M be a compact Riemannian smooth manifold and let μ be a probability Borel measure on M having full topological support. Let $f : (M, \mu) \to (M, \mu)$ be a smooth measure–preserving diffeomorphism. An important question of smooth ergodic theory is the following: whether there is a relation between asymptotic properties of the sequence $\{Df^n\}_{n\in\mathbb{N}}$ and dynamical properties of the dynamical system $f : (M, \mu) \to (M, \mu)$. There are results describing a close relation in the case where M is the torus. For example, if f is homotopic to the identity, the coordinates of the rotation vector of f are rationally independent and the sequence $\{Df^n\}_{n\in\mathbb{N}}$ is uniformly bounded, then f is C^0–conjugate to an ergodic rotation (see [8] p.181). Moreover, if $\{Df^n\}_{n\in\mathbb{N}}$ is bounded in the C^r–norm ($r \in \mathbb{N} \cup \{\infty\}$), then f and the ergodic rotation are C^r–conjugated (see [8] p.182). On the other hand, if $\{Df^n\}_{n\in\mathbb{N}}$ has an "exponential growth", more precisely if f is an Anosov diffeomorphism, then f is C^0–conjugate to an algebraic automorphism of the torus (see [11]).

A natural question is what can happen between the above extreme cases? The aim of this paper is to classify measure–preserving tori diffeomorphisms f for which the sequence $\{Df^n\}_{n\in\mathbb{N}}$ has polynomial growth. The first definition of polynomial growth of the derivative was proposed in [4]. In [4], the following result has been proved.

Proposition 1.1. *Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be an ergodic area–preserving C^2–diffeomorphism. If the sequence $\{n^{-\tau} Df^n\}_{n\in\mathbb{N}}$ converges a.e. ($\tau > 0$) to a nonzero function, then $\tau = 1$ and f is algebraically (i.e. via a group automorphism) conjugate to the skew product of an irrational rotation on the circle and a circle cocycle with nonzero topological degree.*

1991 *Mathematics Subject Classification.* 37A05, 37C05, 37C40.

Key words and phrases. Area–preserving diffeomorphisms, polynomial growth of the derivative, random diffeomorphisms.

Research partly supported by KBN grant 1 P03A 038 26.

Moreover, the author in [5] showed that if $f : \mathbb{T}^2 \to \mathbb{T}^2$ is an ergodic area–preserving C^3–diffeomorphism for which the sequence $\{n^{-1}Df^n\}_{n\in\mathbb{N}}$ is C^0–separated from 0 and ∞ and it is bounded in the C^2–norm, then f is also algebraically conjugate to the skew product of an irrational rotation on the circle and a circle cocycle with nonzero topological degree.

We also recall the main result of [13] asserting that if $f : \mathbb{T}^2 \to \mathbb{T}^2$ is a homotopic to the identity symplectic diffeomorphism with a fixed point, then f is equals the identity map or there exists $c > 0$ such that

$$\max(\|Df^n\|_\infty, \|Df^{-n}\|_\infty) \geq cn$$

for any natural n (see [14] for some generalizations).

In the present paper some versions of Proposition 1.1 are discussed. In Section 2 we consider the random case. In Section 3 we classify area–preserving ergodic C^2–diffeomorphisms of a polynomial uniform growth of the derivative on the 3–torus, i.e. diffeomorphisms for which the sequence $\{n^{-\tau}Df^n\}_{n\in\mathbb{N}}$ converges uniformly to a non–zero function. It is shown that if the limit function is of class C^1, then τ is 1 or 2, and the diffeomorphism is C^2–conjugate to a 2–step skew product. We indicate why there is no 4–dimensional analogue of Proposition 1.1 in Section 4.

2. **Random diffeomorphism on the** 2**–torus.** Throughout this section we will consider smooth random dynamical systems over an abstract dynamical system $(\Omega, \mathcal{F}, P, T)$, where $(\Omega, \mathcal{F}, P)$ is a Lebesgue space and $T : (\Omega, \mathcal{F}, P) \to (\Omega, \mathcal{F}, P)$ is an ergodic measure–preserving automorphism. We will consider a compact Riemannian C^∞–manifold M equipped with its Borel σ–algebra $\mathcal{B}$ as a phase space for smooth random diffeomorphisms. A measurable map f

$$\mathbb{Z} \times \Omega \times M \ni (n, \omega, x) \longmapsto f^n_\omega x \in M$$

satisfying for P–a.e. $\omega \in \Omega$ the following conditions

- $f^0_\omega = \mathrm{Id}_M$, $f^{m+n}_\omega = f^m_{T^n\omega} \circ f^n_\omega$ for all $m, n \in \mathbb{Z}$,
- $f^n_\omega : M \to M$ is a smooth function for all $n \in \mathbb{Z}$,

is called a *smooth random dynamical system* (RDS). Of course, the smooth RDS is generated by the random diffeomorphism $f_\omega = f^1_\omega$ in the sense that

$$f^n_\omega = \begin{cases} f_{T^{n-1}\omega} \circ \ldots \circ f_{T\omega} \circ f_\omega & \text{for} \quad n > 0 \\ \mathrm{Id}_M & \text{for} \quad n = 0 \\ f^{-1}_{T^n\omega} \circ f^{-1}_{T^{n+1}\omega} \circ \ldots \circ f^{-1}_{T^{-1}\omega} & \text{for} \quad n < 0. \end{cases}$$

Consider the skew–product transformation $T_f : (\Omega \times M, \mathcal{F} \otimes \mathcal{B}) \to (\Omega \times M, \mathcal{F} \otimes \mathcal{B})$ induced naturally by f as follows:

$$T_f(\omega, x) = (T\omega, f_\omega x).$$

Then $T^n_f(\omega, x) = (T^n\omega, f^n_\omega x)$ for all $n \in \mathbb{Z}$. We call a probability measure μ on $(\Omega \times M, \mathcal{F} \otimes \mathcal{B})$ f*–invariant* if μ is invariant under T_f and has marginal P on Ω. Such measures can also be characterized in terms of their disintegrations μ_ω, $\omega \in \Omega$ by $f_\omega \mu_\omega = \mu_{T\omega}$ P–a.e. A measure μ is said to be *ergodic* if $T_f : (\Omega \times M, \mathcal{F} \otimes \mathcal{B}, \mu) \to (\Omega \times M, \mathcal{F} \otimes \mathcal{B}, \mu)$ is ergodic. We say that μ has full support, if $\mathrm{supp}(\mu_\omega) = M$ for P–a.e. $\omega \in \Omega$.

In this section we will deal with almost everywhere diffentiable and C^r–random dynamical systems with polynomial growth of the derivative. Suppose that $f : \mathbb{Z} \times \Omega \times M \to M$ is a C^0–RDS and μ is an f–invariant measure on $\Omega \times M$. The

RDS f is called μ–almost everywhere diffentiable if for every integer n and for μ–a.e. $(\omega, x) \in \Omega \times M$ there exists the derivative $Df^n_\omega(x) : T_xM \to T_{f^n_\omega}M$ and

$$\int_M \|Df^n_\omega(x)\|_{n,\omega,x} d\mu_\omega(x) < \infty$$

for every $n \in \mathbb{Z}$ and P–a.e. $\omega \in \Omega$, where $\| \cdot \|_{n,\omega,x}$ is the operator norm in $\mathcal{L}(T_xM, T_{f^n_\omega x}M)$.

In the paper we will discuss in details random diffeomorphisms on tori. Let d be a natural number. By $\mathbb{T}^d$ we denote the d–dimensional torus $\{(z_1, \ldots, z_d) \in \mathbb{C}^d : |z_1| = \ldots = |z_d| = 1\}$ which most often will be treated as the quotient group $\mathbb{R}^d/\mathbb{Z}^d$; $\lambda^{\otimes d}$ will denote Lebesgue measure on $\mathbb{T}^d$. We will identify functions on $\mathbb{T}^d$ with $\mathbb{Z}^d$–periodic functions (i.e. periodic of period 1 in each coordinate) on $\mathbb{R}^d$. Let $f : \mathbb{T}^d \to \mathbb{T}^d$ be a smooth diffeomorphism. We will identify f with a diffeomorphism $f : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$f(x_1, \ldots, x_j + 1, \ldots, x_d) = f(x_1, \ldots, x_d) + (a_{1j}, \ldots, a_{dj})$$

for every $(x_1, \ldots, x_d) \in \mathbb{R}^d$, where $A = [a_{ij}]_{1 \le i,j \le d} \in GL_d(\mathbb{Z})$. We call A the *linear part* of the diffeomorphism f. Then there exist smooth functions $\tilde{f}_i : \mathbb{T}^d \to \mathbb{R}$ such that

$$f_i(x_1, \ldots, x_d) = \sum_{j=1}^{d} a_{ij} x_j + \tilde{f}_i(x_1, \ldots, x_d),$$

where $f_i : \mathbb{R}^d \to \mathbb{R}$ is the i–th coordinate functions of f for $i = 1, \ldots, d$.

Definition 2.1. We say that a μ–almost everywhere diffentiable RDS f on $\mathbb{T}^d$ over $(\Omega, \mathcal{F}, P, T)$ has *τ–polynomial ($\tau > 0$) growth of the derivative* if

$$\frac{1}{n^\tau} Df^n_\omega(x) \to g(\omega, x) \text{ for } \mu\text{–a.e. } (\omega, x) \in \Omega \times \mathbb{T}^d,$$

where $g : \Omega \times \mathbb{T}^d \to M_d(\mathbb{R})$ is μ non–zero, i.e. there exists a set $A \in \mathcal{F} \otimes \mathcal{B}$ such that $\mu(A) > 0$ and $g(x) \neq 0$ for all $x \in A$. Moreover, if additionally Df^n belongs to $L^1((\Omega \times \mathbb{T}^d, \mu), M_d(\mathbb{R}))$ for all $n \in \mathbb{N}$ and the sequence $\{n^{-\tau} Df^n\}$ converges in $L^1((\Omega \times \mathbb{T}^d, \mu), M_d(\mathbb{R}))$ then we say that f has *τ–polynomial L^1–growth of the derivative.*

We now give an example of an ergodic RDS on $\mathbb{T}^2$ with linear L^1–growth of the derivative. Before we do it let us introduce a standard notation. Let $\tau : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be a measure–preserving ergodic automorphism of a standard Borel space and let G be a compact metric Abelian group. Then each measurable map $\varphi : X \to G$ determines a measurable cocycle over τ given by

$$\varphi^{(n)}(x) = \begin{cases} \varphi(x) + \varphi(\tau x) + \ldots + \varphi(\tau^{n-1} x) & \text{for } \quad n > 0 \\ e & \text{for } \quad n = 0 \\ -(\varphi(\tau^n x) + \varphi(\tau^{n+1} x) + \ldots + \varphi(\tau^{-1} x)) & \text{for } \quad n < 0. \end{cases}$$

which will be identified with the function φ. We say that the cocycle φ is a coboundary if there exists a measurable map $g : X \to G$ such that $\varphi = g - g \circ \tau$. We call the cocycle φ ergodic if the skew product

$$\tau_\varphi : (X \times G, \mu \otimes \lambda_G) \to (X \times G, \mu \otimes \lambda_G), \;\; \tau_\varphi(x, g) = (\tau x, g + \varphi(x))$$

is ergodic, where λ_G is the Haar measure on G.

Let us consider an almost everywhere diffentiable RDS f on $\mathbb{T}^2$ over $(\Omega, \mathcal{F}, P, T)$ (called the random Anzai skew product) of the form

$$f_\omega(x_1, x_2) = (x_1 + \alpha(\omega), x_2 + \varphi(\omega, x_1)),$$

where the skew product $T_\alpha : (\Omega \times \mathbb{T}, P \otimes \lambda) \to (\Omega \times \mathbb{T}, P \otimes \lambda)$, $T_\alpha(\omega, x) = (T\omega, x + \alpha(\omega))$ is ergodic and $\varphi : \Omega \times \mathbb{T} \to \mathbb{T}$ is an absolutely continuous random mapping of the circle such that $D\varphi \in L^1(\Omega \times \mathbb{T}, P \otimes \lambda)$ and $\int_\Omega d(\varphi_\omega) dP(\omega) \neq 0$ ($d(\varphi_\omega)$ stands for the topological degree of $\varphi_\omega : \mathbb{T} \to \mathbb{T}$). Then the product measure $P \otimes \lambda^{\otimes 2}$ is f–invariant. The following lemma is a little generalization of Lemma 3 in [9].

Lemma 2.1. *The RDS f is ergodic and has linear L^1–growth of the derivative.*

Proof. First, note that

$$f^n_\omega(x_1, x_2) = (x_1 + \alpha^{(n)}(\omega), x_2 + \varphi^{(n)}(\omega, x_1))$$

for all $n \in \mathbb{N}$. Therefore

$$\frac{1}{n} Df^n_\omega(x_1, x_2) = \left[\begin{array}{cc} 1/n & 0 \\ (1/n)\sum_{k=0}^{n-1} D\varphi(T^k_\alpha(\omega, x_1)) & 1/n \end{array} \right].$$

By the ergodicity of T_α,

$$\frac{1}{n}\sum_{k=0}^{n-1} D\varphi(T^k_\alpha(\omega, x)) \to \int_\Omega \int_\mathbb{T} D\varphi_\omega(y)\, dy\, dP(\omega) = \int_\Omega d(\varphi_\omega)\, dP(\omega) \neq 0$$

for $P \otimes \lambda$–a.e. $(\omega, x) \in \Omega \times \mathbb{T}$ and in the L^1–norm, which implies linear L^1–growth of the derivatives of f.

To proof the ergodicity of f, we consider the family of unitary operators $\{U_m : L^2(\Omega \times \mathbb{T}, P \otimes \lambda) \to L^2(\Omega \times \mathbb{T}, P \otimes \lambda), m \in \mathbb{Z}\}$ given by

$$U_m g(\omega, x) = e^{2\pi i m \varphi(\omega, x)} g(T\omega, x + \alpha(\omega)).$$

We will show that

$$\langle U^n_m g, g\rangle = \int_{\Omega\times\mathbb{T}} e^{2\pi i m \varphi^{(n)}(\omega, x)} g(T^n_\alpha(\omega, x)) \bar{g}(\omega, x)\, dP(\omega)\, dx \to 0 \text{ as } n \to \infty \quad (2.1)$$

for all $g \in L^2(\Omega \times \mathbb{T}, P \otimes \lambda)$ and $m \in \mathbb{Z} \setminus \{0\}$. Let Λ denote the set of all $g \in L^2(\Omega \times \mathbb{T}, P \otimes \lambda)$ satisfying (2.1). It is easy to check that Λ is a closed linear subspace of $L^2(\Omega \times \mathbb{T}, P \otimes \lambda)$. Therefore it suffices to show (2.1) for all functions of the form $g(\omega, x) = h(\omega)e^{2\pi i k x}$, where $h \in L^\infty(\Omega, P)$ and $k \in \mathbb{Z}$. For such g we have

$$\begin{aligned} |\langle U^n_m g, g\rangle| &= \left|\int_\Omega h(T^n\omega)\bar{h}(\omega) e^{2\pi i k \alpha^{(n)}(\omega)} \left(\int_\mathbb{T} e^{2\pi i m \varphi^{(n)}(\omega, x)}\, dx\right) dP(\omega)\right| \\ &\leq \|h\|^2_{L^\infty} \int_\Omega \left|\int_\mathbb{T} e^{2\pi i m \varphi^{(n)}(\omega, x)}\, dx\right| dP(\omega). \end{aligned}$$

Let $\tilde{\varphi} : \Omega \times \mathbb{T} \to \mathbb{R}$ be an absolutely continuous random function such that $\varphi(\omega, x) = \tilde{\varphi}(\omega, x) + d(\varphi_\omega)\, x$. Without loss of generality we can assume that $\int_\Omega d(\varphi_\omega)\, dP(\omega) = a > 0$. For any natural n let $A_n = \{\omega \in \Omega : (d(\varphi_\omega))^{(n)}/n > a/2\}$. By the ergodicity of T, $P(\Omega\setminus A_n)\to 0$ as $n\to\infty$. Applying integration by parts we obtain

$$\begin{aligned}
&\frac{1}{\|h\|^2_{L^\infty}}|\langle U_m^n g,g\rangle|\\
&\le P(\Omega\setminus A_n)+\int_{A_n}\Big|\int_{\mathbb{T}} e^{2\pi i m\tilde\varphi^{(n)}(\omega,x)}d\frac{e^{2\pi i m(d(\varphi_\omega))^{(n)}x}}{2\pi i m(d(\varphi_\omega))^{(n)}}\Big|\,dP(\omega)\\
&\le P(\Omega\setminus A_n)+\frac{1}{\pi|m|an}\int_{A_n}\Big|\int_{\mathbb{T}} e^{2\pi i m(d(\varphi_\omega))^{(n)}x}de^{2\pi i m\tilde\varphi^{(n)}(\omega,x)}\Big|\,dP(\omega)\\
&\le P(\Omega\setminus A_n)+\frac{2}{\pi an}\int_{A_n}\Big|\int_{\mathbb{T}} D\tilde\varphi^{(n)}(\omega,x)\,dx\Big|\,dP(\omega)\\
&\le P(\Omega\setminus A_n)+\frac{2}{\pi a}\int_{\Omega\times\mathbb{T}}|D\tilde\varphi^{(n)}(\omega,x)/n|\,dP(\omega)\,dx.
\end{aligned}$$

As $\int_{\Omega\times\mathbb{T}}D\tilde\varphi(\omega,x)\,dP(\omega)dx=0$, applying the Birkhoff ergodic theorem for T_α we conclude that $\int_{\Omega\times\mathbb{T}}|D\tilde\varphi^{(n)}(\omega,x)/n|\,dP(\omega)\,dx$ tends to zero, which proves our claim.

