Diffeomorphisms with polynomial growth of the derivative

(Selfreport)

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This selfreport consists of two parts. The dissertation entitled "Diffeomorphisms with polynomial growth of the derivative" is presented in the first part. The second part is devoted to the results the author has obtained after being awarded the degree of Ph.D. These results were not included in the dissertation.

Contents

A Dissertation

A.1 Subject matter of the dissertation

The habilitation thesis entitled

"Diffeomorphisms with polynomial growth of the derivative"

includes results published in the following articles:

[R1] *Linear growth of the derivative for measure–preserving diffeomorphisms*, item [17] in the list of the quoted literature,

[R2] *Measure–preserving diffeomorphisms of the torus*, item [19],

[R3] *On diffeomorphisms with polynomial growth of the derivative on surfaces*, item [20],

[R4] *Polynomial growth of the derivative for diffeomorphisms on tori*, item [22],

[R5] *On cocycles with values in the group* SU(2), item [18],

[R6] *On the degree of cocycles with values in the group* SU(2), item [21].

In the further part of the selfreport, we will use the denotations [R1],...,[R6] to refer to the above articles.

A.2 Introduction

Automorphisms of standard probability Borel space $(X, \mathcal{B}, \mu)$, i.e. measurable automorphisms preserving measure μ, are the fundamental object of research in ergodic theory. Space $(X, \mathcal{B}, \mu)$ is then treated as a state space (a phase space), while the automorphism T reflects the changes of states in time, i.e. it describes the dynamics of the phenomena. Therefore, the system $(X, \mathcal{B}, \mu, T)$ (or simply T for short) is often called a dynamical system. One of the fundamental questions of ergodic theory is whether two given dynamical systems have the same dynamics, i.e. whether they are isomorphic? We say that the two dynamical systems $(X_1, \mathcal{B}_1, \mu_1, T_1)$ and $(X_2, \mathcal{B}_2, \mu_2, T_2)$ are metrically isomorphic if there exists a measurable isomorphism $S : (X_1, \mathcal{B}_1, \mu_1) \to (X_2, \mathcal{B}_2, \mu_2)$, such that $S_*(\mu_1) = \mu_2$ (the image of measure μ_1 is equal to μ_2 via S) and $S \circ T_1 = T_2 \circ S$.

Ergodic theory also deals with the examination of such asymptotic properties of the sequence of iterations $\{T^n\}_{n\in\mathbb{N}}$ of automorphism T as: ergodicity, weak mixing, strong mixing, etc.

Definition 1. Automorphism $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is

- *ergodic* if
$$\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k \to \int_X f\, d\mu \quad \text{w } L^2(X, \mathcal{B}, \mu)$$
for any $f \in L^2(X, \mathcal{B}, \mu)$;

- *weakly mixing* if
$$\frac{1}{n}\sum_{k=0}^{n-1} \left| \int_X f(T^k x)g(x)\, d\mu(x) - \int_X f(x)\, d\mu(x) \int_X g(x)\, d\mu(x) \right| \to 0$$
for any $f, g \in L^2(X, \mathcal{B}, \mu)$;

- *strongly mixing* (or *mixing* for short) if
$$\int_X f(T^n x)g(x)\, d\mu(x) \to \int_X f(x)\, d\mu(x) \int_X g(x)\, d\mu(x)$$
for any $f, g \in L^2(X, \mathcal{B}, \mu)$.

Some of the more subtle properties of automorphisms come from the spectral theory of unitary operators. Every measure–preserving automorphism $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is associated with its Koopman operator on $U_T : L^2(X, \mathcal{B}, \mu) \to L^2(X, \mathcal{B}, \mu)$ given by the formula:

$$(U_T(f))(x) = f(Tx)$$

for any $f \in L^2(X, \mathcal{B}, \mu)$ (instead of space $L^2(X, \mathcal{B}, \mu)$, Koopman operator is often considered on its subspace $L^2_0(X, \mathcal{B}, \mu)$ of functions with zero integral).

Let us make a general assumption that U is a unitary operator of some separable Hilbert space H. For any $f \in H$, we denote by $\mathbb{Z}(f)$ the cyclic space generated by f, i.e. the smallest U–invariant closed subspace of H containing f, while by σ_f we denote spectral measure of f, i.e. Borel measure on the circle $\mathbb{T}$ determined by the equations

$$\langle U^n f, f\rangle = \hat{\sigma}_f(n) = \int_{\mathbb{T}} e^{2\pi i n x}\, d\sigma_f(x)$$

for all $n \in \mathbb{Z}$. Then, the spectral theorem says that there exists a sequence of elements $\{f_n\}_{n\in\mathbb{N}}$ of H such that

$$H = \bigoplus_{n=1}^{\infty} \mathbb{Z}(f_n) \quad \text{oraz} \quad \sigma_{f_1} \gg \sigma_{f_2} \gg \dots$$

Moreover, the sequence $\{\sigma_{f_n}\}_{n\in\mathbb{N}}$ is uniquely determined modulo equivalent relation of the measures. Then, the spectral type σ_U of the measure σ_{f_1} (the class of the equivalence relations of the measures) is called a maximal spectral type of the operator U. Here are some of spectral properties examined in ergodic theory:

- the operator U has Lebesgue spectrum (continuous, singular or discrete) if U is a type of Lebesgue measure (continuous, singular or discrete);
- the operator U is mixing if σ_U is a type of Rajchman measure, i.e. $\hat{\sigma}_f(n) \to 0$ for any $f \in H$;
- the operator U has homogeneous spectrum if $\sigma_{f_n} \equiv \sigma_{f_1}$ lub $\sigma_{f_n} \equiv 0$ for any natural n;
- the spectrum of the operator U has Lebesgue component of infinite multiplicity if $\lambda \ll \sigma_{f_n}$ for all $n \in \mathbb{N}$, or equivalently, if there exists a sequence $\{g_n\}_{n\in\mathbb{N}}$ such that $U^k g_n \perp U^l g_m$ when $m \neq n$ or $k \neq l$.

Of course, ergodicity as well as weak or strong mixing of automorphisms are also spectral properties, because the automorphism T is ergodic iff 1 is not an eigenvalue of the Koopman operator $U_T : L_0^2(X, \mathcal{B}, \mu) \to L_0^2(X, \mathcal{B}, \mu)$, T is weakly mixing iff $\sigma_T := \sigma_{U_T}$ is a type of continuous measure and T is strongly mixing iff σ_T is a type of Rajchman measure.

In physical considerations, the state space, in addition to the measure–theoretical structure, is often equipped with a topological structure or simply with a differential structure (especially in classical mechanics). In that case, the automorphism describing dynamics usually preserves a given structure, too. Then, the situation is the following: the dynamics in the state space M (M is a differential manifold here) are governed by the laws described by an autonomous differential equation of the following form

$$\frac{dx}{dt} = X(x), \tag{1}$$

where $X : M \to TM$ is a tangent vector field. Equation (1) is associated with flow $\{\psi_t\}_{t\in\mathbb{R}}$, i.e. a smooth mapping $\mathbb{R}\times M \ni (t,x) \mapsto \psi_t x \in M$ fulfilling the conditions

$$\begin{aligned} \psi_{t_1+t_2}(x) &= \psi_{t_1}(\psi_{t_2}(x)) \text{ for any } t_1, t_2 \in \mathbb{R}, \\ \psi_0(x) &= x, \end{aligned}$$

determined by the following solutions of the equation (1):

$$\begin{cases} \frac{d}{dt}\psi_t(x) &= X(\psi_t x) \\ \psi_0(x) &= x \end{cases}$$

for any $t \in \mathbb{R}$ and $x \in M$. Then, we can consider discretizations of the flow $\{\psi_t\}_{t\in\mathbb{R}}$, i.e. diffeomorphisms $\psi_t : M \to M$ for $t \neq 0$. Such diffeomorphisms are called times of the flow. Another approach consists in finding a submanifold $N \subset M$ of codimension 1, which is transversal to the orbits of the flow $\{\psi_t\}_{t\in\mathbb{R}}$ and such that each point $x \in N$ returns to N after the positive time $\tau_+(x) > 0$ and the negative time $\tau_-(x) < 0$. Then, the diffeomorphism $f : N \to N$ given by the formula $f(x) = \psi_{\tau_+(x)}(x)$ and known as *Poincaré transformation* is a significant element of the research in the understanding of the dynamics of flow $\{\psi_t\}_{t\in\mathbb{R}}$.

In the context of smooth dynamical systems, the concept of isomorphism of two systems is much stronger. We will say that two C^r–diffeomorphisms $f_1 : M_1 \to M_1$, $f_2 : M_2 \to M_2$ ($r \in \mathbb{N}\cup\{\infty\}$) are C^r–conjugated, when there exists a C^r–isomorphism $g : M_1 \to M_2$ such that $g \circ f_1 = f_2 \circ g$.