Now suppose, contrary to our assertion, that f is not ergodic. Since the skew product T_α is ergodic, there exists a measurable function $g:\Omega\times\mathbb{T}\to\mathbb{T}$ and $m\in\mathbb{Z}\setminus\{0\}$ such that $e^{2\pi i m\varphi(\omega,x)}=g(\omega,x)\bar g(T_\alpha(\omega,x))$. Then $\langle U_m^n g,g\rangle=1$ for all $n\in\mathbb{N}$, contrary to (2.1). □

The aim of this section is to classify C^r–random dynamical systems on the 2–torus that have polynomial (L^1) growth of the derivative and are ergodic with respect to an invariant measure having full support. We say that two random dynamical systems f and g on $\mathbb{T}^d$ over $(\Omega,\mathcal{F},P,T)$ are smoothly conjugate if there exists a smooth random diffeomorphism $h:\Omega\times\mathbb{T}^d\to\mathbb{T}^d$ such that $f_\omega\circ h_\omega=h_{T\omega}\circ g_\omega$ for P–a.e. $\omega\in\Omega$. If additionally there exists a group automorphism $A:\mathbb{T}^d\to\mathbb{T}^d$ such that $h_\omega=A$ for P–a.e. $\omega\in\Omega$, we say that f and g are algebraically conjugate. Given a smooth RDS f on $\mathbb{T}^2$ over $(\Omega,\mathcal{F},P,T)$ let us denote by $\varepsilon:\Omega\to\mathbb{Z}_2$ the measurable cocycle over the automorphism $T:\Omega\to\Omega$ given by

$$\varepsilon_\omega=\begin{cases}1 & \text{if } f \text{ preserves orientation,}\\ -1 & \text{otherwise.}\end{cases}$$

We will prove the following theorems.

Theorem 2.2. *Let f be a C^r–random dynamical system on $\mathbb{T}^2$ over $(\Omega,\mathcal{F},P,T)$ ($r\ge 1$). Let μ be an f–invariant ergodic measure having full support on $\Omega\times\mathbb{T}^2$. Suppose that f has τ–polynomial growth of the derivative. Then $\tau\ge 1$ and f is algebraically conjugate to a random skew product of the form*

$$\hat f_\omega(x_1,x_2)=(F_\omega(x_1),x_2+\varphi_\omega(x_1)),$$

where $F:\Omega\times\mathbb{T}\to\mathbb{T}$ is a C^r–random diffeomorphism of the circle. Moreover, there exist a random homeomorphism of the circle $\xi:\Omega\times\mathbb{T}\to\mathbb{T}$ and a measurable function $\alpha:\Omega\to\mathbb{T}$ such that

$$\xi_{T\omega}\circ F_\omega(x)=\varepsilon_\omega\xi_\omega(x)+\alpha_\omega \quad P\text{–a.e.}$$

and consequently f is topologically conjugate to the random skew product

$$\mathbb{T}^2\ni(x_1,x_2)\longmapsto(\varepsilon_\omega x_1+\alpha_\omega,x_2+\varphi_\omega\circ\xi_\omega^{-1}(x_1))\in\mathbb{T}^2.$$

Theorem 2.3. *Under the hypothesis of Theorem 2.2, if additionally f has τ–polynomial L^1–growth of the derivative and μ is equivalent to the measure $P\otimes\lambda^{\otimes 2}$ with $d\mu/d(P\otimes\lambda^{\otimes 2})$, $d(P\otimes\lambda^{\otimes 2})/d\mu\in L^\infty(\Omega\times\mathbb{T}^2)$, then*

- $\tau = 1$,
- *there exist a Lipschitz random diffeomorphism of the circle* $\xi : \Omega \times \mathbb{T} \to \mathbb{T}$ *with* $D\xi, D\xi^{-1} \in L^\infty(\Omega \times \mathbb{T}, P \otimes \lambda)$ *and a measurable function* $\alpha : \Omega \to \mathbb{T}$ *such that*
$$\xi_{T\omega} \circ F_\omega(x) = \xi_\omega(x) + \alpha_\omega \ \ P\text{–a.e. and}$$
- $\int_\Omega d(\varphi_\omega \circ \xi_\omega^{-1})\, dP(\omega) \neq 0.$

For convenience of the reader the proofs of the above theorems are divided into a sequence of lemmas. Let f be a C^r–random dynamical system on $\mathbb{T}^d$ over $(\Omega, \mathcal{F}, P, T)$. Let μ be an f–invariant ergodic measure having full support on $\Omega \times \mathbb{T}^d$. Suppose that f has τ–polynomial growth of the derivative. Let $g : \Omega \times \mathbb{T}^d \to M_d(\mathbb{R})$ denote the limit of the sequence $\{n^{-\tau} Df^n\}$.

Lemma 2.4. *For* μ*–a.e.* $(\omega, x) \in \Omega \times \mathbb{T}^d$ *and all* $n \in \mathbb{Z}$ *we have*

$$g(\omega, x) \neq 0, \quad g(\omega, x)^2 = 0 \text{ and} \tag{2.2}$$

$$g(\omega, x) = g(T^n\omega, f^n_\omega x)\, Df^n_\omega(x). \tag{2.3}$$

For $\mu \otimes \mu$*–a.e.* $(\omega, x, \upsilon, y) \in \Omega \times \mathbb{T}^d \times \Omega \times \mathbb{T}^d$ *we have*

$$g(\omega, x)\, g(\upsilon, y) = 0 \text{ and } g(\omega, x) = Df_\upsilon(y)\, g(\omega, x). \tag{2.4}$$

Proof. Let $A \subset \Omega \times \mathbb{T}^d$ be a T_f–invariant subset having full μ–measure such that $(\omega, x) \in A$ implies $\lim_{n\to\infty} n^{-\tau} Df^n_\omega(x) = g(\omega, x)$. Assume that $(\omega, x) \in A$. Since

$$\left(\frac{m+n}{m}\right)^\tau \frac{1}{(m+n)^\tau} Df^{m+n}_\omega(x) = \frac{1}{m^\tau} Df^m_{T^n\omega}(f^n_\omega x)\, Df^n_\omega(x)$$

and $(T^n\omega, f^n_\omega x) \in A$ for all $m, n \in \mathbb{N}$, letting $m \to \infty$ we obtain

$$g(\omega, x) = g(T^n\omega, f^n_\omega x)\, Df^n_\omega(x) \quad \text{for all } (\omega, x) \in A \text{ and } n \in \mathbb{N}.$$

Let $B = \{(\omega, x) \in A : g(\omega, x) \neq 0\}$. By the above remark, B is T_f–invariant. Since g is μ non–zero, $\mu(B) = 1$, by the ergodicity of T_f.

By the Jewett–Krieger theorem, we can assume that Ω is a compact metric space, $T : \Omega \to \Omega$ is a uniquely ergodic homeomorphism and P is the unique T–invariant measure. Now choose a sequence $\{A_k\}_{k\in\mathbb{N}}$ of measurable subsets of A such that the functions $g, Df : A_k \to M_d(\mathbb{R})$ are continuous, all non-empty open subsets of A_k (in the induced topology) have positive measure and $\mu(A_k) > 1 - 1/k$ for any natural k. Since the transformation $(T_f)_{A_k} : (A_k, \mu_{A_k}) \to (A_k, \mu_{A_k})$ induced by T_f on A_k is ergodic, for every natural k we can find a measurable subset $B_k \subset A_k$ such that every orbit $\{(T_f)^n_{A_k}(\omega, x)\}_{n\in\mathbb{N}}$, $(\omega, x) \in B_k$, is dense in A_k in the induced topology and $\mu(B_k) = \mu(A_k)$.

Assume that $(\omega, x), (\upsilon, y) \in B_k$. Then there exists an increasing sequence $\{m_i\}_{i\in\mathbb{N}}$ of natural numbers such that $(T_f)^{m_i}_{A_k}(\omega, x) \to (\upsilon, y)$. Hence there exists an increasing sequence $\{n_i\}_{i\in\mathbb{N}}$ of natural numbers such that $T_f^{n_i}(\omega, x) \to (\upsilon, y)$ and $T_f^{n_i}(\omega, x) \in A_k$ for all $i \in \mathbb{N}$. Since $g, Df : A_k \to M_d(\mathbb{R})$ are continuous, $g(T^{n_i}\omega, f^{n_i}_\omega x) \to g(\upsilon, y)$ and $Df_{T^{n_i}\omega}(f^{n_i}_\omega x) \to g(\upsilon, y)$. Since

$$\frac{1}{n_i^\tau} g(\omega, x) = g(T^{n_i}\omega, f^{n_i}_\omega x)\, \frac{1}{n_i^\tau} Df^{n_i}_\omega(x),$$

letting $i \to \infty$ we obtain $g(\upsilon, y)\, g(\omega, x) = 0$. Since

$$\frac{1}{n_i^\tau} Df^{n_i+1}_\omega(x) = Df_{T^{n_i}\omega}(f^{n_i}_\omega x)\, \frac{1}{n_i^\tau} Df^{n_i}_\omega(x),$$

letting $i \to \infty$ we obtain $g(\omega, x) = Df_\upsilon(y)\, g(\omega, x)$. Therefore

$$\mu \otimes \mu\{(\omega, x, \upsilon, y) \in \Omega \times \mathbb{T}^d \times \Omega \times \mathbb{T}^d :\ g(\upsilon, y)\, g(\omega, x) = 0\}) > \left(1 - \frac{1}{k}\right)^2,$$

$$\mu\{(\omega, x) \in \Omega \times \mathbb{T}^d :\ g(\omega, x)^2 = 0\}) > 1 - \frac{1}{k}$$

and

$$\mu \otimes \mu\{(\omega, x, \upsilon, y) \in \Omega \times \mathbb{T}^d \times \Omega \times \mathbb{T}^d :\ g(\omega, x) = Df_\upsilon(y)\, g(\omega, x)\}) > \left(1 - \frac{1}{k}\right)^2$$

for any natural k, which proves the lemma. □

Let us return to case $d = 2$. Suppose that A, B are non–zero real 2×2–matrixes such that $A^2 = B^2 = AB = 0$. Then (see Lemma 4 in [4]) there exist real numbers $a, b \neq 0$ and c such that

$$A = a \begin{bmatrix} c \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -c \end{bmatrix} \quad \text{and} \quad B = b \begin{bmatrix} c \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -c \end{bmatrix}$$

or

$$A = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \quad \text{and} \quad B = b \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

It follows that g can be represented as

$$g = h \begin{bmatrix} c \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -c \end{bmatrix},$$

where $h : \Omega \times \mathbb{T}^2 \to \mathbb{R}$ is a measurable function which is non–zero at μ–a.e. point and $c \in \mathbb{R}$. We can omit the second case where

$$g = h \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix},$$

because it reduces to case $c = 0$ after interchanging the coordinates, which is an algebraic isomorphism. Then by (2.4) we obtain

$$\begin{bmatrix} c \\ 1 \end{bmatrix} = Df_\omega(x) \begin{bmatrix} c \\ 1 \end{bmatrix} \tag{2.5}$$

for P–a.e. $\omega \in \Omega$ and for all $x \in \mathbb{T}^2$, because μ has full support. From (2.3) we obtain

$$h(\omega, x) \begin{bmatrix} 1 & -c \end{bmatrix} = h(T\omega, f_\omega x) \begin{bmatrix} 1 & -c \end{bmatrix} Df_\omega(x) \tag{2.6}$$

for μ–a.e. $(\omega, x) \in \Omega \times \mathbb{T}^2$.

Lemma 2.5. *If c is irrational, then $f_\omega(x_1, x_2) = (x_1 + \alpha(\omega), x_2 + \gamma(\omega))$, where $\alpha, \gamma : \Omega \to \mathbb{T}$ are measurable functions. Consequently, the sequence $n^{-\tau} Df^n$ tends uniformly to zero.*

Proof. From (2.5) we have

$$c = c\frac{\partial (f_\omega)_1}{\partial x_1} + \frac{\partial (f_\omega)_1}{\partial x_2} \text{ and } 1 = c\frac{\partial (f_\omega)_2}{\partial x_1} + \frac{\partial (f_\omega)_2}{\partial x_2}$$

for P–a.e. $\omega \in \Omega$. It follows that for $i = 1, 2$ there exists a C^{r+1}–random function $u_i : \Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$f_i(\omega, x_1, x_2) = x_i + u_i(\omega, x_1 - cx_2).$$

Represent f as

$$\begin{aligned} f_1(\omega,x_1,x_2) &= a_{11}(\omega)x_1+a_{12}(\omega)x_2+\widetilde{f}_1(\omega,x_1,x_2),\\ f_2(\omega,x_1,x_2) &= a_{21}(\omega)x_1+a_{22}(\omega)x_2+\widetilde{f}_2(\omega,x_1,x_2),\end{aligned}$$

where $\{a_{ij}(\omega)\}_{i,j=1,2}\in GL_2(\mathbb{Z})$ and $\widetilde{f}_1,\widetilde{f}_2:\Omega\times\mathbb{T}^2\to\mathbb{R}$. Then

$$u_1(\omega,x+1)=(a_{11}(\omega)-1)(x+1)+\widetilde{f}_1(\omega,x+1,0)=u_1(\omega,x)+a_{11}(\omega)-1$$

and

$$u_1(\omega,x+c)=(a_{11}(\omega)-1)x-a_{12}(\omega)+\widetilde{f}_1(\omega,x,-1)=u_1(\omega,x)-a_{12}(\omega).$$

Therefore $a_{11}(\omega)-1=\lim_{x\to+\infty}u_1(\omega,x)/x=-a_{12}(\omega)/c$ for μ–a.e. $\omega\in\Omega$. Since c is irrational, we conclude that $a_{11}(\omega)-1=a_{12}(\omega)=0$, hence that $u_1(\omega,\cdot)$ is 1 and c periodic, and finally $u_1(\omega,\cdot)$ is a constant for μ–a.e. $\omega\in\Omega$. It is clear that the same conclusion can be obtained for u_2, which completes the proof. □

Lemma 2.6. *If c is rational, then there exist a group automorphism $A:\mathbb{T}^2\to\mathbb{T}^2$, a C^r–random diffeomorphism of the circle $F:\Omega\times\mathbb{T}\to\mathbb{T}$ and a C^r–random function $\varphi:\Omega\times\mathbb{T}\to\mathbb{T}$ such that*

$$A\circ f_\omega\circ A^{-1}(x_1,x_2)=(F_\omega x_1,x_2+\varphi_\omega(x_1)).$$

Moreover,

$$h_{T\omega}\circ A^{-1}(F_\omega(x_1),x_2+\varphi_\omega(x_1))\cdot DF_\omega(x_1)=h_\omega\circ A^{-1}(x_1,x_2)\qquad(2.7)$$

for $\hat{\mu}$–a.e. $(\omega,x_1,x_2)\in\Omega\times\mathbb{T}^2$, where $\hat{\mu}:=(\mathrm{Id}_\Omega\times A)\,\mu$ and $h_\omega\circ A^{-1}:\mathbb{T}^2\to\mathbb{R}$ depends only on the first coordinate.