Sometimes, as in Hamiltonian dynamics for example, there exist natural smooth measures on the state space, which are invariant with respect to the action of a given diffeomorphism. The dissertation deals with such a situation. Let M (a state space) be a compact, connected and finite dimension differential C^∞–manifold and let $(U_i, \varphi_i)_{i\in I}$ be a certain atlas of this manifold. We say that probability Borel measure on M is a positive C^∞–measure, if for any $i \in I$ the image of the measure $\mu|_{U_i}$ is equivalent to the Lebesgue measure on $\varphi_i(U_i)$ via the mapping φ_i, and its density is a positive function of C^∞ class. In this dissertation, the dynamical properties of diffeomorphisms of manifold preserving certain positive probability C^∞–measures are being examined from the point of view of ergodic theory.

The linearization or the examination of linear transformations $Df^n(x) : T_xM \to T_{f^nx}M$ is one of the basic methods of the examination of the dynamical properties of diffeomorphisms $f : M \to M$. The asymptotic behaviour of such linearizations is described from the point of view of ergodic theory by the following fundamental Oseledec's theorem:

Theorem 1. *Let f be a C^1–diffeomorphism of a compact differential manifold M. Then, there exists a Borel full measure subset $M' \subset M$ (measure 1 for any f–invariant probability measure) with the following properties: If $x \in M'$, then there exists a sequence of numbers $\lambda_1(x) > \lambda_2(x) > \ldots > \lambda_m(x)$ and a decomposition*

$$T_xM = E_1(x) \oplus \ldots \oplus E_m(x)$$

such that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|Df^n(x)u\|_{f^n x} = \lambda_j(x)$$

for any $0 \neq u \in E_j(x)$ and any $1 \leq j \leq m$. Moreover, $E_j(fx) = Df(x)E_j(x)$ and $\lambda_j(Tx) = \lambda_j(x)$ for any $1 \leq j \leq m$.

The norm in Oseledec's theorem comes from any established Riemannian form on M. Since all such forms are equivalent, the numbers $\lambda_j(x)$, $j = 1, \ldots, m$ do not depend on the choice of the form. The numbers $\lambda_j(x)$, $j = 1, \ldots, m$ are called Lyapunov exponents. Then, $E^u(x) = \bigoplus_{i:\lambda_i(x)>0} E_i(x)$ is a subspace of unstable directions, while $E^s(x) = \bigoplus_{i:\lambda_i(x)<0} E_i(x)$ is a subspace of stable directions in T_xM.

Taking the linearization of diffeomorphisms into consideration makes it possible to introduce a natural distinction between *elliptic*, *parabolic* and *hyperbolic* diffeomorphisms. This distinction, discussed in more detail in [28], is not fully formal and, briefly (but in a very informal way), it can be presented in the following way:

- Linearizations of hyperbolic diffeomorphisms possess only eigenvalues with absolute values different from unity, which means that the sequence $\{Df^n\}_{n\in\mathbb{N}}$ grows exponentially in some directions, and in some directions it also decreases exponentially. In more formal terms, it means that all Lyapunov exponents are non zero ones.

- All the eigenvalues of the linearization of elliptic diffeomorphisms have modules equal to one and do not have non–trivial Jordan blocks. Then, the sequence $\{\|Df^n\|\}_{n\in\mathbb{N}}$ does not grow too fast.

- All the eigenvalues of the linearization of parabolic diffeomorphisms have modules equal to one and possess non–trivial Jordan blocks. Then, the sequence $\{\|Df^n\|\}_{n\in\mathbb{N}}$ grows much faster than in the case of elliptic diffeomorphisms, but significantly slower than in the case of exponential growth.

Of course, not all diffeomorphisms can be classified in this way. However, a significant part of them belongs to one of those classes. In the further part of this chapter, we will present some examples of the most representative diffeomorphisms for each class and we will discuss their properties.

Anosov diffeomorphisms are the best known and the best examined kinds of hyperbolic diffeomorphisms. We say that $f : M \to M$ is an Anosov diffeomorphism if there exist constants $K > 0$, $0 < \lambda < 1$ such that the tangent space T_xM in every point decomposes into a direct sum of a stable subspace E^s_x and an unstable subspace E^u_x and at the same time, these subspaces fulfill the following conditions:

$$Df(x)E^s_x = E^s_{f(x)}, \qquad Df(x)E^u_x = E^u_{f(x)},$$

$$\|Df^n(x)|E^s_x\| \le K\lambda^n, \qquad \|Df^{-n}(x)|E^u_x\| \le K\lambda^n$$

for all $x \in M$ and $n \in \mathbb{N}$. Anosov diffeomorphisms have clear classification when a finite dimension torus is a state space. As it was proved by Manning in [44], any Anosov diffeomorphism of a torus is C^0–conjugate with a hyperbolic group automorphism of this torus. A similar theorem is true for infra-nil-manifold diffeomorphisms. So far, no general answer has been given to the question of on which manifolds Anosov diffeomorphisms exit. Up till now, this type of diffeomorphisms have been successfully constructed only on infra-nil-manifolds. The problem becomes much simpler if we narrow the scope of our interest down to two–dimensional manifolds, i.e. to surfaces. Then, as it was proved by Franks in [16], the two–dimensional torus is the only surface allowing Anosov diffeomorphisms.

Rotations on tori, which appear naturally in the theory of Hamiltonian systems, are the best known and the best examined class of elliptic diffeomorphisms. Let us consider a broader class $\mathrm{Diff}^r_0(\mathbb{T}^d)$ of C^r $(r \in \mathbb{N} \cup \{\infty\})$ diffeomorphisms of the d–dimensional torus $\mathbb{T}^d$, which are homotopic to identity. Let us assume that $f \in \mathrm{Diff}^r_0(\mathbb{T}^d)$ and μ is a probability f–invariant and ergodic measure. Then, the lift of the diffeomorphism f has the form of

$\tilde{f} = \mathrm{Id} + \varphi$, where $\varphi \in C^r(\mathbb{T}^d, \mathbb{R}^d)$. Moreover,

$$\frac{1}{n}(\tilde{f}^n(x) - x) = \frac{1}{n}\sum_{k=0}^{n-1} \varphi(f^k(x)) \to \mu(\varphi) = \int_{\mathbb{T}^d} \varphi \, d\mu$$

for μ–a.e. $x \in \mathbb{T}^d$. In this case, vector $\mu(\varphi)$ is called a vector of rotation for f, while the simplex

$$\rho(f) = \{\mu(\varphi) : \mu \text{ is a probability } f\text{–invariant measure}\}$$

is a set of rotation vectors for f. The set $\rho(f)$ is a dynamical invariant, which has a significant impact on whether diffeomorphism f is conjugate to the rotation. The asymptotic behaviour of the sequence $\{Df^n\}_{n\in\mathbb{N}}$ is another important factor influencing the properties of f.

Theorem 2 (see [52] and [29])**.** *Let $f \in \mathrm{Diff}^1_0(\mathbb{T}^d)$ and let the sequence $\{Df^n\}_{n\in\mathbb{N}}$ be uniformly bounded. Then, $\rho(f)$ is a one–element set and if the coordinates of the vector $\alpha \in \rho(f)$ are rationally independent (i.e. if $n_1\alpha_1 + \ldots + n_d\alpha_d = m$ for integer m, n_i, $i = 1, \ldots, d$, then $n_1 = n_2 = \ldots = n_d = 0$), then f is C^0–conjugate with the rotation by α. Moreover, if $f \in \mathrm{Diff}^r_0(\mathbb{T}^d)$ ($r \in \mathbb{N} \cup \{\infty\}$) and the sequence $\{Df^n\}_{n\in\mathbb{N}}$ is bounded in the C^r norm, then f is C^r–conjugate with the rotation.*

The times of horocycle flows on surfaces with a constant negative curvature and skew Anzai products are the best–known examples of parabolic diffeomorphism. Before we move on to discuss those examples, we will introduce several concepts regarding groups of matrices. Let $G \subset \mathrm{GL}(d, \mathbb{R})$ ($\mathrm{GL}(d, \mathbb{C})$) be a closed Lie matrix group. Let $\mathfrak{g}$ denote the Lie algebra of group the G. Then, the mapping $\exp : \mathfrak{g} \to G$ is locally inversible around $0 \in \mathfrak{g}$. Let $f : G \to G$ be a diffeomorphism. Now, we determine the derivative f at point $g \in G$ using the maps coming from the mapping $\exp(\,\cdot\,) \cdot g$, i.e. let us consider the mapping

$$\mathfrak{g} \ni X \longmapsto \exp^{-1}[f(\exp(X)g)f(g)^{-1}] \in \mathfrak{g},$$

which is well defined in a certain neighbourhood $0 \in \mathfrak{g}$. The derivative of this mapping at $0 \in \mathfrak{g}$ will be denoted by $L(f)(g) : \mathfrak{g} \to \mathfrak{g}$. Let us assume now that f is a left rotation on the group, i.e. $f(g) = ag$. Then, $L(f)(g)X = aXa^{-1}$. Next, let us assume that $\Gamma \subset G$ is a discrete co–compact subgroup of G. Then the homogeneous space G/Γ is a compact C^∞–manifold of the dimension $\dim(G)$. Let $f : G/\Gamma \to G/\Gamma$ be a left rotation with the form of $f(g\Gamma) = ag\Gamma$. Then, if we calculate the derivative f at point $g\Gamma \in G/\Gamma$ using local inverses of the mappings in the form of $\mathfrak{g} \ni X \mapsto \pi(\exp(X)g) \in G/\Gamma$, where $\pi : G \to G/\Gamma$ is a natural covering, we will obtain a linear transformation $L(f)(g\Gamma) : \mathfrak{g} \to \mathfrak{g}$ in the form $L(f)(g\Gamma)X = aXa^{-1}$.