Proof. Let p and q be integers such that $q>0$, $\gcd(p,q)=1$ and $c=p/q$. Choose $a,b\in\mathbb{Z}$ such that $ap-bq=1$. Consider the group automorphism $A:\mathbb{T}^2\to\mathbb{T}^2$ associated to the matrix $A=\begin{bmatrix} q & -p\\ -b & a\end{bmatrix}$. Then $A^{-1}=\begin{bmatrix} a & p\\ b & q\end{bmatrix}$. Set $\hat{f}_\omega:=A\circ f_\omega\circ A^{-1}$. Then $\hat{\mu}$ is an $\hat{f}$–invariant measure and

$$D\hat{f}_\omega(x)=A\cdot(Df_\omega(A^{-1}x))\cdot A^{-1}.$$

>From (2.5) we have

$$\begin{bmatrix} p\\ q\end{bmatrix}=Df_\omega(x)\begin{bmatrix} p\\ q\end{bmatrix}$$

for P–a.e. $\omega\in\Omega$ and all $x\in\mathbb{T}^2$. Consequently,

$$\begin{bmatrix} 0\\ 1\end{bmatrix}=D\hat{f}_\omega(x)\begin{bmatrix} 0\\ 1\end{bmatrix}$$

for P–a.e. $\omega\in\Omega$ and all $x\in\mathbb{T}^2$. From (2.6) we have

$$h_\omega(x)\begin{bmatrix} q & -p\end{bmatrix}=h_{T\omega}(f_\omega x)\begin{bmatrix} q & -p\end{bmatrix}Df_\omega(x)$$

for μ–a.e. $(\omega,x)\in\Omega\times\mathbb{T}^2$. Consequently,

$$h_\omega\circ A^{-1}(x)\begin{bmatrix} 1 & 0\end{bmatrix}=h_{T\omega}\circ A^{-1}(\hat{f}_\omega x)\begin{bmatrix} 1 & 0\end{bmatrix}D\hat{f}_\omega(x)$$

for $\hat{\mu}$–a.e. $(\omega,x)\in\Omega\times\mathbb{T}^2$. It follows that $\partial(\hat{f}_\omega)_1/\partial x_2=0$ and $\partial(\hat{f}_\omega)_2/\partial x_2=1$ for P–a.e. $\omega\in\Omega$ and

$$\left(h_{T\omega}\circ A^{-1}\circ\hat{f}_\omega\right)(x)\,\frac{\partial(\hat{f}_\omega)_1}{\partial x_1}(x)=h_\omega\circ A^{-1}(x)$$

for $\hat{\mu}$–a.e. $(\omega, x) \in \Omega \times \mathbb{T}^2$. Therefore

$$\hat{f}_\omega(x_1, x_2) = (F_\omega x_1, x_2 + \varphi_\omega(x_1)),$$

where $F, \varphi : \Omega \times \mathbb{T} \to \mathbb{T}$ are C^r–random functions and

$$h_{T\omega} \circ A^{-1}(F_\omega(x_1), x_2 + \varphi_\omega(x_1)) \cdot DF_\omega(x_1) = h_\omega \circ A^{-1}(x_1, x_2)$$

for $\hat{\mu}$–a.e. $(\omega, x_1, x_2) \in \Omega \times \mathbb{T}^2$. Since $\hat{f}_\omega : \mathbb{T}^2 \to \mathbb{T}^2$ is a C^r–diffeomorphism, we conclude that $F_\omega : \mathbb{T} \to \mathbb{T}$ is a C^r–diffeomorphism for P–a.e. $\omega \in \Omega$. Since

$$\frac{1}{n^\tau} Df^n_\omega(x) \to h_\omega(x) \begin{bmatrix} p/q \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -p/q \end{bmatrix}$$

for μ–a.e. $(\omega, x_1, x_2) \in \Omega \times \mathbb{T}^2$,

$$\frac{1}{n^\tau} D\hat{f}^n_\omega(x) \to h_\omega(A^{-1}x)/q^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

for $\hat{\mu}$–a.e. $(\omega, x_1, x_2) \in \Omega \times \mathbb{T}^2$. Set $\hat{h}_\omega := h_\omega \circ A^{-1}$. Then

$$\frac{1}{n^\tau} \sum_{k=0}^{n-1} D\varphi_{T^k\omega}(F^k_\omega(x_1)) \cdot DF^k_\omega(x_1) \to \hat{h}_\omega(x_1, x_2)/q^2$$

for $\hat{\mu}$–a.e. $(\omega, x_1, x_2) \in \Omega \times \mathbb{T}^2$. It follows that $\hat{h}_\omega$ depends only on the first coordinate. □

Proof of Theorem 2.2. By Lemmas 2.5 and 2.6, to prove the first claim of the theorem it is enough to show that $\tau \geq 1$. Suppose that $\tau < 1$. Let $\nu := (\mathrm{Id}_\Omega \times \pi)\hat{\mu}$, where $\pi : \mathbb{T}^2 \to \mathbb{T}$ is the projection onto the first coordinate. Then ν is an F–invariant ergodic measure of full support on $\Omega \times \mathbb{T}$. By Lemma 2.6,

$$\hat{h}_{T^k\omega}(F^k_\omega(x)) \cdot DF^k_\omega(x) = \hat{h}_\omega(x)$$

and

$$\frac{1}{n^\tau} \sum_{k=0}^{n-1} D\varphi_{T^k\omega}(F^k_\omega(x)) \cdot DF^k_\omega(x) \to \hat{h}_\omega(x)/q^2 \tag{2.8}$$

for ν–a.e. $(\omega, x) \in \Omega \times \mathbb{T}$. Therefore

$$\frac{1}{n^\tau} \sum_{k=0}^{n-1} D\varphi_{T^k\omega}(F^k_\omega(x))/\hat{h}_{T^k\omega}(F^k_\omega(x)) \to 1/q^2 \tag{2.9}$$

and consequently

$$\frac{1}{n} \sum_{k=0}^{n-1} D\varphi_{T^k\omega}(F^k_\omega(x))/\hat{h}_{T^k\omega}(F^k_\omega(x)) \to 0$$

for ν–a.e. $(\omega, x) \in \Omega \times \mathbb{T}$. It follows that the measurable cocycle $D\varphi/\hat{h} : \Omega \times \mathbb{T} \to \mathbb{R}$ over the skew product T_F is recurrent (see [15]). Consequently, for ν–a.e. $(\omega, x) \in \Omega \times \mathbb{T}$ there exists an increasing sequence of natural numbers $\{n_i\}_{i \in \mathbb{N}}$ such that

$$|\sum_{k=0}^{n_i-1} D\varphi_{T^k\omega}(F^k_\omega(x))/\hat{h}_{T^k\omega}(F^k_\omega(x))| \leq 1.$$

It follows that

$$\frac{1}{n_i^\tau} \sum_{k=0}^{n_i-1} D\varphi_{T^k\omega}(F^k_\omega(x))/\hat{h}_{T^k\omega}(F^k_\omega(x)) \to 0,$$

contrary to (2.9).

Now let us decompose $\nu_\omega = \nu_\omega^d + \nu_\omega^c$, where ν_ω^d is the discrete and ν_ω^c is the continuous part of the measure ν_ω. As this decomposition is measurable we can consider the measures $\nu^d = \int_\Omega \nu_\omega^d\, dP(\omega)$ and $\nu^c = \int_\Omega \nu_\omega^c\, dP(\omega)$ on $\Omega \times \mathbb{T}$. It is easy to check that ν^d and ν^c are F–invariant. By the ergodicity of ν, either $\nu = \nu^d$ or $\nu = \nu^c$.

We now show that $\nu = \nu^c$. Suppose the contrary, that $\nu = \nu^d$. Let $\Delta : \Omega \times \mathbb{T} \to [0,1]$ denote the measurable function given by $\Delta(\omega, x) = \nu_\omega(\{x\})$. As ν is F–invariant we have

$$\Delta(T\omega, F_\omega x) = \nu_{T\omega}(\{F_\omega x\}) = F_\omega^{-1}\nu_{T\omega}(\{x\}) = \nu_\omega(\{x\}) = \Delta(\omega, x)$$

and consequently Δ is T_F–invariant. By the ergodicity of T_F, the function Δ is ν constant. It follows that the measure ν_ω has only finitely many of atoms for P–a.e. $\omega \in \Omega$, which contradicts the fact that ν has full support.

Define $\xi_\omega(x) := \int_0^x d\nu_\omega$ for all $x \in \mathbb{R}$. Then $\xi_\omega(x+1) = \xi_\omega(x) + 1$, because $\int_x^{x+1} d\nu_\omega = 1$. Since ν_ω is continuous and ν has full support, the function $\xi_\omega : \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing. Therefore $\xi : \Omega \times \mathbb{T} \to \mathbb{T}$ is a random homeomorphism. As ν is F–invariant we have

$$\begin{aligned} \xi_{T\omega}(F_\omega x) &= \int_0^{F_\omega x} d\nu_{T\omega} = \int_0^{F_\omega 0} d\nu_{T\omega} + \int_{F_\omega 0}^{F_\omega x} dF_\omega \nu_\omega \\ &= \alpha_\omega + \varepsilon_\omega \int_0^x d\nu_\omega = \varepsilon_\omega \xi_\omega(x) + \alpha_\omega \end{aligned}$$

for P–a.e. $\omega \in \Omega$, where $\alpha_\omega = \int_0^{F_\omega 0} d\nu_{T\omega}$. □

Proof of Theorem 2.3. Suppose that f has τ–polynomial L^1–growth of the derivative and μ is equivalent to $P \otimes \lambda^{\otimes 2}$. Then $DF, D\varphi \in L^1(\Omega \times \mathbb{T}, \nu)$ and $\hat{\mu}$ is equivalent to $P \otimes \lambda^{\otimes 2}$. Let $\theta \in L^1(\Omega \times \mathbb{T}^2, P \otimes \lambda^{\otimes 2})$ denote the Radon–Nikodym derivative of $\hat{\mu}$ with respect to $P \otimes \lambda^{\otimes 2}$. Then

$$\varepsilon_\omega \cdot \theta_{T\omega}(F_\omega(x_1), x_2 + \varphi_\omega(x_1)) \cdot DF_\omega(x_1) = \theta_\omega(x_1, x_2)$$

for $P \otimes \lambda^{\otimes 2}$–a.e. $(\omega, x_1, x_2) \in \Omega \times \mathbb{T}^2$. By (2.7), there exists a non–zero constant C such that $\theta_\omega(x_1, x_2) = C|\hat{h}_\omega(x_1)|$ for $P \otimes \lambda^{\otimes 2}$–a.e. $(\omega, x_1, x_2) \in \Omega \times \mathbb{T}^2$. Then the random homeomorphism $\xi_\omega : \mathbb{T} \to \mathbb{T}$ given by $\xi_\omega(x) := \int_0^x d\nu_\omega = \int_0^x \theta_\omega(t)dt$ is a Lipschitz random diffeomorphism, because θ and $1/\theta$ are bounded. It follows that f is Lipschitz conjugate to the random skew product

$$(T_{\alpha,\varepsilon,\psi})_\omega(x_1, x_2) = (\varepsilon_\omega x_1 + \alpha_\omega, x_2 + \psi_\omega(x_1)),$$

where $\psi_\omega := \varphi_\omega \circ \xi_\omega^{-1}$. From (2.8) we conclude that $T_{\alpha,\varepsilon,\psi}$ has τ–polynomial L^1–growth of the derivative and

$$\frac{1}{n^\tau} \sum_{k=0}^{n-1} \varepsilon_\omega^{(k)} D\psi(T_{\alpha,\varepsilon}^k(\omega, x)) \to \tilde{h}_\omega(x) \neq 0 \tag{2.10}$$

in $L^1(\Omega \times \mathbb{T}, P \otimes \lambda)$, where

$$\tilde{h}_\omega(x) = \hat{h}_\omega \circ \xi_\omega^{-1}(x) \cdot D\xi_\omega^{-1}(x)/q^2 \quad \text{and} \quad (T_{\alpha,\varepsilon})_\omega(x) = (\varepsilon_\omega x + \alpha_\omega).$$

It follows immediately that $\tau = 1$.

Now suppose that ε is a coboundary over T. Then there exists a measurable function $\eta : \Omega \to \mathbb{Z}_2$ such that $\varepsilon = \eta/(\eta \circ T)$ and the random diffeomorphism

$$\Omega \times \mathbb{T} \ni (\omega, x) \longmapsto (\omega, \eta_\omega x) \in \Omega \times \mathbb{T}$$

C^∞–conjugates the skew products $T_{\alpha,\varepsilon}$ and $T_{(\eta\circ T)\cdot\alpha,1}$, which is just our assertion.

Otherwise, the cocycle ε is ergodic over T. Then the cocycle $\varepsilon : \Omega\times\mathbb{T} \to \mathbb{Z}_2$ must be a coboundary over the automorphism $T_{\alpha,\varepsilon} : \Omega \times \mathbb{T} \to \Omega \times \mathbb{T}$. Indeed, suppose, contrary to our claim, that the skew product

$$\Omega \times \mathbb{T} \times \mathbb{Z}_2 \ni (\omega, x, y) \longmapsto (T\omega, \varepsilon_\omega x + \alpha_\omega, \varepsilon_\omega y) \in \Omega \times \mathbb{T} \times \mathbb{Z}_2$$

is ergodic. By the Birkhoff ergodic theorem,

$$\frac{1}{n}\sum_{k=0}^{n-1} \varepsilon_\omega^{(k)} \cdot y \cdot D\psi(T^k_{\alpha,\varepsilon}(\omega, x)) \to \int_{\Omega\times\mathbb{T}\times\mathbb{Z}_2} y' \cdot D\psi(\omega', t)\, dP(\omega')\, dt\, d\lambda_{\mathbb{Z}_2}(y') = 0$$

in $L^1(\Omega \times \mathbb{T} \times \mathbb{Z}_2, P \otimes \lambda \otimes \lambda_{\mathbb{Z}_2})$, contrary to (2.10). Consequently, there exists a measurable function $g : \Omega \times \mathbb{T} \to \mathbb{Z}_2$ such that $\varepsilon_\omega\, g(\omega, x) = g(T\omega, \varepsilon_\omega x + \alpha_\omega)$. It follows that $\varepsilon_\omega \int_{\mathbb{T}} g(\omega, t)\, dt = \int_{\mathbb{T}} g(T\omega, t)\, dt$. By the ergodicity of ε over T, we have $\int_{\mathbb{T}} g(\omega, t) dt = 0$. Let $G : \Omega \times \mathbb{T} \to [-1, 1]$ be given by $G_\omega(x) := \int_0^x g(\omega, t)\, dt$. Then

$$DG_{T\omega}(\varepsilon_\omega x + \alpha_\omega) = g(T\omega, \varepsilon_\omega x + \alpha_\omega) = \varepsilon_\omega g(\omega, x) = \varepsilon_\omega DG_\omega(x).$$

Consequently, there exists a measurable function $\beta : \Omega \to \mathbb{R}$ such that

$$G_{T\omega}(\varepsilon_\omega x + \alpha_\omega) = G_\omega(x) + \beta_\omega.$$

Therefore $\int_{\mathbb{T}} G_{T\omega}(t)\, dt = \int_{\mathbb{T}} G_\omega(t)\, dt + \beta_\omega$ and

$$G(T_{\alpha,\varepsilon}(\omega, x)) - \int_{\mathbb{T}} G_{T\omega}(t)\, dt = G(\omega, x) - \int_{\mathbb{T}} G_\omega(t)\, dt.$$

Consequently, $G(\omega, x) = \int_{\mathbb{T}} G_\omega(t)dt + c$, by the ergodicity of $T_{\alpha,\varepsilon}$. It follows that $0 = DG_\omega(x) = g(\omega, x) = \pm 1$ for a.e. $(\omega, x) \in \Omega \times \mathbb{T}$, which is impossible. Therefore ε is a coboundary over T, and the proof is complete. □

3. **Area–preserving diffeomorphisms of the** 3**–torus.** In this section we give a classification of area–preserving ergodic diffeomorphisms of a polynomial uniform growth of the derivative on the 3–torus. A C^1–diffeomorphism $f : \mathbb{T}^3 \to \mathbb{T}^3$ has τ–*polynomial uniform growth of the derivative* if the sequence $\{n^{-\tau} Df^n\}_{n\in\mathbb{N}}$ converges uniformly to a non–zero function. We first present a sequence of essential examples of such diffeomorphisms. We will consider 2–step skew products $T_{\alpha,\beta,\gamma,\varepsilon} : \mathbb{T}^3 \to \mathbb{T}^3$ given by

$$T_{\alpha,\beta,\gamma,\varepsilon}(x_1, x_2, x_3) = (x_1 + \alpha, \varepsilon x_2 + \beta(x_1), x_3 + \gamma(x_1, x_2)),$$

where α is irrational, $\varepsilon = \pm 1$ and $\beta : \mathbb{T} \to \mathbb{T}$, $\gamma : \mathbb{T}^2 \to \mathbb{T}$ are of class C^1. We will denote by $d_i(\gamma)$ the topological degree of γ with respect to the i-th coordinate for $i = 1, 2$. Here and subsequently, h_{x_i} stands for the partial derivative $\partial h/\partial x_i$.

Example 3.1. Assume that $\varepsilon = 1$, β is a constant function, $\alpha, \beta, 1$ are rationally independent and $(d_1(\gamma), d_2(\gamma)) \neq 0$. Then

$$\frac{1}{n} DT^n_{\alpha,\beta,\gamma,1} \to \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d_1(\gamma) & d_2(\gamma) & 0 \end{bmatrix} \neq 0$$

uniformly and $T_{\alpha,\beta,\gamma,1}$ is ergodic, by Lemma 2.1.

Example 3.2. Assume that $\varepsilon = 1$, $d(\beta) \neq 0$ and $d_2(\gamma) \neq 0$. By Lemma 2.1, $T_{\alpha,\beta,\gamma,1}$ is ergodic. Moreover, $T_{\alpha,\beta,\gamma,1}$ has square uniform growth of the derivative, more precisely,

$$\frac{1}{n^2} DT^n_{\alpha,\beta,\gamma,1} \to \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d(\beta)d_2(\gamma)/2 & 0 & 0 \end{bmatrix} \neq 0$$

uniformly.

Example 3.3. Assume that $\varepsilon = -1$, γ depends only on the first coordinate, $d(\gamma) \neq 0$ and the factor map $\mathbb{T}^2 \ni (x_1, x_2) \longmapsto (x_1 + \alpha, -x_2 + \beta(x_1)) \in \mathbb{T}^2$ is ergodic. Then

$$\frac{1}{n} DT^n_{\alpha,\beta,\gamma,-1} \to \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d(\gamma) & 0 & 0 \end{bmatrix} \neq 0$$

uniformly and $T_{\alpha,\beta,\gamma,-1}$ is ergodic, by Lemma 2.1.

The main result of this section is the following theorem.

Theorem 3.1. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be an area–preserving ergodic C^2–diffeomorphism with τ–polynomial uniform growth of the derivative ($\tau > 0$). Suppose that the limit function $\lim_{n\to\infty} n^{-\tau} Df^n$ is of class C^1. Then τ is 1 or 2, and f is C^2–conjugate to a diffeomorphism of the form*

$$\mathbb{T}^3 \ni (x_1, x_2, x_3) \longmapsto (x_1 + \alpha, \varepsilon x_2 + \beta(x_1), x_3 + \gamma(x_1, x_2)) \in \mathbb{T}^3,$$

where $\varepsilon = \det Df = \pm 1$.

As in the previous section, the proof of the main theorem is divided into several lemmas. Suppose that $f : \mathbb{T}^3 \to \mathbb{T}^3$ is an area–preserving ergodic diffeomorphism with τ–polynomial growth of the derivative. Assume that the limit of the sequence $\{n^{-\tau} Df^n\}_{n\in\mathbb{N}}$, denoted by $g : \mathbb{T}^3 \to M_3(\mathbb{R})$, is of class C^1. By Lemma 2.4, $g(\bar{x})\, g(\bar{y}) = 0$ and $g(\bar{x})^2 = 0$ for all $\bar{x}, \bar{y} \in \mathbb{T}^3$.

Lemma 3.2. *Suppose that A, B are non–zero real 3×3–matrixes such that $A^2 = B^2 = AB = BA = 0$. Then there exist three non–zero vectors (real 1×3–matrixes) $\bar{a}$, $\bar{b}$, $\bar{c}$ such that*

- *$A = \bar{a}^T\, \bar{b}$ and $B = \bar{a}^T\, \bar{c}$, where $\bar{b}\, \bar{a}^T = 0$ and $\bar{c}\, \bar{a}^T = 0$ or*
- *$A = \bar{a}^T\, \bar{c}$ and $B = \bar{b}^T\, \bar{c}$, where $\bar{c}\, \bar{a}^T = 0$ and $\bar{c}\, \bar{b}^T = 0$.*

Proof. Suppose that $\bar{x} \in \mathbb{C}^3$ is an eigenvector of A with the eigenvalue $\lambda \in \mathbb{C}$. Then $\lambda^2 \bar{x} = A^2 \bar{x} = 0$ and consequently $\lambda = 0$. It follows that the Jordan canonical form of A equals either

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

But the latter case can not occur because the square of the latter matrix is non–zero. It follows that there exists $C \in GL_3(\mathbb{R})$ such that

$$A = C \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} C^{-1} = \begin{bmatrix} c_{12} \\ c_{22} \\ c_{32} \end{bmatrix} \begin{bmatrix} c^{-1}_{11} & c^{-1}_{12} & c^{-1}_{13} \end{bmatrix}.$$

Therefore we can find non–zero real 1×3–matrixes $\bar{a}_1$, $\bar{a}_2$ such that $A = \bar{a}_1^T\, \bar{a}_2$. As $A^2 = 0$ we have $\bar{a}_1 \perp \bar{a}_2$. Similarly, we can find non–zero real 1×3–matrixes $\bar{b}_1$, $\bar{b}_2$ such that $B = \bar{b}_1^T\, \bar{b}_2$ and $\bar{b}_1 \perp \bar{b}_2$. Let $\bar{o} \in \mathbb{R}^3$ be a non–zero vector orthogonal to both $\bar{a}_1$ and $\bar{a}_2$. As $AB = BA = 0$ we have $\bar{a}_1 \perp \bar{b}_2$ and $\bar{a}_2 \perp \bar{b}_1$. It follows that there exists a real matrix $[d_{ij}]_{i,j=1,2}$ such that

$$\bar{b}_1 = d_{11}\bar{a}_1 + d_{12}\bar{o} \text{ and } \bar{b}_2 = d_{21}\bar{a}_2 + d_{22}\bar{o}.$$

Then $0 = \langle \bar{b}_1, \bar{b}_2 \rangle = d_{12}d_{22}\|\bar{o}\|^2$. If $d_{12} = 0$, then $d_{11} \neq 0$ and we put $\bar{a} := \bar{a}_1$, $\bar{b} := \bar{a}_2$, $\bar{c} := d_{11}\bar{b}_2$. Then $\bar{a}^T \bar{b} = A$ and $\bar{a}^T \bar{c} = B$. If $d_{22} = 0$, then $d_{21} \neq 0$ and we put $\bar{a} := \bar{a}_1/d_{21}$, $\bar{b} := \bar{b}_1$, $\bar{c} := \bar{b}_2$. Then $\bar{a}^T \bar{c} = A$ and $\bar{b}^T \bar{c} = B$, which completes the proof. □

By the above lemma, there exists $\bar{c} \in \mathbb{R}^3$ such that for any two linearly independent vectors $\bar{a}, \bar{b} \in \mathbb{R}^3$ orthogonal to $\bar{c}$ there exist C^1–functions $h_1, h_2 : \mathbb{T}^3 \to \mathbb{R}$ such that $g(\bar{x})$ equals

$$\bar{c}^T(h_1(\bar{x})\bar{a} + h_2(\bar{x})\bar{b}) \text{ or } (h_1(\bar{x})\bar{a} + h_2(\bar{x})\bar{b})^T \bar{c}$$

for all $\bar{x} \in \mathbb{T}^3$. We first treat the special case of Theorem 3.1 where the limit function g is constant.

Lemma 3.3. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be an area–preserving ergodic C^1–diffeomorphism with τ–polynomial uniform growth of the derivative ($\tau > 0$). Suppose that the limit function $g = \lim_{n\to\infty} n^{-\tau} Df^n$ is constant. Then τ is 1 or 2, and f is algebraically conjugate to a diffeomorphism of the form*

$$\mathbb{T}^3 \ni (x_1, x_2, x_3) \longmapsto (x_1 + \alpha, \varepsilon x_2 + \beta(x_1), x_3 + \gamma(x_1, x_2)) \in \mathbb{T}^3,$$

where $\varepsilon = \det Df = \pm 1$.

Before we pass to the proof we introduce some notation. Let $A \in GL_3(\mathbb{R})$. Denote by $\mathbb{T}^3_A$ the quotient group $\mathbb{R}^3/(\mathbb{Z}^3 A^T)$, which is a model of the 3–torus as well. Then the map

$$A : \mathbb{T}^3 \to \mathbb{T}^3_A, \quad A\bar{x} = \bar{x}A^T$$

establishes a smooth isomorphism between $\mathbb{T}^3$ and $\mathbb{T}^3_A$. Suppose that $\xi : \mathbb{T}^3_A \to \mathbb{T}^3_A$ is a diffeomorphism. Then $A^{-1} \circ \xi \circ A$ is a diffeomorphism of the torus $\mathbb{T}^3$. Let $N \in GL_3(\mathbb{Z})$ be its linear part. Then

$$\xi(\bar{x} + \bar{m}A^T) = \xi(\bar{x}) + \bar{m}N^T A^T$$

for all $\bar{m} \in \mathbb{Z}^3$. Moreover, we can write

$$\xi(\bar{x}) = \bar{x}(ANA^{-1})^T + \tilde{\xi}(\bar{x})$$

and ANA^{-1} (resp. $\tilde{\xi}$) we will be called the A–linear (resp. the A–periodic) part of ξ. The name A–periodic is justified by $\tilde{\xi}(\bar{x} + \bar{m}A^T) = \tilde{\xi}(\bar{x})$ for all $\bar{m} \in \mathbb{Z}^3$.

Suppose that $f : \mathbb{T}^3 \to \mathbb{T}^3$ is a smooth diffeomorphism with τ–polynomial uniform growth of the derivative and $g : \mathbb{T}^3 \to M_3(\mathbb{R})$ is the limit of the sequence $\{n^{-\tau} Df^n\}_{n\in\mathbb{N}}$. Let us consider the diffeomorphism $\hat{f} : \mathbb{T}^3_A \to \mathbb{T}^3_A$ given by $\hat{f} := A \circ f \circ A^{-1}$. Then

$$\frac{1}{n^\tau} D\hat{f}^n(\bar{x}) = \frac{1}{n^\tau} A \cdot (Df^n(A^{-1}\bar{x})) \cdot A^{-1} \to A \cdot g(A^{-1}\bar{x}) \cdot A^{-1} \tag{3.11}$$

uniformly on $\mathbb{T}^3_A$. Let us denote by $\hat{g} : \mathbb{T}^3_A \to M_3(\mathbb{R})$ the function $\hat{g}(\bar{x}) := A \cdot g(A^{-1}\bar{x}) \cdot A^{-1}$. Lemma 2.4 now gives

$$g(\bar{x}) = g(f\bar{x}) \cdot Df(\bar{x}) \quad \text{and} \quad g(\bar{y}) = Df(\bar{x}) \cdot g(\bar{y}) \tag{3.12}$$

for all $\bar{x}, \bar{y} \in \mathbb{T}^3$, and consequently

$$\hat{g}(\bar{x}) = \hat{g}(\hat{f}\bar{x}) \cdot D\hat{f}(\bar{x}) \quad \text{and} \quad \hat{g}(\bar{y}) = D\hat{f}(\bar{x}) \cdot \hat{g}(\bar{y}) \tag{3.13}$$

for all $\bar{x}, \bar{y} \in \mathbb{T}^3_A$.

Throughout this paper we denote by $G(\bar{c})$ the subgroup of all $\bar{m} \in \mathbb{Z}^3$ such that $\bar{m} \perp \bar{c}$. Of course, if $\bar{c} \in \mathbb{R}^3 \setminus \{0\}$, then the rank of $G(\bar{c})$ can be equal 0, 1 or 2. The reader can find further useful properties of the group $G(\bar{c})$ in Appendix B.

Suppose that $f : \mathbb{T}^3 \to \mathbb{T}^3$ is an area–preserving ergodic C^1–diffeomorphism with τ–polynomial uniform growth of the derivative and the limit function g is constant. By Lemma 3.2, there exist mutually orthogonal vectors $\bar{a}, \bar{c} \in \mathbb{R}^3$ such that $g = \bar{c}^T \bar{a}$.