Let

$$G = \mathrm{SL}(2,\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc = 1 \right\}$$

and let μ be a right–invariant and left-invariant Haar measure on $\mathrm{SL}(2,\mathbb{R})$. Let $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ be a discrete and co-compact subgroup. Since μ is invariant on the right-hand shifts, it can be considered as a Borel measure on the homogeneous space $M = \mathrm{SL}(2,\mathbb{R})/\Gamma$. In this case, it is finite, thus, we can assume it its probabilistic after it has been normalized. Next, let us consider the flow $\{h_t\}_{t\in\mathbb{R}}$ on M in the form

$$h_t(g\Gamma) = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \cdot g\Gamma.$$

The flow of horocycles on any compact surface with a constant negative curvature is analytically conjugate with a certain flow of this type. Of course, the flow $\{h_t\}_{t\in\mathbb{R}}$ preserves the measure μ. Any flow of horocycles and consequently every non–zero time, i.e. diffeomorphism $h_t : (M,\mu) \to (M,\mu)$, $t \neq 0$, is mixing. Moreover, it has Lebesgue spectrum with infinite multiplicity. Every diffeomorphism h_t, $t \neq 0$ is parabolic, because

$$\begin{aligned} L(h_t^n)(x) \begin{bmatrix} a & b \\ c & -a \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} a - tnb & b \\ c + 2tna - t^2n^2b & -a + tnb \end{bmatrix}. \end{aligned}$$

In addition, the sequence of derivatives $\{L(h_t^n)\}_{n\in\mathbb{N}}$ has a square growth.

The next major example of parabolic diffeomorphisms are skew Anzai products on the two–dimensional torus, i.e. diffeomorphisms $T_\varphi : \mathbb{T}^2 \to \mathbb{T}^2$ with the form

$$T_\varphi(z,\omega) = (Tz, \varphi(z)\omega), \tag{2}$$

where $T : \mathbb{T} \to \mathbb{T}$ is an ergodic rotation and $\varphi : \mathbb{T} \to \mathbb{T}$ is an absolutely continuous mapping. Let us assume that the topological degree $d(\varphi)$ of the mapping φ is different from zero. Then, $\varphi(e^{2\pi ix}) = e^{2\pi i(d(\varphi)x+\tilde{\varphi}(x))}$, where $\tilde{\varphi} : \mathbb{T} \to \mathbb{R}$ is an absolutely continuous function. Moreover,

$$L\left(T_\varphi^n\right)(e^{2\pi ix}, \omega) = \left[\begin{array}{cc} 1 & 0 \\ d(\varphi)n + \sum_{k=0}^{n-1} \tilde{\varphi}'(T^k x) & 1 \end{array}\right].$$

On the basis of Birkhoff ergodic theorem

$$\frac{1}{n}\sum_{k=0}^{n-1} \tilde{\varphi}'(T^k x) \to \int_{\mathbb{T}} \tilde{\varphi}'(t)\, dt = 0$$

for a.e. $x \in \mathbb{T}$. Hence

$$\frac{1}{n} L\left(T_\varphi^n\right)(z, \omega) \to \left[\begin{array}{cc} 0 & 0 \\ d(\varphi) & 0 \end{array}\right] \tag{3}$$

for a.e. $(z, \omega) \in \mathbb{T}^2$. Let us notice that if φ belongs to C^1 class, then, due to the monoergodicity of the rotation T, the convergence in (3) is uniform. The operator U_{T_φ} on the subspace $L^2(dz) \subset L^2(\mathbb{T}^2)$ of the functions dependent only on the first coordinate is unitarily equivalent to the operator U_T, and thus it has pure discrete spectrum. The spectral properties T_φ on the orthogonal space $L^2(dz)^\perp$ were already examined by Anzai in paper [4]; however, the strongest results were obtained by Iwanik, Lemańczyk and Rudolph in paper [31]. If $\varphi : \mathbb{T} \to \mathbb{T}$ is an absolutely continuous function and $d(\varphi) \neq 0$ then U_{T_φ} is a mixing operator on $L^2(dz)^\perp$. If, in addition, the derivative φ' is of bounded variation, then U_{T_φ} has Lebesgue spectrum with infinite multiplicity on $L^2(dz)^\perp$.

This dissertation concentrates on the research of parabolic diffeomorphisms and, more precisely, diffeomorphisms with a polynomial growth of the derivative. Consequently, the following three problems are being considered:

- to provide a formal definition of the notion of polynomial growth of the derivative;
- for a given manifold, to classify ergodic diffeomorphisms with polynomial growth of the derivative (if they exist);
- to examine dynamical and spectral properties of such diffeomorphisms.

A.3 Diffeomorphisms on tori

In this chapter, the results presented in articles [R1], R2] and [R4] will be discussed. These papers introduce various definitions of the notion of polynomial growth of the derivative for diffeomorphisms of the simplest manifolds, such as tori. The papers also present certain results, which classify this type of ergodic diffeomorphisms.

By $\mathbb{T}^d$ we will denote the d–dimensional torus $\{(z_1,\ldots,z_d)\in\mathbb{C}^d : |z_1| = \ldots = |z_d| = 1\}$, which will be identified with the quotient group $\mathbb{R}^d/\mathbb{Z}^d$. λ_d will denote Lebesgue measure on $\mathbb{T}^d$. Let us assume that $f:\mathbb{T}^d\to\mathbb{T}^d$ is a diffeomorphism. Then, any of its lift $\tilde{f}:\mathbb{R}^d\to\mathbb{R}^d$ can be presented as

$$\tilde{f}(x) = Ax + \bar{f}(x),$$

where $A\in \mathrm{GL}(d,\mathbb{Z})$, i.e. A is an integer matrix such that $|\det(A)| = 1$ and $\bar{f}:\mathbb{T}^d\to\mathbb{R}^d$, i.e. $\bar{f}$ is a period function due to each coordinate, its period being one. In paper [R1], the following definition of τ–polynomial growth of the derivative.

Definition 2. A diffeomorphism $f:\mathbb{T}^d\to\mathbb{T}^d$ possesses *τ–polynomial growth of the derivative* ($\tau>0$) if

$$\frac{1}{n^\tau}D\tilde{f}(x)\to g(x) \quad \text{for } \lambda_d\text{–a.e. } x\in\mathbb{T}^d,$$

where $g:\mathbb{T}^d\to \mathrm{M}_{d\times d}(\mathbb{R})$ is a non–zero a.e. function, i.e. there exists a measurable subset $A\subset\mathbb{T}^d$ such that $\lambda_d(A)>0$ and $g(x)\neq 0$ for $x\in A$.

It is worth mentioning here that the original definition from paper [R1] refers to diffeomorphisms of any manifolds and unfortunately it is not correct. Its correct version was presented in paper [R3].

Of course, the skew Anzai products in the form (2) have linear growth of the derivative, if $d(\varphi)\neq 0$. The main result of paper [R1] says that there are no other area–preserving ergodic diffeomorphisms with a polynomial growth of the derivative on the two–dimensional torus.

Theorem 3 ([R1])**.** *If $f:(\mathbb{T}^2,\lambda_2)\to(\mathbb{T}^2,\lambda_2)$ is a measure–preserving ergodic C^2–diffeomorphism with τ–polynomial growth of the derivative, then and f is algebraically conjugate (via the group automorphism) with a certain skew Anzai product T_φ, where $T:\mathbb{T}\to\mathbb{T}$ is an ergodic rotation, while $\varphi:\mathbb{T}\to\mathbb{T}$ is a C^2–mapping with non-zero topological degree.*

The classification of diffeomorphisms (with polynomial growth of the derivative) of the three–dimensional torus is the next step. Of course, in this case, apart from linear growth, one should also expect diffeomorphisms with square growth of the derivative. Indeed, any two–step skew product with the form

$$f(x,y,z) = (x+\alpha, y+\beta(x), z+\gamma(x,y)),$$

where $\alpha \in \mathbb{T}$ is an irrational number, $\beta : \mathbb{T} \to \mathbb{T}$ is a mapping of C^1 class such that $d(\beta) \neq 0$ and $\gamma : \mathbb{T}^2 \to \mathbb{T}$ is a mapping of C^1 class such that $d_2(\gamma) = d(\gamma(x,\cdot)) \neq 0$, is ergodic and has square growth of the derivative, and more precisely

$$\frac{1}{n^2} D\tilde{f}^n \to \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d(\beta)d_2(\gamma)/2 & 0 & 0 \end{bmatrix}$$

uniformly. Roughly speaking, the main result of paper [R4] is that only linear or square growth of the derivative is possible on the three–dimensional torus; moreover, each ergodic diffeomorphism with such growth of the derivative is conjugate with a certain two–step skew product. This time, τ–polynomial growth of the derivative means that the sequence $\{Df^n/n^\tau\}_{n\in\mathbb{N}}$ converges uniformly towards a non–zero function g, which, in addition, is of C^1 class.

Theorem 4 ([R4]). *If $f : (\mathbb{T}^3, \lambda_3) \to (\mathbb{T}^3, \lambda_3)$ is a measure–preserving ergodic C^2–diffeomorphism with τ–polynomial growth of the derivative, then $\tau = 1$ or $\tau = 2$ and f is C^2–conjugate with a certain skew product in the form of*

$$\mathbb{T}^3 \ni (x,y,z) \mapsto (x+\alpha, \varepsilon y + \beta(x), z + \gamma(x,y)) \in \mathbb{T}^3,$$

where $\varepsilon = \det Df = \pm 1$.

The analysis of so–called random diffeomorphisms with polynomial growth of the derivative on the two–dimensional torus contained in [R4] is one of the steps needed to prove Theorem 4. Let T be a measure–preserving ergodic automorphism of a standard probability Borel space $(\Omega, \mathcal{F}, P)$. By $\mathcal{B}$, we denote σ–algebra of Borel sets on $\mathbb{T}^2$. Then, any measurable mapping

$$(\Omega \times \mathbb{T}^2, \mathcal{F} \otimes \mathcal{B}) \ni (\omega, x) \mapsto f_\omega(x) \in (\mathbb{T}^2, \mathcal{B})$$

such that $f_\omega : \mathbb{T}^2 \to \mathbb{T}^2$ is a C^r–diffeomorphism ($r \in \mathbb{N} \cup \{\infty\}$) for P–a.e. $\omega \in \Omega$, is called a *random C^r–diffeomorphism* on $\mathbb{T}^2$ over $(\Omega, \mathcal{F}, P, T)$. The random diffeomorphism built over T is associated with a skew product $T_f : \Omega \times \mathbb{T}^2 \to \Omega \times \mathbb{T}^2$ in the form of

$$T_f(\omega, x) = (T\omega, f_\omega(x)).$$

The probability measure μ on the space $(\Omega \times \mathbb{T}^2, \mathcal{F} \otimes \mathcal{B})$ is called an f–invariant measure if its projection on Ω is equal to P and μ is T_f–invariant, which equivalently means that $f_\omega \mu_\omega = \mu_{T\omega}$ for P–a.e. $\omega \in \Omega$, where $\mu_\omega, \omega \in \Omega$ is the disintegration of measure μ over P. Then, we say that the random diffeomorphism $\{f_\omega : \omega \in \Omega\}$ is ergodic, if the skew product $T_f : (\Omega \times \mathbb{T}^2, \mu) \to (\Omega \times \mathbb{T}^2, \mu)$ is ergodic.

Let us assume that measure μ is equivalent to $P \otimes \lambda_2$.

Definition 3. We say that the random diffeomorphism $\{f_\omega : \omega \in \Omega\}$ has *τ–polynomial growth of the derivative*, if

$$\frac{1}{n^\tau} D(f_{T^{n-1}\omega} \circ f_{T^{n-2}\omega} \circ \ldots \circ f_{T\omega} \circ f_\omega)(x) \to g(\omega, x)$$

both in space $L^1((\Omega \times \mathbb{T}^2, \mu), \mathrm{M}_{2\times 2}(\mathbb{R}))$ and for μ–a.e. $(\omega, x) \in \Omega \times \mathbb{T}^2$, and the function $g : \Omega \times \mathbb{T}^2 \to \mathrm{M}_{2\times 2}(\mathbb{R})$ is μ–non–zero one.

Random skew Anzai products in the form

$$f_\omega(x, y) = (x + \alpha(\omega), y + \varphi(\omega, x))$$

are examples of random diffeomorphisms with linear growth of the derivative. If random rotation $(\omega, x) \mapsto (T\omega, x + \alpha(x))$ is ergodic, $D_x\varphi \in L^1(\Omega \times \mathbb{T}, P \otimes \lambda_1)$ and $\int_\Omega d(\varphi_\omega)\, dP(\omega) \neq 0$, then the random diffeomorphism $\{f_\omega\}_{\omega \in \Omega}$ is ergodic and it has linear growth of the derivative. The following result classifies some random diffeomorphisms with polynomial growth of the derivative and is a generalization of Theorem 3.

Theorem 5 ([R4]). *Let f be a random C^1–diffeomorphism on $\mathbb{T}^2$ over $(\Omega, \mathcal{F}, P, T)$ and let μ be an invariant ergodic measure for f equivalent to $P \otimes \lambda_2$ such that Radon–Nikodym derivatives $d\mu/d(P \otimes \lambda_2)$, $d(P \otimes \lambda_2)/d\mu$ are bounded. Then, if f has τ–polynomial growth of the derivative, then $\tau = 1$ and f is Lipschitz conjugate with some random Anzai skew product, i.e. there* *exists a random homeomorphism* $g : \Omega \times \mathbb{T}^2 \to \mathbb{T}^2$ *such that* $g_\omega, g_\omega^{-1} : \mathbb{T}^2 \to \mathbb{T}^2$ *are Lipschitz mappings for* P*–a.e.* $\omega \in \Omega$ *and*

$$\{g_{T\omega}^{-1} \circ f_\omega \circ g_\omega\}_{\omega \in \Omega}$$

is a random skew Anzai product.

Let us return to ordinary diffeomorphisms (diffeomorphisms devoid of randomness), and more precisely, to diffeomorphisms on the four-dimensional torus. In this case, it is not possible to classify ergodic diffeomorphisms with polynomial growth of the derivative in the same way as we did in the case of the three-dimensional torus. Not all ergodic diffeomorphisms on $\mathbb{T}^4$ with polynomial growth of the derivative are conjugate with multi–step skew products. In order to construct a counterexample, it is enough to find a measure–preserving diffeomorphism $h : (\mathbb{T}^2, \lambda_2) \to (\mathbb{T}^2, \lambda_2)$ which is weakly mixing and such that the sequence $\{Dh^n/n\}_{n\in\mathbb{N}}$ converges uniformly towards zero. Such diffeomorphisms, as it was proved in [R4], include for example non–zero times of certain weakly mixing Hamiltonian flows on $\mathbb{T}^2$. The existence of such flows was proven by Shklover in [53]. Then, the product diffeomorphism $T_\varphi \times h : \mathbb{T}^4 \to \mathbb{T}^4$, in which T_φ is an ergodic skew Anzai product such that $d(\varphi) \neq 0$, is an ergodic diffeomorphism with linear growth of the derivative. In [R4], by using certain theorems on the disjointness of dynamical systems proved by Furstenberg in [27], it was shown that $T_\varphi \times h$ is not conjugate with any multi–step skew, even in the metrical sense.

Of course, Definition 2 is not the only definition of polynomial growth of the derivative. In paper [R2], another approach to the problem was proposed. In this approach, a very restrictive assumption on the convergence of the sequence $\{Df^n/n^\tau\}_{n\in\mathbb{N}}$ is avoided. This time, a diffeomorphism $f : \mathbb{T}^d \to \mathbb{T}^d$ has τ–polynomial growth of the derivative, when there exist real constants c, C such that

$$0 < c \le \frac{\|Df^n(x)\|}{n^\tau} \le C \tag{4}$$

for any $n \in \mathbb{N}$ and $x \in \mathbb{T}^d$. Let us notice that the definition given above does not depends on the choice of the norm on $\mathrm{M}_{d\times d}(\mathbb{R})$ and it can be easily referred to the case of compact differentiable manifolds. Then, this norm will be an operator norm derived from a certain Riemannian form (all such forms are equivalent).