Lemma 3.4. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be an area–preserving C^1–diffeomorphism. Suppose that f preserves orientation, has τ–polynomial uniform growth of the derivative and the limit function $g = \lim_{n\to\infty} n^{-\tau} Df^n$ equals $\bar{c}^T \bar{a}$, where $\bar{a} \perp \bar{c}$. Then the rank of $G(\bar{a})$ equals 2. Moreover, τ equals either 1 or 2.*

Proof. Let $\bar{b} \in \mathbb{R}^3$ be a vector orthogonal to both $\bar{a}$ and $\bar{c}$ such that $\det(A) = 1$, where

$$A = \begin{bmatrix} \bar{a} \\ \bar{b} \\ \bar{c} \end{bmatrix}.$$

Consider $\hat{f} : \mathbb{T}^3_A \to \mathbb{T}^3_A$ given by $\hat{f} := A \circ f \circ A^{-1}$. Then

$$\hat{g} = A \cdot \bar{c}^T \bar{a} \cdot A^{-1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

>From (3.13) we obtain

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} D\hat{f} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = D\hat{f} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Consequently,

$$\begin{array}{lll} \frac{\partial}{\partial x_1}\hat{f}_1(\bar{x}) = 1, & \frac{\partial}{\partial x_2}\hat{f}_1(\bar{x}) = 0, & \frac{\partial}{\partial x_3}\hat{f}_1(\bar{x}) = 0, \\ \frac{\partial}{\partial x_3}\hat{f}_1(\bar{x}) = 0, & \frac{\partial}{\partial x_3}\hat{f}_2(\bar{x}) = 0, & \frac{\partial}{\partial x_3}\hat{f}_3(\bar{x}) = 1 \end{array}$$

for all $\bar{x} \in \mathbb{T}^3_A$. It follows that

$$\hat{f}(x_1, x_2, x_3) = (x_1 + \alpha, x_2 + \beta(x_1), x_3 + \gamma(x_1, x_2)),$$

where $\beta : \mathbb{R} \to \mathbb{R}$, $\gamma : \mathbb{R}^2 \to \mathbb{R}$ are C^1–functions. Let $N \in GL_3(\mathbb{Z})$ stand for the linear part of f. Then the A–linear part of $\hat{f}$ equals

$$ANA^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ K_{21} & 1 & 0 \\ K_{31} & K_{32} & 1 \end{bmatrix}.$$

It follows that

$$\begin{aligned} \bar{a}N &= \bar{a} && (3.14) \\ \bar{b}N &= K_{21}\bar{a} + \bar{b} && (3.15) \\ \bar{c}N &= K_{31}\bar{a} + K_{32}\bar{b} + \bar{c}. && (3.16) \end{aligned}$$

Let $\tilde{f} : \mathbb{T}^3 \to \mathbb{R}^3$ stand for the periodic part of f, i.e. $f(\bar{x}) = \bar{x}N^T + \tilde{f}(\bar{x})$. Then

$$f^n(\bar{x}) = \bar{x}(N^n)^T + \sum_{k=0}^{n-1} \tilde{f}(f^k\bar{x})(N^{n-1-k})^T.$$

Since $\int_{\mathbb{T}^3} D(\tilde{f} \circ f^k)(\bar{x})\, d\bar{x} = 0$ for all natural k,

$$\frac{1}{n^\tau} N^n = \frac{1}{n^\tau} \int_{\mathbb{T}^3} Df^n(\bar{x}) d\bar{x} \to g. \tag{3.17}$$

It follows that

$$\frac{1}{n^\tau} \begin{bmatrix} 1 & 0 & 0 \\ K_{21} & 1 & 0 \\ K_{31} & K_{32} & 1 \end{bmatrix}^n \to \hat{g} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{3.18}$$

Suppose, contrary to our claim, that rank $G(\bar{a}) < 2$.

First, suppose that rank $G(\bar{a}) = 0$. From (3.14) we have $N = \mathrm{Id}$. Consequently, $n^{-\tau}N^n$ tends to zero, contrary to (3.17).

Now suppose that rank $G(\bar{a}) = 1$. Let $\bar{m} \in \mathbb{Z}^3$ be a generator of $G(\bar{a})$. Then there exists a vector $\bar{r} \in \mathbb{Q}^3$ such that $N - \mathrm{Id} = \bar{m}^T\bar{r}$, by (3.14). From (3.15) we have

$$\bar{b}\,\bar{m}^T\,\bar{r} = \bar{b}(N - \mathrm{Id}) = K_{21}\bar{a}.$$

Suppose that $K_{21} \neq 0$. Then rank $G(\bar{a}) =$ rank $G(\bar{r}) = 2$, which contradicts our assumption. Consequently, $K_{21} = 0$. It follows that

$$\begin{bmatrix} 1 & 0 & 0 \\ K_{21} & 1 & 0 \\ K_{31} & K_{32} & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ nK_{31} & nK_{32} & 1 \end{bmatrix}.$$

>From (3.18) it follows that $\tau = 1$ and $K_{31} = 1, K_{32} = 0$. Then

$$\bar{c}\,\bar{m}^T\,\bar{r} = \bar{c}(N - \mathrm{Id}) = \bar{a},$$

by (3.16). It follows that rank $G(\bar{a}) =$ rank $G(\bar{r}) = 2$, which contradicts our assumption.

Finally, we have to prove that τ equals either 1 or 2. >From (3.18) we obtain

$$n^{1-\tau}K_{21} \to 0, \quad n^{1-\tau}K_{31} + \frac{1 - 1/n}{2} n^{2-\tau}K_{21}K_{32} \to 1, \quad n^{1-\tau}K_{32} \to 0.$$

If $K_{21} = 0$, then $\tau = 1$ and $K_{31} = 1$. Otherwise, $\tau = 2$ and $K_{21}K_{32} = 2$, which completes the proof. □

Proof of Lemma 3.3. First, notice that f^2 preserves area and orientation, and $n^{-\tau}Df^{2n}$ tends uniformly to $2^\tau\bar{c}^T\bar{a}$. By Lemma 3.4, rank $G(\bar{a}) = 2$. It follows that $\bar{a} = a\bar{m} \in a\mathbb{Z}^3$, by Lemma B.1 (see Appendix B). Now choose $\bar{n}, \bar{k} \in \mathbb{Z}^3$ such that the determinant of

$$A := \begin{bmatrix} \bar{m} \\ \bar{n} \\ \bar{k} \end{bmatrix}$$

equals 1. Let us consider the diffeomorphism $\hat{f} : \mathbb{T}^3 \to \mathbb{T}^3$ given by $\hat{f} := A \circ f \circ A^{-1}$. Then

$$\hat{g} = A \cdot g \cdot A^{-1} = a \begin{bmatrix} 0 \\ \bar{n}\bar{c}^T \\ \bar{k}\bar{c}^T \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

>From (3.13) we have

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} D\hat{f}(\bar{x}) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ \bar{n}\bar{c}^T \\ \bar{k}\bar{c}^T \end{bmatrix} = D\hat{f}(\bar{x}) \begin{bmatrix} 0 \\ \bar{n}\bar{c}^T \\ \bar{k}\bar{c}^T \end{bmatrix}.$$

It follows that

$$\hat{f}(x_1, x_2, x_3) = (x_1 + \alpha, \varphi_{x_1}(x_2, x_3)),$$

where $\varphi : \mathbb{T} \times \mathbb{T}^2 \to \mathbb{T}^2$ is an area–preserving random diffeomorphism over the rotation by an irrational number α. Then

$$\begin{bmatrix} \bar{n}\bar{c}^T \\ \bar{k}\bar{c}^T \end{bmatrix} = D\varphi_{x_1}(x_2, x_3) \begin{bmatrix} \bar{n}\bar{c}^T \\ \bar{k}\bar{c}^T \end{bmatrix}$$

for all $(x_1, x_2, x_3) \in \mathbb{T}^3$

Suppose that $\bar{n}\bar{c}^T$ and $\bar{k}\bar{c}^T$ are rationally independent. Then by Lemma 2.5, $\varphi_{x_1}(x_2, x_3) = (x_2 + \beta(x_1), x_3 + \gamma(x_1))$, where $\beta, \gamma : \mathbb{T} \to \mathbb{T}$ are C^1–functions, which is our claim.

Otherwise, by Lemma 2.6, there exist a group automorphism $B : \mathbb{T}^2 \to \mathbb{T}^2$ and C^1–functions $\beta : \mathbb{T} \to \mathbb{T}$, $\gamma : \mathbb{T}^2 \to \mathbb{T}$ such that

$$B \circ \varphi_{x_1} \circ B^{-1}(x_2, x_3) = (\varepsilon x_2 + \beta(x_1), x_3 + \gamma(x_1, x_2)),$$

where $\varepsilon = \det Df$, which proves the claim. □

Proof of Theorem 3.1. is divided into a few cases.

Case 1. Suppose that $g = \bar{c}^T(h_1\bar{a} + h_2\bar{b})$, where $\bar{a}$ and $\bar{b}$ are orthogonal to $\bar{c}$ and the matrix

$$A = \begin{bmatrix} \bar{a} \\ \bar{b} \\ \bar{c} \end{bmatrix}$$

is nonsingular. Let $\hat{f} : \mathbb{T}^3_A \to \mathbb{T}^3_A$ be given by $\hat{f} := A \circ f \circ A^{-1}$. Then

$$\hat{g} = A \cdot \bar{c}^T(\hat{h}_1\bar{a} + \hat{h}_2\bar{b}) \cdot A^{-1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} \hat{h}_1 & \hat{h}_2 & 0 \end{bmatrix},$$

where $\hat{h}_i(\bar{x}) := h_i(A^{-1}\bar{x})$ for $i = 1, 2$. From (3.13) we obtain

$$\begin{aligned} \begin{bmatrix} \hat{h}_1(\bar{x}) & \hat{h}_2(\bar{x}) & 0 \end{bmatrix} &= \begin{bmatrix} \hat{h}_1(\hat{f}\bar{x}) & \hat{h}_2(\hat{f}\bar{x}) & 0 \end{bmatrix} D\hat{f}(\bar{x}), \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} &= D\hat{f}(\bar{x}) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned} \tag{3.19}$$

for all $\bar{x} \in \mathbb{T}^3_A$. Consequently, $\partial\hat{f}_1(\bar{x})/\partial x_3 = 0$, $\partial\hat{f}_2(\bar{x})/\partial x_3 = 0$ and $\partial\hat{f}_3(\bar{x})/\partial x_3 = 1$ for all $\bar{x} \in \mathbb{T}^3_A$. It follows that

$$\hat{f}(x_1, x_2, x_3) = (F(x_1, x_2), x_3 + \gamma(x_1, x_2)),$$

where $\gamma : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function and $F : \mathbb{R}^2 \to \mathbb{R}^2$ is the diffeomorphism given by $F(x_1, x_2) = (\hat{f}_1(x_1, x_2), \hat{f}_2(x_1, x_2))$. Let K stand for the A–linear part of $\hat{f}$, $K = A\,N\,A^{-1}$, where $N \in GL_3(\mathbb{Z})$ is the linear part of f. Then $\det K = \det N = \varepsilon' = \pm 1$ and $K_{13} = 0, K_{23} = 0, K_{33} = 1$. Moreover, there exist C^2–functions $\tilde{F} : \mathbb{R}^2 \to \mathbb{R}^2$, $\tilde{\gamma} : \mathbb{R}^2 \to \mathbb{R}$ which are $(\bar{a}\bar{m}^T, \bar{b}\bar{m}^T)$–periodic for all $\bar{m} \in \mathbb{Z}^3$ such that

$$F(\bar{x}) = \tilde{F}(\bar{x}) + \bar{x}K'^T \quad \text{and} \quad \gamma(x_1, x_2) = \tilde{\gamma}(x_1, x_2) + K_{31}x_1 + K_{32}x_2,$$

where $K' = K|_{\{1,2\}\times\{1,2\}} \in GL_2(\mathbb{R})$ and $\det K' = \varepsilon'$. From (3.11) we have

$$\frac{1}{n^\tau} DF^n(x_1,x_2) \to 0 \quad \text{and} \quad \frac{1}{n^\tau}\sum_{k=0}^{n-1} D(\gamma\circ F^k)(x_1,x_2) \to \left[\, \hat h_1(\bar x)\ \hat h_2(\bar x)\,\right]$$

uniformly on $\mathbb{T}^3_A$. Therefore $\hat h_1, \hat h_2$ depend only on the first two coordinates. Let $H : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $H(x_1,x_2) = \left[\, \hat h_1(x_1,x_2,0)\ \ \hat h_2(x_1,x_2,0)\,\right]$. Then H is $(\bar a\bar m^T, \bar b\bar m^T)$–periodic for all $\bar m \in \mathbb{Z}^3$ and is of class C^1. From (3.19) we have

$$H(F\bar x)\cdot DF(\bar x) = H(\bar x) \tag{3.20}$$

for all $\bar x \in \mathbb{R}^2$. Set $\chi_n := n^{-\tau}\sum_{k=0}^{n-1}\gamma\circ F^k$. Since $D\chi_n \to H$ uniformly on $\mathbb{R}^2$, $\chi_n(x_1,x_2) - \chi_n(x_1,0) \to \int_0^{x_2} H_2(x_1,t)\,dt$, $\chi_n(x_1,x_2) - \chi_n(0,x_2) \to \int_0^{x_1} H_1(t,x_2)\,dt$ for all $(x_1,x_2)\in\mathbb{R}^2$. Let $\xi : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\begin{aligned}\xi(x_1,x_2) \ &:= \ \lim_{n\to\infty}(\chi_n(x_1,x_2) - \chi_n(0,0)) = \int_0^{x_1} H_1(t,x_2)\,dt + \int_0^{x_2} H_2(0,t)\,dt \\ &= \ \int_0^{x_2} H_2(x_1,t)\,dt + \int_0^{x_1} H_1(t,0)\,dt.\end{aligned}$$

Then $\partial\xi/\partial x_1 = H_1$, $\partial\xi/\partial x_2 = H_2$ and ξ is of class C^2. By (3.20), there exists $\alpha\in\mathbb{R}$ such that

$$\xi(F\bar x) = \xi(\bar x) + \alpha. \tag{3.21}$$

By Lemma B.1 (see Appendix B), there exists a C^2–function $\tilde\xi : \mathbb{R}^2 \to \mathbb{R}$ which is $(\bar a\bar m^T, \bar b\bar m^T)$–periodic for all $\bar m\in\mathbb{Z}^3$ and $d_1, d_2 \in \mathbb{R}$ such that $\xi(x_1,x_2) = \tilde\xi(x_1,x_2) + d_1x_1 + d_2x_2$. Since $H \neq 0$, it is easy to see that $(d_1,d_2) \neq (0,0)$. Moreover, from (3.21) we have

$$[\, d_1\ d_2\,]\ K' = [\, d_1\ d_2\,] \tag{3.22}$$

and

$$\begin{aligned}&\tilde\xi(\bar x) + \alpha \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.23)\\ &\quad = \ \tilde\xi(\tilde F_1(\bar x) + K_{11}x_1 + K_{12}x_2, \tilde F_2(\bar x) + K_{21}x_1 + K_{22}x_2) + d_1\tilde F_1(\bar x) + d_2\tilde F_2(\bar x).\end{aligned}$$

Case 1a. Suppose that rank $G(\bar c) = 0$. By Lemma B.1, $D\hat f$ is constant. It follows that Df and g are constant. Therefore $g = \bar c^T\bar a$, where $\bar a$ is orthogonal to $\bar c$. From (3.12) we obtain $\bar c^T = Df(\bar x)\,\bar c^T$ for all $\bar x\in\mathbb{T}^3$. As $G(\bar c) = \{0\}$ and $Df(\bar x)\in GL^3(\mathbb{Z})$ we have $Df(x) =$Id for all $\bar x\in\mathbb{T}^3$. Consequently, f is a rotation on the 3–torus, which is impossible.