Let us assume that $f : (\mathbb{T}^2, \lambda_2) \to (\mathbb{T}^2, \lambda_2)$ is a measure–preserving diffeomorphism with linear growth of the derivative. Then, as it was proved in paper [R2], for any $x \in \mathbb{T}^2$, there exist two directions $u(x), v(x) \in \mathbb{S}^1 \subset \mathbb{R}^2$ such that

$$\lim_{n\to\infty} \frac{1}{n} \left(\|Df^n(x)u(x)\| - \|Df^n(x)\| \right) = 0,$$
$$\text{the sequence} \quad \{\|Df^n(x)v(x)\|\}_{n\in\mathbb{N}} \quad \text{is bounded.}$$

The direction $u(x)$ is a distant equivalent of the unstable direction, while $v(x)$ is an equivalent of the stable direction considered in hyperbolic dynamics. In the hyperbolic case, the stable and unstable directions (or rather subspaces) make it possible determine stable and unstable submanifolds (Hadamard–Perron theorem), which make the basis for further analysis. The parabolic case does not create such good conditions as the hyperbolic one and consequently we have to support ourselves with additional assumptions. The property that the sequence $\{Df^n/n\}_{n\in\mathbb{N}}$ is bounded in C^2 norm is such an additional assumption in paper [R2]. This assumption allows us to prove that the functions $u, v : \mathbb{T}^2 \to \mathbb{S}^1$ are of C^1 class. Then, it allows us to prove the main result of this paper:

Theorem 6 ([R2])**.** *Let $f : (\mathbb{T}^2, \lambda_2) \to (\mathbb{T}^2, \lambda_2)$ be a measure–preserving ergodic C^3–diffeomorphism. If f has linear growth of the derivative* (*i.e. inequality* (4) *occurs for* $\tau = 1$) *and the sequence $\{Df^n/n\}_{n\in\mathbb{N}}$ is bounded in the C^2 norm, then f is algebraically conjugate with a certain skew Anzai product T_φ such that $d(\varphi) \neq 0$.*

A.4 Polynomial growth of the derivative for diffeomorphisms of any differential manifolds

Let us assume that M is a compact connected and k–dimensional C^∞–manifold. Let $f : M \to M$ be a C^∞–diffeomorphism preserving a certain positive probability C^∞–measure μ on M. Then, it is possible to define the notion of polynomial growth of the derivative of f imitating the phenomenon described in Definition 2. This time, we observe polynomial growth for the derivative of the sequence $\{f^n\}_{n\in\mathbb{N}}$ "perceived" via charts of a certain atlas $\{U_i, \varphi_i\}_{i\in I}$ of the manifold M.

Definition 4. ([R3]) We say that the pair $(f, \{U_i, \varphi_i\}_{i\in I})$ has *τ–polynomial growth of the derivative*, when, for any $i, j \in I$, there exists a measurable function $A_{ji} : U_i \to \mathrm{M}_{k\times k}(\mathbb{R})$ μ–non–zero and such that for μ–almost every$x \in U_i$, if $\{n_l\}_{l\in\mathbb{N}}$ is a sequence of natural numbers divergent towards infinity and $f^{n_l}x \in U_j$ for $l \in \mathbb{N}$, then

$$\lim_{l\to\infty} \frac{1}{{n_l}^{\tau}} D(\varphi_j \circ f^{n_l} \circ \varphi_i^{-1})(\varphi_i(x)) = A_{ji}(x).$$

We will say that the diffeomorphism $f : (M, \mu) \to (M, \mu)$ has *τ–polynomial growth of the derivative*, when there exista an atlas on M suitable for f, i.e. f together with this atlas have suitable growth.

Let us note that suitable skew products on tori described in Chapter A.3 are diffeomorphisms with polynomial growth of the derivative in the sense of Definition 4. We obtain the adequate atlases from local inverses of the coverings.

As in the case of Anosov diffeomorphisms, the issue of determining the manifolds, on which ergodic diffeomorphisms exist with polynomial growth of the derivative, is significant. For dimensions greater than two, this problem seems to be extremely difficult. Already in the third dimension, in this class there is the Cartesian product of Klein bottle and the circle. Probably, in higher dimensions, other infra-nil-manifolds are also permissible, the two–dimensional case being very poor, which was proved in paper [R3].

Theorem 7 ([R3])**.** *Torus is the only compact and connected two–dimensional C^∞–surface, which permits the existence of an ergodic C^∞–diffeomorphism with polynomial growth of the derivative* (*which preserves a positive probability C^∞–measure*)*. Moreover, any diffeomorphism of this kind is C^∞–conjugate with a certain skew Anzai product $T_\varphi : \mathbb{T}^2 \to \mathbb{T}^2$, where $d(\varphi) \neq 0$.*

The proof of this theorem is based on the following general result.

Theorem 8 ([R3])**.** *Let M be a compact and connected C^∞–manifold and let μ be a positive probability C^∞–measure on M. If $f : (M, \mu) \to (M, \mu)$ is an ergodic C^∞–diffeomorphism with polynomial growth of the derivative, then there exists a C^∞–flow $\{\psi_t\}_{t\in\mathbb{R}}$ on M such that f commutes with $\{\psi_t\}_{t\in\mathbb{R}}$, i.e. $f \circ \psi_t = \psi_t \circ f$ for any $t \in \mathbb{R}$, and $\{\psi_t\}_{t\in\mathbb{R}}$ does not have any fixed points. In addition, $\{\psi_t\}_{t\in\mathbb{R}}$ preserves measure μ.*

Using Poincaré-Hopf theorem, we conclude that the Euler characteristic $\chi(M)$ of manifold M equals zero. If $\dim(M) = 2$, then M must be the torus or Klein bottle. Since, Klein bottle possesses a two–point covering provided by the torus, then the analysis of diffeomorphisms, when M is the torus, is the essence of the elimination of the case of Klein bottle. Then, it is easy to notice that the flow $\{\psi_t\}_{t\in\mathbb{R}}$ can be treated as a Hamiltonian flow on the torus, i.e. a flow associated with the equation

$$\begin{aligned}\frac{dx}{dt} &= \frac{\partial H}{\partial y}(x,y)\\ \frac{dy}{dt} &= -\frac{\partial H}{\partial x}(x,y),\end{aligned}$$

where $H(x,y) = \tilde{H}(x,y) + \gamma_1 x + \gamma_2 y$ and $\tilde{H} : \mathbb{T}^2 \to \mathbb{R}^2$ is a function of C^∞ class.

In paper [R3], the following result was proved:

Theorem 9 ([R3])**.** *Let $\{\psi_t\}_{t\in\mathbb{R}}$ be a Hamiltonian C^∞–flow on the torus $\mathbb{T}^2$, which does not have any fixed points. Let $f : (\mathbb{T}^2, \lambda_2) \to (\mathbb{T}^2, \lambda_2)$ be an ergodic C^∞–diffeomorphism commuting with the flow $\{\psi_t\}_{t\in\mathbb{R}}$. If the flow $\{\psi_t\}_{t\in\mathbb{R}}$ is ergodic, then there exists the constant $C > 0$ such that*

$$\liminf_{n\to\infty} \|Df^n(x)\| \le C$$

for a.e. $x \in \mathbb{T}^2$. Whereas, if the flow $\{\psi_t\}_{t\in\mathbb{R}}$ is not ergodic, then f is C^∞–conjugate with a certain skew Anzai product.

The result presented above makes it possible to complete the proof of Theorem 7. The proof of Theorem 9 makes use of the special representation of the flow $\{\psi_t\}_{t\in\mathbb{R}}$ as well as a certain condition implying the recurrence of cocycles over $\mathbb{Z}^2$–actions, which was proved by Depauw in [9].

A.5 Skew products on $\mathbb{T}^d \times \mathrm{SU}(2)$

In this chapter, we will consider diffeomorphisms of certain compact Lie matrix groups. Let $G \subset \mathrm{GL}(d, \mathbb{C})$ be a compact matrix group, let $\mathfrak{g}$ be its Lie algebra, while ν its Haar measure. Then, we say that diffeomorphism $f : (G, \nu) \to (G, \nu)$ has τ–polynomial growth of the derivative if

$$\frac{1}{n^\tau} L(f^n)(g) \to H(g)$$

for ν–a.e. while $H(g) : \mathfrak{g} \to \mathfrak{g}$ is a non–zero linear transformation on the set of positive ν–measure. This definition is another generalization of Definition 2.

In the further part of the selfreport, we will discuss the properties of some diffeomorphisms on $\mathbb{T}^d \times \mathrm{SU}(2)$ with linear growth of the derivative, which were proved in papers [R5] and [R6]. We will consider skew products of ergodic rotations on $\mathbb{T}^d$ with rotations by cocycles with values in the group $\mathrm{SU}(2)$.

Let us make a general assumption that T is a measure–preserving ergodic automorphism of standard probability Borel space $(X, \mathcal{B}, \mu)$. Let G be a closed Lie matrix group, while ν a right–invariant Haar measure on G. Then, any measurable mapping $\varphi : X \to G$ determines the skew product $T_\varphi : (X \times G, \mu \otimes \nu) \to (X \times G, \mu \otimes \nu)$ given by the formula

$$T_\varphi(x, g) = (Tx, g \cdot \varphi(x)).$$

By a measurable cocycle over the action of automorphism T, we will denote any measurable mapping $\mathbb{Z} \times X \ni (n, x) \mapsto \psi^{(n)}(x) \in G$ such that

$$\psi^{(n+m)}(x) = \psi^{(n)}(x) \cdot \psi^{(m)}(T^n x)$$

for any $m, n \in \mathbb{Z}$ and $x \in X$. Then, any mapping $\varphi : X \to G$ determines a measurable cocycle over automorphism T given by the formula

$$\varphi^{(n)}(x) = \begin{cases} \varphi(x) \cdot \varphi(Tx) \cdot \ldots \cdot \varphi(T^{n-1}x) \text{ for } & n > 0 \\ e \text{ for } & n = 0 \\ \left(\varphi(T^n x) \cdot \varphi(T^{n+1}x) \cdot \ldots \cdot \varphi(T^{-1}x)\right)^{-1} \text{ for } & n < 0. \end{cases} \tag{5}$$

This correspondence between cocycles and mappings is one-to-one; therefore, we will identify the cocycle $\varphi^{(\cdot)}(\cdot)$ with the function φ. Moreover, let us notice that $T_\varphi^n(x, g) = (T^n x, g \cdot \varphi^{(n)}(x))$ for any integer n. We say that two cocycles $\varphi, \psi : X \to G$ are *cohomologous* when there exists a measurable mapping $p : X \to G$ such that

$$\varphi(x) = p(x)^{-1} \cdot \psi(x) \cdot p(Tx).$$

Then, the mapping $X \times G \ni (x, g) \mapsto (x, g \cdot p(x)) \in X \times G$ establishes a measurable isomorphism between skew products T_φ and T_ψ. If, in addition, X is a C^r–manifold ($r \in \mathbb{N} \cup \{\infty\}$) and all the functions φ, ψ, p are of C^r class, then we say that φ and ψ are C^r–cohomologous.

The examination of skew products is extremely interesting due to their association with the theory of linear differential equations. Let $G \subset \mathrm{GL}(d, \mathbb{C})$

be a closed Lie matrix group. By $\mathfrak{g} \subset \mathfrak{g}(d, \mathbb{C})$, we denote its Lie algebra. Let us consider a differential equation on $\mathbb{C}^d$ with the following form

$$\frac{d}{dt} x(t) = x(t) A(t), \tag{6}$$

where $A : \mathbb{R} \to \mathfrak{g}$ is a function of C^r class ($r \in \mathbb{N} \cup \{\infty\}$). By $\Phi : \mathbb{R} \to G$, we denote the fundamental solution (the fundamental matrix) for (6), i.e.

$$\begin{cases} \frac{d}{dt}\Phi(t) &= \Phi(t)A(t) \\ \Phi(0) &= \mathrm{Id}\,. \end{cases}$$

In the simplest case, when the function A is periodic (with period equal to 1), then, on the basis of the Floquet theorem (see [48], for example), equation (6) is reduced to an equation with constant coefficients, which means that there exists function $c : \mathbb{T} \to G$ and $B \in \mathfrak{g}$ such that

$$\Phi(t) = c(0)^{-1} e^{tB} c(t).$$

However, the phenomenon described here does not occur when A is an almost periodic function. Let us assume that function A has the following form

$$A(t) = \bar{A}(S_t 0),$$

where $\bar{A} : \mathbb{T}^{k+1} \to \mathfrak{g}$ is a function of C^r class and $\{S_t\}_{t \in \mathbb{R}}$ is an ergodic linear flow on torus $\mathbb{T}^{k+1}$ with the form

$$S_t(x_1, \ldots, x_{k+1}) = (x_1 + t\omega_1, \ldots, x_{k+1} + t\omega_{k+1}).$$

Then, we say that equation (6) is C^r–reducible to an equation with constant coefficients if there exists function

$$\Phi(t) = c(0)^{-1} e^{tB} c(S_t 0). \tag{7}$$

Moreover, let us consider function $\phi : \mathbb{R} \times \mathbb{T}^{k+1} \to G$ determined by

$$\begin{cases} \frac{d}{dt}\phi(t,\theta) &= \phi(t,\theta)\bar{A}(S_t\theta) \\ \phi(0,\theta) &= \mathrm{Id} \end{cases}$$

for all $t \in \mathbb{R}$ and $\theta \in \mathbb{T}^{k+1}$. Then ϕ is a cocycle over the flow $\{S_t\}_{t \in \mathbb{R}}$, i.e.

$$\phi(t_1 + t_2, \theta) = \phi(t_1, \theta)\phi(t_2, S_{t_1}\theta) \tag{8}$$

for any $t_1, t_2 \in \mathbb{R}$ and $\theta \in \mathbb{T}^{k+1}$. Moreover, the condition of C^r–reducibility (7) is equivalent to condition

$$\phi(t,\theta) = c(\theta)^{-1} e^{tB} c(S_t\theta), \tag{9}$$

which means that the cocycle ϕ is C^r–cohomologous with cocycle $(t,\theta) \mapsto e^{tB}$. The cocycle ϕ is associated with C^r–flow (skew product) $\{S_t^\phi\}_{t\in\mathbb{R}}$ on $\mathbb{T}^{k+1} \times G$ given by the formula

$$S_t^\phi(\theta, g) = (S_t\theta, g \cdot \phi(t,\theta)).$$

Let $M \simeq \mathbb{T}^k \times G$ be a submanifold with the form $\{(x_1,\ldots,x_k,0,g) \in \mathbb{T}^{k+1} \times G : (x_1,\ldots,x_k,g) \in \mathbb{T}^k \times G\}$. Then, M is a transversal manifold to the orbits of the flow $\{S_t^\phi\}_{t\in\mathbb{R}}$, while Poincaré transformation on M is naturally conjugate with the skew product $T_\varphi : \mathbb{T}^k \times G \to \mathbb{T}^k \times G$, where $T : \mathbb{T}^k \to \mathbb{T}^k$ is an ergodic rotation $Tx = x + \alpha$ ($\alpha = (\omega_1/\omega_{k+1},\ldots,\omega_k/\omega_{k+1})$) and $\varphi : \mathbb{T}^k \to G$, $\varphi(x) = \phi(1/\omega_{k+1}, x, 0)$. Moreover, condition (9) is equivalent with the C^r–cohomologousness of the cocycle φ with a constant cocycle. Indeed, let us assume that

$$\varphi(x) = c(x)^{-1} e^{B} c(Tx),$$

where $c : \mathbb{T}^k \to G$ is a function of C^r class and $B \in \mathfrak{g}$. Without loss of generality of the reasoning, we can assume that that $\omega_{k+1} = 1$. Then, the function $\tilde{c} : \mathbb{R}^k \times \mathbb{R} \to G$ given by the formula

$$\tilde{c}(x,y) = e^{-By} c(x - y\alpha)\phi(y, x - y\alpha, 0)$$

is $\mathbb{Z}^{k+1}$–periodic and of C^r class. Using formula (8), one can conclude that

$$\phi(t,x,y) = \tilde{c}(x,y)^{-1} e^{tB} \tilde{c}(x + t\alpha, y + t).$$

To sum up, equation (6) is C^r–reducible iff the cocycle φ associated with it is C^r–cohomologous with a constant cocycle. Moreover, in some cases, for example when $k = 1$ and $G = \mathrm{SU}(2)$ (see Rychlik [50]), the mapping $\bar{A} \mapsto \varphi_{\bar{A}} = \varphi$ is a surjection on the whole set of C^r–cocycles $C^r(\mathbb{T}^k, G)$.

More information on the reducibility of cocycles and some linear equations as well as the connection of this subject matter with quasi–periodic Schrödinger equations can be found in papers [11, 12, 13, 14, 39, 40, 41] published in recent years.