Case 1b. Suppose that rank $G(\bar c) = 1$. By Lemma B.1, there exist real numbers l_1, l_2 such that $\bar m = l_1\bar a + l_2\bar b$ generates $G(\bar c)$ and C^2–functions $\bar F : \mathbb{T}\to\mathbb{R}^2$, $\bar\xi : \mathbb{T}\to\mathbb{R}$, $\bar\gamma : \mathbb{T}\to\mathbb{R}$ such that

$$\tilde F(x_1,x_2) = \bar F(l_1x_1 + l_2x_2),\ \tilde\xi(x_1,x_2) = \bar\xi(l_1x_1+l_2x_2) \text{ and } \tilde\gamma(x_1,x_2) = \bar\gamma(l_1x_1+l_2x_2).$$

From (3.23) we obtain

$$\begin{aligned}\bar\xi(l_1x_1 + l_2x_2) + \alpha \ &= \ \bar\xi(l_1\bar F_1(l_1x_1+l_2x_2) + l_2\bar F_2(l_1x_1+l_2x_2) + s_1x_1 + s_2x_2) \\ &\quad + d_1\bar F_1(l_1x_1+l_2x_2) + d_2\bar F_2(l_1x_1+l_2x_2),\end{aligned}$$

where $[\, s_1\ s_2\,] = [\, l_1\ l_2\,]\,K'$. If (s_1,s_2) and (l_1,l_2) are linearly independent, then $\bar\xi$ is constant. It follows that H is constant which reduces the problem to Lemma 3.3. Otherwise, there exists a real number s such that $(s_1,s_2) = s(l_1,l_2)$ and

$$\bar\xi(x) + \alpha = \bar\xi(l_1\bar F_1(x) + l_2\bar F_2(x) + sx) + d_1\bar F_1(x) + d_2\bar F_2(x)$$

for any real x. Since f preserves area $\det DF(\bar{x}) = \varepsilon = \pm 1$ for all $\bar{x} \in \mathbb{T}^3$. It follows that

$$\begin{aligned} \varepsilon &= \det \begin{bmatrix} l_1 D\bar{F}_1(x) + K_{11} & l_2 D\bar{F}_1(x) + K_{12} \\ l_1 D\bar{F}_2(x) + K_{21} & l_2 D\bar{F}_2(x) + K_{22} \end{bmatrix} \\ &= (l_1 K_{22} - l_2 K_{21}) D\bar{F}_1(x) + (-l_1 K_{12} + l_2 K_{11}) D\bar{F}_2(x) + \det K \\ &= (l_1 D\bar{F}_1(x) + l_2 D\bar{F}_2(x)) \det K/s + \det K \end{aligned}$$

for any real x. Since $\bar{F}_1, \bar{F}_2$ are 1–periodic, we have $l_1 D\bar{F}_1(x) + l_2 D\bar{F}_2(x) = 0$ and $\det K = \varepsilon$. Therefore the function $l_1\bar{F}_1 + l_2\bar{F}_2$ is constant. Let us choose real numbers r_1, r_2 such that the determinant of the matrix

$$L = \begin{bmatrix} l_1 & l_2 & 0 \\ r_1 & r_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

equals 1. Now consider the diffeomorphism $\check{f} : \mathbb{T}^3_{LA} \to \mathbb{T}^3_{LA}$ given by $\check{f} = L \circ \hat{f} \circ L^{-1}$. Then

$$\check{f}(x_1, x_2, x_3) = (sx_1 + \alpha, \varepsilon/s x_2 + r x_1 + r_1 \bar{F}_1(x_1) + r_2 \bar{F}_2(x_1), x_3 + \bar{\gamma}(x_1) + p_1 x_1 + p_2 x_2).$$

As $\partial \check{f}_1^n / \partial x_1 = s^n$ and $\partial \check{f}_2^n / \partial x_2 = (\varepsilon/s)^n$ we obtain $s = \pm 1$, because $\check{f}$ has polynomial uniform growth of the derivative. Moreover,

$$LA = \begin{bmatrix} \bar{m} \\ r_1 \bar{a} + r_2 \bar{b} \\ \bar{c} \end{bmatrix}$$

and $L \circ A \circ f = \check{f} \circ L \circ A$. Therefore $f(\bar{x})\, \bar{m}^T = s\, \bar{x}\, \bar{m}^T + \alpha$. Observe that $s = 1$. Indeed, suppose, contrary to our claim, that $s = -1$. Consider the smooth function $\kappa : \mathbb{T}^3 \to \mathbb{C}$ given by $\kappa(\bar{x}) = e^{2\pi i \bar{x} \bar{m}^T}$. Then $\kappa \circ f^2 = \kappa$. Since κ is smooth, we conclude that it is constant, by the ergodicity of f. Consequently, $\bar{m} = 0$, which is impossible. Now choose $\bar{n}, \bar{k} \in \mathbb{Z}^3$ such that the determinant of

$$A := \begin{bmatrix} \bar{m} \\ \bar{n} \\ \bar{k} \end{bmatrix}$$

equals 1. Let us consider the diffeomorphism $\hat{f} : \mathbb{T}^3 \to \mathbb{T}^3$ given by $\hat{f} := A \circ f \circ A^{-1}$. From (3.13) we have

$$\begin{bmatrix} 0 \\ \bar{n}\bar{c}^T \\ \bar{k}\bar{c}^T \end{bmatrix} = D\hat{f}(\bar{x}) \begin{bmatrix} 0 \\ \bar{n}\bar{c}^T \\ \bar{k}\bar{c}^T \end{bmatrix}.$$

Moreover,

$$\hat{f}_1(\bar{x}) = f(\bar{x}(A^{-1})^T)\bar{m}^T = \bar{x}(A^{-1})^T \bar{m}^T + \alpha = x_1 + \alpha.$$

Our claim now follows by the same arguments as in the proof of Lemma 3.3.

Case 1c. Suppose that rank $G(\bar{c}) = 2$. Then we can assume that $\bar{a}, \bar{b}, \bar{c} \in \mathbb{Z}^3$ and $\bar{a}, \bar{b}$ generates $G(\bar{c})$. Set $q = \det A \in \mathbb{N}$. Then the A–linear part of $\hat{f}$ (which is equal $K = A\, N\, A^{-1}$) belongs to $M_3(q^{-1}\mathbb{Z})$. Moreover, the functions $\tilde{F} : \mathbb{R}^2 \to \mathbb{R}^2$, $\tilde{\gamma} : \mathbb{R}^2 \to \mathbb{R}$ and $\tilde{\xi} : \mathbb{R}^2 \to \mathbb{R}$ are $\mathbb{Z}^2$–periodic, by Lemma B.2 (see Appendix B).

Case 1c(i). Suppose that d_1/d_2 is irrational. From (3.22) we obtain $K' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Set

$$L := \begin{bmatrix} 1/q & 0 & 0 \\ 0 & 1/q & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Consider the diffeomorphism $\check{f} : \mathbb{T}^3_{LA} \to \mathbb{T}^3_{LA}$ given by $\check{f} = L \circ \hat{f} \circ L^{-1}$. Then

$$\check{f}(x_1, x_2, x_3) = (\check{F}(x_1, x_2), x_3 + \check{\gamma}(x_1, x_2)),$$

where $\check{F}(x_1, x_2) = q^{-1}F(qx_1, qx_2)$ and $\check{\gamma}(x_1, x_2) = \gamma(qx_1, qx_2)$. Then

$$\check{F}(\bar{x} + \bar{m}) - \check{F}(\bar{x}) = \bar{m} \quad \text{and} \quad \check{\gamma}(\bar{x} + \bar{m}) - \check{\gamma}(\bar{x}) = qK_{31}m_1 + qK_{32}m_2 \in \mathbb{Z}$$

for all $\bar{m} \in \mathbb{Z}^2$. Therefore, $\check{f}$ can also be treated as a diffeomorphism of the torus $\mathbb{T}^3$. Let $\check{\xi}(x_1, x_2) = \xi(qx_1, qx_2)$. Then

$$\check{\xi} \circ \check{F} = \check{\xi} + \alpha, \tag{3.24}$$

$D\check{\xi} : \mathbb{R}^2 \to \mathbb{R}$ is $\mathbb{Z}^2$–periodic and non–zero at each point. Moreover, $\check{f} : \mathbb{T}^3 \to \mathbb{T}^3$ has τ–polynomial uniform growth of the derivative. More precisely,

$$\frac{1}{n^\tau} D\check{f}^n \to \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \multicolumn{2}{c}{D\check{\xi}} & 0 \end{bmatrix} \tag{3.25}$$

uniformly.

Let us denote by φ^t the Hamiltonian C^2–flow on $\mathbb{T}^2$ defined by the Hamiltonian equation

$$\frac{d}{dt}\varphi^t(\bar{x}) = \begin{bmatrix} \check{\xi}_{x_2}(\varphi^t(\bar{x})) \\ -\check{\xi}_{x_1}(\varphi^t(\bar{x})) \end{bmatrix}.$$

Since φ^t has no fixed point and $\int_{\mathbb{T}^2} \check{\xi}_{x_1}(\bar{x})d\bar{x} / \int_{\mathbb{T}^2} \check{\xi}_{x_2}(\bar{x})d\bar{x} = d_1/d_2$ is irrational, it follows that φ^t is C^2–conjugate to the special flow constructed over the rotation by an irrational number a and under a positive C^2–function $b : \mathbb{T} \to \mathbb{R}$, (see for instance [2, Ch. 16]) i.e. there exists an area–preserving C^2–diffeomorphism $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ and a matrix $N \in GL_2(\mathbb{Z})$ such that

$$\det D\rho \equiv -\hat{b} = -\int_{\mathbb{T}} b(x)\, dx, \qquad \sigma^t \circ \rho = \rho \circ \varphi^t,$$

where $\sigma^t(x_1, x_2) = (x_1, x_2 + t)$ and

$$\rho(\bar{x} + \bar{m}) = (\rho_1(\bar{x}) + (\bar{m}N)_1 + (\bar{m}N)_2 a, \rho_2(\bar{x}) - b^{((\bar{m}N)_2)}(\rho_1(\bar{x})))$$

for all $\bar{m} \in \mathbb{Z}^2$. Let $T_{a,-b} : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ denote by the skew product given by $T_{a,-b}(x_1, x_2) = (x_1 + a, x_2 - b(x_1))$. Let us consider the quotient space $M = M_{a,b} = \mathbb{T} \times \mathbb{R}/\sim$, where the relation $\sim$ is defined by $(x_1, x_2) \sim (y_1, y_2)$ if and only if $(x_1, x_2) = T^k_{a,-b}(y_1, y_2)$ for an integer k. Then the quotient flow $\sigma^t_{a,b}$ of the action σ^t modulo the relation $\sim$ is the special flow constructed over the rotation by a and under the function b. Moreover, $\rho : \mathbb{T}^2 \to M$ conjugates flows φ^t and $\sigma^t_{a,b}$. Let $\bar{F} : M \to M$ stand for the C^2–diffeomorphism $\bar{F} := \rho \circ \check{F} \circ \rho^{-1}$. Since the map $\mathbb{R} \ni t \longmapsto \check{\xi}(\varphi^t\bar{x}) \in \mathbb{R}$ is constant for each $\bar{x} \in \mathbb{R}^2$ we see that the map

$$\mathbb{R} \ni t \longmapsto \check{\xi} \circ \rho^{-1}(\sigma^t(x_1, x_2)) = \check{\xi} \circ \rho^{-1}(x_1, x_2 + t) \in \mathbb{R}$$

is constant for each $(x_1,x_2)\in\mathbb{R}^2$. It follows that the function $\check{\xi}\circ\rho^{-1}:\mathbb{R}^2\to\mathbb{R}$ depends only on the first coordinate. Moreover,

$$\begin{aligned} D\rho^{-1}(\bar{x})\begin{bmatrix}0\\1\end{bmatrix} &= \frac{d}{dt}\rho^{-1}\circ\sigma^t(\bar{x})|_{t=0}=\frac{d}{dt}\varphi^t\circ\rho^{-1}(\bar{x})|_{t=0}\\ &= \begin{bmatrix}\check{\xi}_{x_2}(\rho^{-1}(\bar{x}))\\ -\check{\xi}_{x_1}(\rho^{-1}(\bar{x}))\end{bmatrix}. \end{aligned}$$

Consequently, $\partial\rho_1^{-1}/\partial x_2=(\partial\check{\xi}/\partial x_2)\circ\rho^{-1}$ and $\partial\rho_2^{-1}/\partial x_2=-(\partial\check{\xi}/\partial x_1)\circ\rho^{-1}$. It follows that

$$\frac{d}{dx_1}(\check{\xi}\circ\rho^{-1})=\frac{\partial\check{\xi}}{\partial x_1}\circ\rho^{-1}\cdot\frac{\partial\rho_1^{-1}}{\partial x_1}+\frac{\partial\check{\xi}}{\partial x_2}\circ\rho^{-1}\cdot\frac{\partial\rho_2^{-1}}{\partial x_1}=-\det D\rho^{-1}=\hat{b}^{-1}.$$

Therefore

$$\check{\xi}\circ\rho^{-1}(x_1,x_2)=\hat{b}^{-1}\delta x_1+c. \tag{3.26}$$

We see by (3.24) that $\check{\xi}\circ\rho^{-1}\circ\bar{F}=\check{\xi}\circ\rho^{-1}+\alpha$ and consequently $\bar{F}_1(x_1,x_2)=x_1+\hat{b}\alpha$. For abbreviation, we will write α instead of $\hat{b}\alpha$. Since $\bar{F}:\mathbb{R}^2\to\mathbb{R}^2$ preserves area, we conclude that

$$\bar{F}(x_1,x_2)=(x_1+\alpha,\varepsilon x_2+\beta(x_1)),$$

where $\beta:\mathbb{R}\to\mathbb{R}$ is a C^2–function and $\varepsilon=\det D\bar{F}=\pm1$. As $\bar{F}$ is a diffeomorphism of M, there exist $m_1,m_2\in\mathbb{Z}$ such that

$$\begin{aligned} &(x_1+1+\alpha,\varepsilon x_2+\beta(x_1+1))\\ &\quad= \bar{F}(x_1+1,x_2)=T^{m_2}_{a,-b}\bar{F}(x_1,x_2)+(m_1,0)\\ &\quad= (x_1+\alpha+m_1+m_2a,\varepsilon x_2+\beta(x_1)-b^{(m_2)}(x_1+\alpha)). \end{aligned}$$

It follows that $m_1=1$, $m_2=0$, hence $\beta:\mathbb{T}\to\mathbb{R}$. Moreover, there exist $n_1,n_2\in\mathbb{Z}$ such that

$$\begin{aligned} &(x_1+a+\alpha,\varepsilon x_2-\varepsilon b(x_1)+\beta(x_1+a))\\ &\quad= \bar{F}\circ T_{a,-b}(x_1,x_2)=T^{n_2}_{a,-b}\bar{F}(x_1,x_2)+(n_1,0)\\ &\quad= (x_1+\alpha+n_1+n_2a,\varepsilon x_2+\beta(x_1)-b^{(n_2)}(x_1+\alpha)). \end{aligned}$$

It follows that $n_1=0$, $n_2=1$, hence $\beta(x)-b(x+\alpha)=-\varepsilon b(x)+\beta(x+a)$. Consequently,

$$(1-\varepsilon)\hat{b}=\int_{\mathbb{T}}(b(x+\alpha)-\varepsilon b(x))dx=\int_{\mathbb{T}}(\beta(x)-\beta(x+a))dx=0.$$

Therefore $\bar{F}(x_1,x_2)=(x_1+\alpha,x_2+\beta(x_1))$ and the skew products $\bar{F}$ and $T_{a,-b}$ commute. Let $\bar{f}:M\times\mathbb{T}\to M\times\mathbb{T}$ denote by the diffeomorphism

$$\bar{f}:=(\rho\times\mathrm{Id}_{\mathbb{T}})\circ\check{f}\circ(\rho\times\ \mathrm{Id}_{\mathbb{T}})^{-1}.$$

Then

$$\bar{f}(x_1,x_2,x_3)=(\bar{F}(x_1,x_2),x_3+\bar{\gamma}(x_1,x_2)),$$

where $\bar{\gamma}:M\to\mathbb{T}$ is given by $\bar{\gamma}=\check{\gamma}\circ\rho^{-1}$. Therefore there exist $k_1,k_2\in\mathbb{Z}$ such that

$$\bar{\gamma}(x_1+1,x_2)=\bar{\gamma}(x_1,x_2)+k_1 \text{ and } \bar{\gamma}(x_1+a,x_2-b(x_1))=\bar{\gamma}(x_1,x_2)+k_2.$$