Let us then return to the considerations on skew products over rotations. Let us assume that $\mathbb{T}^k \ni x \mapsto Tx = x + \alpha \in \mathbb{T}^k$ is an ergodic rotation, while

$\varphi : \mathbb{T}^k \to G$ is a C^1–cocycle over T, where G is a compact (semisimple) Lie matrix group. Then, the linear transformation $L\left(T_\varphi^n\right)(x,g) : \mathbb{R}^k \times \mathfrak{g} \to \mathbb{R}^k \times \mathfrak{g}$ has the form

$$L\left(T_\varphi^n\right)(x,g)(r,X) = (r, X + g(r \cdot L(\varphi^{(n)})(x))g^{-1}),$$

where

$$L(\psi)(x) = (L_{x_1}(\psi)(x), \ldots, L_{x_k}(\psi)(x)) \in \mathfrak{g}^k$$

and

$$L_{x_k}(\psi)(x) = \frac{\partial}{\partial x_j}\psi(x)(\psi(x))^{-1} \in \mathfrak{g}$$

for any $\psi : \mathbb{T}^k \to G$ of class C^1 and $j = 1, \ldots, k$. For any $g \in G$ and $X \in \mathfrak{g}$, let us denote by $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$, $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ linear transformations $\mathrm{Ad}(g)Y = gYg^{-1}$, $\mathrm{ad}(X)Y = [X,Y]$. Then, the Cartan–Killing form

$$\langle X, Y \rangle = -\mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$$

is symmetric, positive and Ad–invariant, therefore, it defines Ad–invariant norm on $\mathfrak{g}$. By applying this form, it is possible to provide a good definition of space $L^2(\mathbb{T}^k, \mathfrak{g})$ with the norm

$$\|\psi\| = \sqrt{\int_{\mathbb{T}^k} \|\psi(x)\|^2 \, dx},$$

which, in addition, is a Hilbert space. On $L^2(\mathbb{T}^k, \mathfrak{g})$, let us consider the unitary operator

$$(U\psi)(x) = \mathrm{Ad}(\varphi(x))\psi(Tx).$$

Then

$$L_{x_j}(\varphi^{(n)})(x) = \sum_{l=0}^{n-1} \left(U^l\left(L_{x_j}(\varphi)\right)\right)(x)$$

for $1 \le j \le k$. Thus, on the basis of the von Neumann ergodic theorem we obtain:

Theorem 10 ([R5]). *For any C^1–cocycle $\varphi : \mathbb{T}^k \to G$ and $j = 1, \ldots, k$, there exists $\psi_j \in L^2(\mathbb{T}^k, \mathfrak{g})$ such that*

$$\frac{1}{n} L_{x_j}(\varphi^{(n)}) \to \psi_j \ w \ L^2(\mathbb{T}^k, \mathfrak{g}). \tag{10}$$

Moreover, the function $\|\psi_j(\cdot)\|$ is a.e. constant and $\mathrm{Ad}(\varphi(x))\psi_j(Tx) = \psi_j(x)$ for a.e. $x \in \mathbb{T}^k$ and $j = 1, \ldots, k$.

Let us add that it is possible to show that the convergence in (10) is also a.e.

Definition 5. Vector

$$d(\varphi) = \frac{1}{2\pi}(\|\psi_1\|, \ldots, \|\psi_k\|) \in \mathbb{R}^k$$

is called a *degree of cocycle* φ over the rotation T.

Of course, if $d(\varphi) \neq 0$, then diffeomorphism T_φ has linear growth of the derivative.

Papers [R5] and [R6] contain an analysis of dynamical properties of cocycles with non–zero degree if

$$G = \mathrm{SU}(2) = \left\{ \begin{bmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{bmatrix} : z_1, z_2 \in \mathbb{C},\ |z_1|^2 + |z_2|^2 = 1 \right\}.$$

Then, an important class of cocycles will include cocycles with values in the maximal torus

$$\mathfrak{T} = \left\{ \begin{bmatrix} z & 0 \\ 0 & \overline{z} \end{bmatrix} : z \in \mathbb{C},\ |z| = 1 \right\},$$

i.e. cocycles with the form

$$\mathrm{diag}_\gamma(x) = \begin{bmatrix} \gamma(x) & 0 \\ 0 & \overline{\gamma(x)} \end{bmatrix},$$

where $\gamma : \mathbb{T} \to \mathbb{T}$. In paper [R5], it was proved that $d(\varphi) \neq 0$ implies non–ergodicity of the skew product T_φ, which is in contrast with the properties of skew products if $G = \mathbb{T}$ (see [31] or the remarks concerning the spectral properties of skew Anzai products in Chapter A.2).

Theorem 11 ([R5])**.** *If $\varphi : \mathbb{T}^k \to \mathrm{SU}(2)$ is a cocycle of C^1 class over an ergodic rotation T and $d(\varphi) \neq 0$, then φ is cohomologous with a cocycle with values in the subgroup $\mathfrak{T}$. Moreover, the skew product $T_\varphi : \mathbb{T}^k \times \mathrm{SU}(2) \to \mathbb{T}^k \times \mathrm{SU}(2)$ is not ergodic.*

Paper [R5] describes ergodic components of skew product T_φ and it presents their spectral analysis in the case where $k = 1$.

Theorem 12 ([R5])**.** *Let $\varphi : \mathbb{T} \to \mathrm{SU}(2)$ be a cocycle of C^1 class over an ergodic rotation T. If $d(\varphi) \neq 0$ and φ is cohomologous with a cocycle with the form $\mathrm{diag}_\gamma : \mathbb{T} \to \mathfrak{T}$, where $\gamma : \mathbb{T} \to \mathbb{T}$, then, all the ergodic components T_φ are metrically isomorphic with the skew product $T_\gamma : \mathbb{T} \times \mathbb{T} \to \mathbb{T} \times \mathbb{T}$. Moreover, Koopman operator U_{T_γ} is mixing on $L^2(dz)^\perp$.*

This result implies that if the degree of a cocycle is non–zero, then it cannot be cohomologous with a constant cocycle. With some additional assumptions, we will find Lebesgue components in the spectrum U_{T_φ}.

Theorem 13 ([R5]). *Let $\varphi : \mathbb{T} \to \mathrm{SU}(2)$ be a cocycle of C^2 class over an ergodic rotation T such that $d(\varphi) \neq 0$ and φ is cohomologous with a cocycle with values w in the subgroup $\mathfrak{T}$ via a function of bounded variation, whose derivative is in $L^2(\mathbb{T}, \mathfrak{su})$. Then, Lebesgue component in the spectrum U_{T_φ} has infinite multiplicity.*

Describing the possible values of the cocycle degree is another important problem raised in [R5] and [R6]. If a cocycle $\varphi : \mathbb{T}^k \to \mathrm{SU}(2)$ is C^1–cohomologous with a cocycle whose values are in the subgroup $\mathfrak{T}$, then, an easy calculation shows that $d(\varphi) \in \mathbb{Z}^k$. A similar property is generally true for C^2–cocycles if $k = 1$. The proof of this fact for the rotation by the gold number, based on the procedure of renormalization introduced by Rychlik in [50], is found in paper [R5]. The proof of the following general version, based on the procedure of renormalization introduced by Krikorian in [41], was presented in paper [R6].

Theorem 14 ([R6]). *For any cocycle $\varphi : \mathbb{T} \to \mathrm{SU}(2)$ of C^2 class over any ergodic rotation T, we have $d(\varphi) \in \mathbb{Z}$.*

This property is not true when $k \geq 2$.

Theorem 15 ([R5]). *For any ergodic rotation $T : \mathbb{T}^2 \to \mathbb{T}^2$ of the form $T(x, y) = (x + \alpha, y + \beta)$, there exists a cocycle $\varphi : \mathbb{T}^2 \to \mathrm{SU}(2)$ of C^∞ class such that $d(\varphi) = (|\beta|, |\alpha|)$.*

Another problem considered in [R6] regards the dependence of the value of the cocycle degree on the changes of rotation in its base with the established function which forms the cocycle. If $G = \mathbb{T}$ and $k = 1$, the degree of the cocycle is equal to the absolute value of the topological degree of the function which determines it, and thus it does not depend on the choice of the rotation in the base. In paper [R6], it was shown that this phenomenon vanishes when $G = \mathrm{SU}(2)$. In this paper, the function $\varphi : \mathbb{T} \to \mathrm{SU}(2)$ of C^∞ class and two irrational rotations, such that the degree φ amounts to 0 and to 1 over the first and second rotation respectively, were determined.