Moreover,

$$\begin{aligned}
&\frac{1}{n^\tau} D\bar{f}^n \\
&= \begin{bmatrix} (D\rho)\circ \check{F}^n \circ \rho^{-1} & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n^{-\tau}(D\check{F}^n)\circ\rho^{-1} & 0 \\ & 0 \\ n^{-\tau}(D(\check{\gamma}^{(n)}))\circ\rho^{-1} & n^{-\tau} \end{bmatrix} \begin{bmatrix} D(\rho^{-1}) & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&\to \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (D\check{\xi})\circ\rho^{-1} & & 0 \end{bmatrix} \begin{bmatrix} D\rho^{-1} & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ D(\check{\xi}\circ\rho^{-1}) & & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hat{b} & 0 & 0 \end{bmatrix}
\end{aligned}$$

uniformly on $M \times \mathbb{T}$, by (3.25) and (3.26). It follows that

$$\frac{1}{n^\tau}\sum_{k=0}^{n-1}(\bar{\gamma}_{x_1}(\bar{F}^k(x_1,x_2)) + \bar{\gamma}_{x_2}(\bar{F}^k(x_1,x_2)) \cdot D\beta^{(k)}(x_1)) \to \hat{b}$$

and $\frac{1}{n^\tau}\sum_{k=0}^{n-1}\bar{\gamma}_{x_2}(\bar{F}^k(x_1,x_2)) \to 0$ uniformly for $(x_1,x_2)\in M$. Consequently,

$$\frac{1}{n^\tau}\sum_{k=0}^{n-1}\int_M (\bar{\gamma}_{x_1}(\bar{F}^k(x_1,x_2)) + \bar{\gamma}_{x_2}(\bar{F}^k(x_1,x_2))\ D\beta^{(k)}(x_1))\, dx_1\, dx_2 \to 1,$$

$$\frac{1}{n^\tau}\sum_{k=0}^{n-1}\int_M \bar{\gamma}_{x_2}(\bar{F}^k(x_1,x_2))\, dx_1\, dx_2 \to 0. \tag{3.27}$$

We now show that

$$\frac{1}{n}\sum_{k=0}^{n-1}\int_M (\bar{\gamma}_{x_1}(\bar{F}^k(x_1,x_2)) + \bar{\gamma}_{x_2}(\bar{F}^k(x_1,x_2)) \cdot D\beta^{(k)}(x_1)) dx_1 dx_2 \to k_1\hat{b}.$$

This implies $\tau = 1$ and $k_1 \neq 0$. To prove this, note that

$$\begin{aligned}
&\frac{1}{n}\sum_{k=0}^{n-1}\int_M \bar{\gamma}_{x_1}(\bar{F}^k(x_1,x_2))\, dx_1\, dx_2 \\
&= \int_0^1\int_0^{b(x_1)} \bar{\gamma}_{x_1}(x_1,x_2)\, dx_2\, dx_1 \\
&= \int_0^1 \frac{d}{dx_1}\left(\int_0^{b(x_1)}\bar{\gamma}(x_1,x_2)\, dx_2\right) dx_1 - \int_0^1 Db(x_1)\bar{\gamma}(x_1,b(x_1))\, dx_1 \\
&= \int_0^{b(1)}\bar{\gamma}(1,x_2)\, dx_2 - \int_0^{b(0)}\bar{\gamma}(0,x_2)\, dx_2 - \int_0^1 Db(x_1)(\bar{\gamma}(x_1+a,0) - k_2)\, dx_1 \\
&= b(0)k_1 - \int_0^1 Db(x_1)\bar{\gamma}(x_1+a,0)\, dx_1.
\end{aligned}$$

Let $u:\mathbb{T}\to\mathbb{R}$ be given by $u(x)=\bar\gamma(x)-k_1x$. Now observe that

$$\begin{aligned}
&\frac{1}{n}\sum_{k=0}^{n-1}\int_M \bar\gamma_{x_2}(\bar F^k(x_1,x_2))\cdot D\beta^{(k)}(x_1)\,dx_1\,dx_2\\
&= \frac{1}{n}\sum_{k=0}^{n-1}\int_0^1\int_0^{b(x_1)}\bar\gamma_{x_2}(x_1,x_2)\cdot D\beta^{(k)}(x_1-k\alpha)\,dx_2\,dx_1\\
&= \frac{1}{n}\sum_{k=0}^{n-1}\int_0^1(\bar\gamma(x_1,b(x_1))-\bar\gamma(x_1,0))\cdot D\beta^{(k)}(x_1-k\alpha)\,dx_1\\
&= \frac{1}{n}\sum_{k=0}^{n-1}\int_0^1(\bar\gamma(x_1+a,0)-k_2-\bar\gamma(x_1,0))\cdot D\beta^{(k)}(x_1-k\alpha)\,dx_1\\
&= \frac{1}{n}\sum_{k=0}^{n-1}\int_0^1 u(x_1+a)(D\beta^{(k)}(x_1-k\alpha)-D\beta^{(k)}(x_1-k\alpha+a))\,dx_1\\
&= \frac{1}{n}\sum_{k=0}^{n-1}\int_0^1 u(x_1+a)(Db(x_1)-Db(x_1-k\alpha))\,dx_1\\
&= \int_0^1 u(x_1+a)Db(x_1)dx_1-\int_0^1 u(x_1+a)\frac{1}{n}\sum_{k=0}^{n-1}Db(x_1-k\alpha)\,dx_1\\
&\to \int_0^1 u(x_1+a)Db(x_1)\,dx_1\\
&= \int_0^1\bar\gamma(x_1+a,0)Db(x_1)\,dx_1-k_1\int_0^1 x_1\cdot Db(x_1)\,dx_1.
\end{aligned}$$

Moreover, $\int_0^1 x\cdot Db(x)\,dx=b(1)-\int_{\mathbb{T}}b(x)\,dx$, which proves the required conclusion. From (3.27) we have $\int_0^1\int_0^{b(x_1)}\bar\gamma_{x_2}(x_1,x_2)dx_2dx_1=0$. However

$$\begin{aligned}
\int_0^1\int_0^{b(x_1)}\bar\gamma_{x_2}(x_1,x_2)dx_2dx_1 &= \int_0^1(\bar\gamma(x_1,b(x_1))-\bar\gamma(x_1,0))dx_1\\
&= \int_0^1(\bar\gamma(x_1+a,0)-\bar\gamma(x_1,0)-k_2)dx_1\\
&= k_1a-k_2.
\end{aligned}$$

It follows that $k_1a=k_2$, which contradicts the fact that $k_1\neq 0$ and a is irrational. Consequently, d_1/d_2 must be rational.

Case 1c(ii). Suppose that $(d_1,d_2)=d(l_1,l_2)$, where l_1,l_2 are relatively prime integers. Since $K'\in M(q^{-1}\mathbb{Z})$ and $\det k'=\varepsilon=\pm1$, there exist $M\in GL_2(\mathbb{Z})$ and $m\in\mathbb{Z}$ such that

$$K'=M^{-1}\begin{bmatrix}1&0\\ m/q&\varepsilon\end{bmatrix}M,$$

by (3.22). Then there exists an even number $r>0$ such that $K'^r\in GL_2(\mathbb{Z})$. Therefore the diffeomorphism $F^r:\mathbb{R}^2\to\mathbb{R}^2$ can be treated as an area–preserving diffeomorphism of the torus $\mathbb{T}^2$. Let $\check\xi:\mathbb{T}^2\to\mathbb{T}$ be given by $\check\xi(x_1,x_2)=d^{-1}\xi(x_1,x_2)$. It follows by (3.21) that

$$\check\xi\circ F^r=\check\xi+r\alpha/d\quad\text{ and }\quad(d_1(\check\xi),d_2(\check\xi))=(l_1,l_2)\neq0.$$

Note that α/d is irrational. Indeed, suppose that $\alpha/d = k/l$, where $k \in \mathbb{Z}$ and $l \in \mathbb{N}$. Let $\Xi : \mathbb{T}^3_A \to \mathbb{C}$ be defined by $\Xi(x_1, x_2, x_3) = \exp 2\pi i l \check{\xi}(x_1, x_2)$. As $\check{\xi} \circ F = \check{\xi} + k/l$ we have

$$\Xi(\hat{f}(x_1, x_2, x_3)) = \exp 2\pi i l \check{\xi}(F(x_1, x_2)) = \Xi(x_1, x_2, x_3).$$

By the ergodicity of $\hat{f}$, Ξ and also $\check{\xi}$ is constant, which is impossible.

By Theorem A.1 (see Appendix A), there is an area–preserving C^2–diffeomorphism $\psi : \mathbb{T}^2 \to \mathbb{T}^2$ such that

$$\psi^{-1} \circ F^r \circ \psi : \mathbb{T}^2 \to \mathbb{T}^2$$

is a skew product and $\check{\xi} \circ \psi(x_1, x_2) = kx_1 + c$, where $k \in \mathbb{N}$ and $c \in \mathbb{R}$. Therefore $D(\xi \circ \psi) = [\, dk \;\; 0\,]$. Let $L \in GL_2(\mathbb{Z})$ stand for the linear part of ψ. Set

$$\bar{L} := \begin{bmatrix} \multicolumn{2}{c}{L} & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 & 0 & 1 \end{bmatrix} \in GL_3(\mathbb{Z}).$$

Let us consider the area–preserving C^2–isomorphism $\rho : \mathbb{T}^3_A \to \mathbb{T}^3_{\bar{L}^{-1}A}$ defined by

$$\rho(x_1, x_2, x_3) = (\psi^{-1}(x_1, x_2), x_3).$$

Let $\check{f} : \mathbb{T}^3_{\bar{L}^{-1}A} \to \mathbb{T}^3_{\bar{L}^{-1}A}$ be given by $\check{f} = \rho \circ \hat{f} \circ \rho^{-1}$. Then

$$\begin{aligned}
&\frac{1}{n^\tau} D\check{f}^n \\
&= \begin{bmatrix} (D\psi^{-1}) \circ F^n \circ \psi & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 \quad 0 & 1 \end{bmatrix} \begin{bmatrix} n^{-\tau}(DF^n) \circ \psi & \begin{matrix} 0 \\ 0 \end{matrix} \\ n^{-\tau}(D(\gamma^{(n)})) \circ \psi & n^{-\tau} \end{bmatrix} \begin{bmatrix} D\psi & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 \quad 0 & 1 \end{bmatrix} \\
&\to \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \multicolumn{2}{c}{D\xi \circ \psi} & 0 \end{bmatrix} \begin{bmatrix} D\psi & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 \quad 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \multicolumn{2}{c}{D(\xi \circ \psi)} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ dk & 0 & 0 \end{bmatrix}
\end{aligned}$$

uniformly. Let $\bar{f} : \mathbb{T}^3 \to \mathbb{T}^3$ stand for the diffeomorphism $\bar{f} := A^{-1} \circ \bar{L} \circ \check{f} \circ \bar{L}^{-1} \circ A$. It is easy to see that

$$\frac{1}{n^\tau} D\bar{f}^n \to A^{-1} \cdot \bar{L} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d & 0 & 0 \end{bmatrix} \cdot \bar{L}^{-1} \cdot A$$

uniformly and that $\bar{f}$ and f are conjugate via the area–preserving C^2–diffeomorphism $A^{-1} \circ \bar{L} \circ \rho \circ A : \mathbb{T}^3 \to \mathbb{T}^3$. An application of Lemma 3.3 for $\bar{f}$ proves the claim.

Case 2. Suppose that $g = (h_1 \bar{a}^T + h_2 \bar{b}^T)\bar{c}$, where $\bar{a}$ and $\bar{b}$ are orthogonal to $\bar{c}$ and the determinant of the matrix $A^{-1} = [\, \bar{c}^T \;\; \bar{a}^T \;\; \bar{b}^T \,]$ equals 1. Let $\hat{f} : \mathbb{T}^3_A \to \mathbb{T}^3_A$ be given by $\hat{f} := A \circ f \circ A^{-1}$. Then

$$\hat{g} = A \cdot (h_1 \bar{a}^T + h_2 \bar{b}^T)\bar{c} \cdot A^{-1} = \begin{bmatrix} 0 \\ \hat{h}_1 \\ \hat{h}_2 \end{bmatrix} [\, 1 \;\; 0 \;\; 0 \,],$$

where $\hat{h}_i(\bar{x}) := h_i(A^{-1}\bar{x})$ for $i = 1, 2$. From (3.13) we get

$$\begin{bmatrix} 0 \\ \hat{h}_1(\bar{x}) \\ \hat{h}_2(\bar{x}) \end{bmatrix} [\, 1 \;\; 0 \;\; 0 \,] = \begin{bmatrix} 0 \\ \hat{h}_1(\hat{f}\bar{x}) \\ \hat{h}_2(\hat{f}\bar{x}) \end{bmatrix} [\, 1 \;\; 0 \;\; 0 \,] D\hat{f}(\bar{x})$$

for all $\bar{x} \in \mathbb{T}^3_A$. Consequently

$$\frac{\partial}{\partial x_1}\hat{f}_1(\bar{x})\hat{h}_i(\hat{f}\bar{x}) = \hat{h}_i(\bar{x}), \quad \frac{\partial}{\partial x_2}\hat{f}_1(\bar{x}) = \frac{\partial}{\partial x_3}\hat{f}_1(\bar{x}) = 0 \quad \text{and} \quad \frac{\partial}{\partial x_1}\hat{f}_1(\bar{x}) \neq 0$$

for all $\bar{x} \in \mathbb{T}^3_A$ and $i = 1, 2$. Now observe that $\hat{h}_1, \hat{h}_2$ are linearly dependent. Indeed, without loss of generality we can assume that $\hat{h}_2$ is $A\lambda^{\otimes 3}$–non–zero. Then $\hat{h}_2(\bar{x}) \neq 0$ for a.e. $\bar{x} \in \mathbb{T}^3_A$, by the ergodicity of $\hat{f}$. Therefore the measurable function $\hat{h}_1/\hat{h}_2 : \mathbb{T}^3_A \to \mathbb{R}$ is $\hat{f}$–invariant. Hence there is a real constant c such that $\hat{h}_1(\bar{x}) = c\hat{h}_2(\bar{x})$ for a.e. $\bar{x} \in \mathbb{T}^3_A$, by ergodicity. Consequently, $h_1 = ch_2$, which reduces the consideration to Case 1, and the proof is complete. □

4. 4–**dimensional case.** In this section we indicate why there is no 4–dimensional analogue of classifications of area–preserving diffeomorphisms of polynomial growth of the derivative presented in previous sections. More precisely, we construct an ergodic area–preserving diffeomorphism of the 4–dimensional torus with linear uniform growth of the derivative which is not even metrically isomorphic to any 3–step skew product, i.e. to any automorphism of $\mathbb{T}^4$ of the form

$$(x_1, x_2, x_3, x_4) \longmapsto (x_1 + \alpha, \varepsilon_1 x_2 + \beta(x_1), \varepsilon_2 x_3 + \gamma(x_1, x_2), \varepsilon_3 x_4 + \delta(x_1, x_2, x_3)),$$

where $\varepsilon_i = \pm 1$ for $i = 1, 2, 3$. Before we pass to the construction we should mention area–preserving diffeomorphisms of the 2–torus with a sublinear growth of the derivative. We say that a C^1–diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ has *sublinear growth of the derivative* if the sequence $\{Df^n/n\}$ tends uniformly to zero.

Suppose that $f : \mathbb{T}^2 \to \mathbb{T}^2$ is an area–preserving weakly mixing C^∞–diffeomorphism with sublinear growth of the derivative. The examples of such diffeomorphisms will be given later. Let $T_\varphi : \mathbb{T}^2 \to \mathbb{T}^2$ be an Anzai skew product of an ergodic rotation $Tx = x + \alpha$ on the circle and a C^∞–function $\varphi : \mathbb{T} \to \mathbb{T}$ with non–zero topological degree.

Theorem 4.1. *The product diffeomorphism $f \times T_\varphi : \mathbb{T}^4 \to \mathbb{T}^4$ is ergodic and has linear uniform growth of the derivative. Moreover, it is not metrically isomorphic to any 3–step skew product.*

Proof. The former claim of the theorem is obvious. Now suppose, contrary to latter claim, that $f \times T_\varphi$ is metrically isomorphic to a 3–step skew product. Then $f \times T_\varphi$ is measure theoretically distal (has generalized discrete spectrum in the terminology of [17]). However, $f \times T_\varphi$ has a weakly mixing factor, which contradicts the fact that measure theoretically distal are disjoint from all weakly mixing dynamical systems (see [7]). □

In the remainder of this section we present two examples of area–preserving weakly mixing diffeomorphisms with sublinear growth of the derivative.

Given $\alpha \in \mathbb{T}$ and $\beta : \mathbb{T} \to \mathbb{R}$ we denote by $T_{\alpha,\beta} : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ the skew product $T_{\alpha,\beta}(x_1, x_2) = (x_1 + \alpha, x_2 + \beta(x_1))$. Let $a \in \mathbb{T}$ be an irrational number and let $b : \mathbb{T} \to \mathbb{R}$ be a positive C^∞–function. By Lemma 2 in [3] and Theorem 1 in [12], the special flow $\sigma^t_{a,b}$ built over the rotation by a and under the function b is C^∞–conjugate to a Hamiltonian C^∞–flow φ^t which has no fixed point on the torus. Therefore there exists a C^∞–diffeomorphism $\rho : M_{a,b} \to \mathbb{T}^2$ such that $\varphi^t = \rho \circ \sigma^t_{a,b} \circ \rho^{-1}$ and there exists C^∞–function $\xi : \mathbb{R}^2 \to \mathbb{R}$ such that $D\xi$ is

$\mathbb{Z}^2$–periodic, non–zero at each point and

$$\frac{d}{dt}\varphi^t(\bar{x}) = \begin{bmatrix} \xi_{x_2}(\varphi^t(\bar{x})) \\ -\xi_{x_1}(\varphi^t(\bar{x})) \end{bmatrix}.$$

We will identify ρ with a diffeomorphism $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\begin{aligned} \rho(x_1+1, x_2) &= \rho(x_1, x_2) + (N_{11}, N_{12}), \\ \rho(x_1+a, x_2 - b(x_1)) &= \rho(x_1, x_2) + (N_{21}, N_{22}) \end{aligned}$$

for any $(x_1, x_2) \in \mathbb{R}^2$, where $N \in GL_2(\mathbb{Z})$. Then

$$D\rho(x_1+1, x_2) = D\rho(x_1, x_2) \tag{4.28}$$

$$D\rho(T^n_{a,-b}(x_1, x_2)) \begin{bmatrix} 1 & 0 \\ -Db^{(n)}(x_1) & 1 \end{bmatrix} = D\rho(x_1, x_2) \tag{4.29}$$

for any integer n.

Let $T_{\alpha,\beta} : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ be a skew product commuting with $T_{a,-b}$, where $\beta : \mathbb{T} \to \mathbb{R}$ is of class C^∞. Then $T_{\alpha,\beta}$ can be treated as a C^∞–diffeomorphism of $M_{a,b}$. Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ stand for the area–preserving C^∞–diffeomorphism $f := \rho \circ T_{\alpha,\beta} \circ \rho^{-1}$.

Lemma 4.2. *The diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ has sublinear growth of the derivative.*

Proof. Since

$$Df^n(\bar{x}) = D\rho(T^n_{\alpha,\beta} \circ \rho^{-1}(\bar{x})) \begin{bmatrix} 1 & 0 \\ D\beta^{(n)}(\rho_1^{-1}(\bar{x})) & 1 \end{bmatrix} D\rho^{-1}(\bar{x}),$$

it suffices to show that

$$\frac{1}{n} D\rho(T^n_{\alpha,\beta}(x_1, x_2)) \begin{bmatrix} 1 & 0 \\ D\beta^{(n)}(x_1) & 1 \end{bmatrix} \to 0$$

uniformly on the set $M' = \{(x_1, x_2) : x_1 \in \mathbb{R}, 0 \le x_2 \le b(x_1)\}$. Given $(x_1, x_2) \in \mathbb{R}^2$ let us denote by $n(x_1, x_2)$ the unique integer such that $T^{n(x_1,x_2)}_{a,-b}(x_1, x_2) \in M'$, i.e. $b^{(n(x_1,x_2))}(x_1) \le x_2 \le b^{(n(x_1,x_2)+1)}(x_1)$. Let c, C be positive constants such that $0 < c \le b(x) \le C$ for every $x \in \mathbb{T}$. Then

$$c|n(x_1, x_2)| \le |x_2| \le C|n(x_1, x_2)| + C.$$

Since

$$\begin{aligned} &\frac{1}{n} D\rho(T^n_{\alpha,\beta}(x_1, x_2)) \begin{bmatrix} 1 & 0 \\ D\beta^{(n)}(x_1) & 1 \end{bmatrix} \\ &= D\rho(T^{n(T^n_{\alpha,\beta}(x_1,x_2))}_{a,-b}(T^n_{\alpha,\beta}(x_1, x_2))) \times \\ &\quad \frac{1}{n} \begin{bmatrix} 1 & 0 \\ -Db^{(n(T^n_{\alpha,\beta}(x_1,x_2)))}(x_1 + n\alpha) + D\beta^{(n)}(x_1) & 1 \end{bmatrix} \end{aligned}$$

(by (4.29)), $D\rho$ is bounded on M' (by (4.28)) and $n^{-1}D\beta^{(n)}$ tends uniformly to zero, it suffices to show that

$$\frac{1}{n} Db^{(n(T^n_{\alpha,\beta}(x_1,x_2)))}(x_1 + n\alpha) \to 0$$

uniformly on M'. To prove this, observe that

$$|n(T^n_{\alpha,\beta}(x_1, x_2))| \le c^{-1}|x_2 + \beta^{(n)}(x_1)| \le k_1 + k_2 n,$$

for any natural n and every $(x_1, x_2) \in M'$, where $k_1 = C/c$ and $k_2 = \|\beta\|_\infty/c$. Fix $\varepsilon > 0$. Let n_0 be a natural number such that $|n| \geq n_0$ implies

$$\frac{1}{|n|}\|Db^{(n)}\|_\infty < \varepsilon/2k_2 \quad \text{and} \quad k_1 + k_2 n \leq 2k_2$$

for any integer n. Assume that n is a natural number such that $n \geq \|b\|_{C^1} n_0/\varepsilon$. Let $(x_1, x_2) \in M'$. If $|n(T^n_{\alpha,\beta}(x_1, x_2))|\|b\|_{C^1}/n < \varepsilon$, then

$$|\frac{1}{n}Db^{(n(T^n_{\alpha,\beta}(x_1,x_2)))}(x_1 + n\alpha)| \leq \frac{|n(T^n_{\alpha,\beta}(x_1, x_2))|}{n}\|b\|_{C^1} < \varepsilon.$$

Otherwise, $|n(T^n_{\alpha,\beta}(x_1, x_2))| \geq \varepsilon n/\|b\|_{C^1} \geq n_0$. Then

$$\begin{aligned} &|\frac{1}{n}Db^{(n(T^n_{\alpha,\beta}(x_1,x_2)))}(x_1 + n\alpha)| \\ &\quad \leq \quad \frac{|n(T^n_{\alpha,\beta}(x_1, x_2))|}{n}\frac{1}{|n(T^n_{\alpha,\beta}(x_1, x_2))|}\|Db^{(n(T^n_{\alpha,\beta}(x_1,x_2)))}\|_\infty \\ &\quad < \quad \frac{k_1 + k_2 n}{n}\frac{\varepsilon}{2k_2} \leq \varepsilon, \end{aligned}$$

which completes the proof. □

Proposition 4.3. *(see* [1]*) For every C^2–function $\beta : \mathbb{T} \to \mathbb{R}$ with zero mean, which is not a trigonometric polynomial there exists a dense G_δ set of irrational numbers $\alpha \in \mathbb{T}$ such that the corresponding skew product $T_{\alpha,\beta} : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ is ergodic.*

>From the proof of the Main Theorem in [16] and the nature of the weak mixing property, we have the following:

Proposition 4.4. *For every positive real analytic function $b : \mathbb{T} \to \mathbb{R}$ which is not a trigonometric polynomial there exists a dense G_δ set of irrational numbers $a \in \mathbb{T}$ such that the corresponding special flow $\sigma^t_{a,b}$ is weakly mixing.*

Example 4.1. Suppose that $\sigma^t_{a,b}$ is a weakly mixing special flow whose roof function is real analytic. Let φ^t be a Hamiltonian flow on $\mathbb{T}^2$ which is C^∞–conjugate to the special flow $\sigma^t_{a,b}$. Then the area–preserving diffeomorphism $\varphi^1 : \mathbb{T}^2 \to \mathbb{T}^2$ is weakly mixing and has sublinear growth of the derivative, by Lemma 4.2.

Example 4.2. By Propositions 4.3 and 4.4, there exist a C^∞–function $\beta : \mathbb{T} \to \mathbb{R}$ with zero mean and an irrational numbers $\alpha \in \mathbb{T}$ such that the corresponding skew product $T_{\alpha,\beta} : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ is ergodic and there is no real $r \neq 0$ for which there exist $c \in \mathbb{T}$ and a measurable function $c_r : \mathbb{T} \to \mathbb{T}$ satisfying

$$c_r(x + \alpha) \cdot e^{2\pi i r\beta(x)} = c \cdot c_r(x).$$

Using a standard construction we can find in the weak closer of $\{T^n_{\alpha,\beta} : n \in \mathbb{Z}\}$ a skew product T_{a,b_1} such that a is an irrational number with $a \neq n\alpha$ for all $n \in \mathbb{Z}$ and $b_1 : \mathbb{T} \to \mathbb{R}$ is a C^∞–function. Let us consider the special flow $\sigma^t_{a,b}$ on $M_{a,b}$, where $b = -b_1 + \|b_1\|_\infty + 1$. Since $T_{\alpha,\beta}$ commutes with $T_{a,-b}$, it can be treated as a C^∞–diffeomorphism of $M_{a,b}$. Moreover, $T_{\alpha,\beta} : M_{a,b} \to M_{a,b}$ is ergodic, by the ergodicity of $T_{\alpha,\beta} : \mathbb{T}\times\mathbb{R} \to \mathbb{T}\times\mathbb{R}$. It is quite easy to prove that $T_{\alpha,\beta} : M_{a,b} \to M_{a,b}$ is also weakly mixing (see [6]). Let φ^t be a Hamiltonian flow on $\mathbb{T}^2$ which is C^∞–conjugate to the special flow $\sigma^t_{a,b}$, via a C^∞–diffeomorphism $\rho : M_{a,b} \to \mathbb{T}^2$. Then the area–preserving C^∞–diffeomorphism $\rho \circ T_{\alpha,\beta} \circ \rho^{-1}$ of $\mathbb{T}^2$ is weakly mixing and has sublinear growth of the derivative, by Lemma 4.2.

Appendix A.

Theorem A.1. *Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be an area–preserving C^2–diffeomorphism. Suppose that there exist an irrational number α and a C^2–function $\xi : \mathbb{T}^2 \to \mathbb{T}$ such that*

$$D\xi(\bar{x}) \neq 0 \text{ for any } \bar{x} \in \mathbb{T}^2, \tag{A.30}$$

$$\xi \circ f = \xi + \alpha. \tag{A.31}$$

Then there exist an area–preserving C^2–diffeomorphism $\psi : \mathbb{T}^2 \to \mathbb{T}^2$, $k \in \mathbb{N}$, $c \in \mathbb{R}$ and a C^2–cocycle $\varphi : \mathbb{T} \to \mathbb{T}$ such that $\xi \circ \psi(x_1, x_2) = kx_1 + c$ and

$$\psi^{-1} \circ f \circ \psi(x_1, x_2) = (x_1 + \alpha, \varepsilon x_2 + \varphi(x_1)),$$

where $\varepsilon = \det Df$.

Proof. By (A.30), ξ is a submersion of $\mathbb{T}^2$ to $\mathbb{T}$ and therefore defines a fibration with the circle as a fiber. Moreover, the cohomology class defined by the closed form $D\xi$ is $p_1\,dx_1 + p_2\,dx_2$, where p_1, p_2 are integers such that $(p_1, p_2) \neq (0, 0)$. By taking $\xi/\gcd(p_1, p_2)$ instead of ξ, we can assume that p_1 and p_2 are relatively prime. Let us consider the symplectic vector field X associated to $D\xi$ by the symplectic form $dx_1 \wedge dx_2$. Its orbits are the levels curves of ξ. Consider now the symplectic vector field X' associated to $D(\xi \circ f)$. The fact that f is a canonical change of coordinates (f preserves the area) implies that the flows of X is conjugate via f to the flow of X' (or to the flow reversed in the time). Therefore (A.31) asserts that the level curves $\xi^{-1}(c)$ and $\xi^{-1}(c+\alpha)$ are periodic curves of X with the same period. Consequently, by irrationality of α, one remarks that the level curves of ξ all have the same period τ. By taking a closed curve transverse to the foliation, parametrized by the value of ξ, and then using the flow of X, one gets a natural diffeomorphism $\mathbb{T} \times \mathbb{R}/\tau\mathbb{Z} \ni (s, t) \mapsto \psi(s, t) \in \mathbb{T}^2$. Then $\psi^*(dx_1 \wedge dx_2) = ds \wedge dt$ and therefore $\tau = 1$. One deduces then that ψ satisfies the asked conditions. □

Appendix B. The proofs of the following lemmas are straightforward and can be found in [6].

Lemma B.1. *Let $\bar{c} \in \mathbb{R}^3$ be a non–zero vector and let $h : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Assume that $\bar{a}, \bar{b} \in \mathbb{R}^3$ are linearly independent vectors orthogonal to $\bar{c}$. Suppose that there exists a vector $\bar{d} \in \mathbb{R}^3$ such that*

$$h(x_1 + \bar{a}\bar{m}^T, x_2 + \bar{b}\bar{m}^T) = h(x_1, x_2) + \bar{d}\bar{m}^T$$

for all $\bar{m} \in \mathbb{Z}^3$. Then there exist $k_1, k_2 \in \mathbb{R}$ such that $\bar{d} = k_1\bar{a} + k_2\bar{b}$ and the function $\tilde{h}(x_1, x_2) = h(x_1, x_2) - k_1x_1 - k_2x_2$ is $(\bar{a}\bar{m}^T, \bar{b}\bar{m}^T)$–periodic for all $\bar{m} \in \mathbb{Z}^3$. Moreover,

- *if* rank $G(\bar{c})=0$*, then $\tilde{h}$ is constant;*
- *if* rank $G(\bar{c})=1$*, then there exist $l_1, l_2 \in \mathbb{R}$ and a continuous function $\rho : \mathbb{T} \to \mathbb{R}$ such that $\tilde{h}(x_1, x_2) = \rho(l_1x_1 + l_2x_2)$ and $l_1\bar{a} + l_2\bar{b} \in \mathbb{Z}^3$ generates $G(\bar{c})$;*
- *if* rank $G(\bar{c})=2$*, then $\bar{c} \in c\mathbb{Z}^3$ where $c \neq 0$.* □

Lemma B.2. *Let $\bar{c} \in \mathbb{Z}^3$ be a non–zero vector. Then for any pair of generators $\bar{a}, \bar{b} \in \mathbb{Z}^3$ of $G(\bar{c})$ we have $\Lambda(\bar{a}, \bar{b}) = \{(\bar{a}\bar{m}^T, \bar{b}\bar{m}^T) \in \mathbb{Z}^2 : \bar{m} \in \mathbb{Z}^3\} = \mathbb{Z}^2$.* □

Acknowledgement. I would like to thank the referee for some remarks that improved the presentation.

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Received August 2002; revised July 2003; final version February 2004.

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