Other consequences of the application of Krikorian procedure proved in [R6] include, firstly, the invariancy of the degree due to the relation of measurable cohomology, secondly, a classification of C^∞–cocycles with non–zero degree if the rotation in the base of the cocycle is slowly approximated by rational numbers. By $\mathrm{G} : \mathbb{T} \to \mathbb{T}$, we denote the Gauss transformation, i.e. $\mathrm{G}(x) = \{1/x\}$. For any $\gamma > 0$ oraz $\sigma > 1$, let us denote

$$\mathrm{CD}(\gamma, \sigma) = \{\alpha \in \mathbb{T} : \forall_{0 \neq k \in \mathbb{N}} \forall_{l \in \mathbb{Z}} |k\alpha - l| > \frac{1}{\gamma k^{\sigma}}\}.$$

Next, by Σ, let us denote the set of those $\alpha \in \mathbb{T}$, for which there exist $\gamma > 0$ and $\sigma > 1$ such that $\mathrm{G}^k(\alpha) \in \mathrm{CD}(\gamma, \sigma)$ for infinitely many $k \in \mathbb{N}$. Due to the ergodicity of the Gauss transformation, the set Σ is a full Lebesgue measure. For any $r \in \mathbb{N}$ and $w \in \mathbb{R}$, let

$$\exp_{r,w}(x) = \begin{bmatrix} e^{2\pi i(rx+w)} & 0 \\ 0 & e^{-2\pi i(rx+w)} \end{bmatrix}.$$

The degree of cocycle $\exp_{r,w}$ is equal to r. Moreover, $\exp_{r,w}$ is stable, in a sense, with respect to C^{∞}–disturbances if the rotation is slowly approximated by rational numbers.

Theorem 16 ([R6]). *For any $\gamma > 0$, $\sigma > 1$ and $r > 0$ there exist $s_0 \in \mathbb{N}$ and $\varepsilon_0 > 0$ such that for any $\alpha \in \mathrm{CD}(\gamma, \sigma)$ and $\varphi \in C^{\infty}(\mathbb{T}, \mathrm{SU}(2))$, if $\|\varphi - \exp_{r,0}\|_{C^{s_0}} < \varepsilon_0$ and $d(\varphi) = r$, then, the cocycle φ is C^{∞}–cohomologous with a cocycle of the form $\exp_{r,w}$.*

This result, proved in [R6], is almost a direct conclusion drawn from a very profound Theorem 9.1 in [41]. Moreover, by using the Krikorian procedure of renormalization, it makes it possible to prove the following result which classifies cocycles with non–zero degree:

Theorem 17 ([R6]). *If $\alpha \in \Sigma$ and $\varphi : \mathbb{T} \to \mathrm{SU}(2)$ is a C^{∞}–cocycle such that $d(\varphi) > 0$, then φ is C^{∞}–cohomologous with a cocycle of the form $\exp_{d(\varphi),w}$.*

However, Theorem 17 is not true if we assume that the degree of a cocycle is equal to zero.

Finally, it should be mentioned that the subject matter connected with the examination of the type of the growth of the sequence of the derivatives of diffeomorphism has been flourishing recently. The subject matter was initiated by Polterovich in paper [45] and various authors continued to discuss it in papers [6, 7, 46, 47]. In [45], Polterovich considers the following sequence for any diffeomorphism $f : M \to M$

$$\Gamma_n(f) = \max\left(\max_{x\in M} \|Df^n(x)\|, \max_{x\in M} \|Df^{-n}(x)\|\right).$$

The considerations presented in the above–mentioned papers mainly refer to the class $\mathrm{Symp}_0(M,\omega)$ of diffeomorphisms M preserving a certain symplectic structure ω determined on M and isotropic with identity. Then, the growth of the sequence $\{\Gamma_n(f)\}_{n\in\mathbb{N}}$ in different cases is described in [45] and [47] in the following way:

Theorem 18. *Let $f \in \mathrm{Symp}_0(M,\omega) \setminus \{\mathrm{Id}\}$.*

- *Let $M = \mathbb{T}^2$ and let f have a fixed point, then there exists $c > 0$ such that $\Gamma_n \geq cn$.*
- *If M is a compact orientable surface with genus larger than one, then there exists $c > 0$ such that $\Gamma_n \geq cn$.*
- *If $M = \mathbb{T}^{2d}$ is the standard symplectic torus and f has a fixed point, then there exists $c > 0$ such that $\Gamma_n \geq c\sqrt{n}$.*
- *If $f \in \mathrm{Diff}^2_0([0,1])$, then either $(\log\Gamma_n(f))/n \to \gamma(f) > 0$, or there exists $C > 0$ such that $\Gamma_n \leq Cn^2$.*

B Scientific achievements in addition to the dissertation

The papers whose results were not included in the dissertation refer to the properties of disjointness of some special flows derived from differential equations on two–dimensional surfaces. These results are included in the following publications:

[D1] *A class of special flows over irrational rotations which is disjoint from mixing flows*, item [23] in the list of the quoted literature,

[D2] *On symmetric logarithm and some old examples in smooth ergodic theory*, item [24].

In the further part of the report, we will use the denotations [D1] and [D2] to refer to these papers.

B.1 The problem of smooth realization

The results of the research published in articles [D1] and [D2] are connected with the so–called problem of smooth realization for flows. A flow on a standard probability Borel space $(X, \mathcal{B}, \mu)$ is any one–parameter group $\{T_t\}_{t\in\mathbb{R}}$ of the automorphisms of space $(X, \mathcal{B}, \mu)$, i.e. measurable mapping $X \times \mathbb{R} \ni (x, t) \mapsto T_t x \in X$ such that

1^o automorphisms T_t preserve measure μ for any $t \in \mathbb{R}$;

2^o for any $t_1, t_2 \in \mathbb{R}$ we have $T_{t_1+t_2}x = T_{t_1}(T_{t_2}x)$ for μ–a.e. $x \in X$.

Let us remember that $\{T_t\}_{t\in\mathbb{R}}$ is a C^r–flow if X is a compact differential C^r–manifold, $\mathcal{B}$ is the σ–algebra of Borel sets, the mapping $X \times \mathbb{R} \ni (x, t) \mapsto T_t x \in X$ is of C^r class and the equation 2^o occurs for all $x \in X$. All the ergodic properties described in Chapter A.2 for automorphisms can also be defined for the action of flows (more details can be found in [8]). The problem of smooth realization can be formulated in the following way: does there exist, for a given ergodic property, a compact C^∞–manifold with a C^∞–flow preserving a positive probability C^∞–measure fulfilling a given property on it? A more specific question can also be the following: on which manifolds is it possible to realize a given property? More details regarding smooth realization can be found in [32]. The problem of the realization of ergodic properties on the simplest manifolds, i.e. two–dimensional surface, seems to be especially interesting. The problem of the existence of ergodic flows on surfaces was solved by Blohin in [5]. He constructed ergodic C^∞–flows on all the compact surfaces with the exception of sphere, projective plain and Klein bottle, on which such flows do not exist. Moreover, Kochergin showed in [37] that there exist mixing C^∞–flows on all the surfaces with the exception of the three surfaces mentioned above. However, when we ask about the existence of weakly mixing flows and non–mixing ones, there appears a problem, which is among the issues raised in paper [D2]. Shklover [53] gave an example of such a flow on the torus. For the manifolds of at least the third dimension, the situation is simpler, because for any such compact manifold, there exists a weakly mixing and non–mixing C^∞–flow (see [3]). The question about a smooth realization of Gaussian flows is another interesting problem. A flow $\{T_t\}_{t\in\mathbb{R}}$ on a space $(X, \mathcal{B}, \mu)$ is called a *Gaussian flow* if there exists a real subspace $H \subset L^2_0(X, \mathcal{B}, \mu)$ such that

- subspace H is $\{T_t\}_{t\in\mathbb{R}}$–invariant,

- subspace H generates $\mathcal{B}$, i.e. the smallest σ–algebra including σ–algebras $h^{-1}(\mathcal{B}_{\mathbb{R}})$, $h \in H$ ($\mathcal{B}_{\mathbb{R}}$ is the σ–algebra of Borel spaces on $\mathbb{R}$) is equal to $\mathcal{B}$,

- each non–zero element from H has Gaussian distribution.

More information on Gaussian automorphisms and flows can be found in [8]. The conjecture regarding Gaussian flows is the following: it is not possible to realize such flows on compact surfaces. This conjecture refers to a wider class of flows, which can be informally called flows with probabilistic origin. They were referred to with the formal name of ELF flows in [D1] (ELF is an acronym of the full French name of *ergodicité des limites faibles* proposed by F. Parreau). The property of ELF is defined as a property of certain joinings of the flow.

B.2 Joinings and properties of ELF

Let $\{T_t\}_{t\in\mathbb{R}}$ and $\{S_t\}_{t\in\mathbb{R}}$ be ergodic flows on standard probability Borel spaces $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$ respectively. By *joining* of flows $\{T_t\}_{t\in\mathbb{R}}$ and $\{S_t\}_{t\in\mathbb{R}}$, we will call any probability measure ρ on $(X \times Y, \mathcal{B} \otimes \mathcal{C})$ invariant for the action of the flow $\{T_t \times S_t\}_{t\in\mathbb{R}}$ and such that their projections on X and Y are equal to μ and ν respectively. The set of all such joinings is denoted by $J(\{T_t\}, \{S_t\})$, while the subset of ergodic joinings, i.e. joinings such that the flow $\{T_t \times S_t\}_{t\in\mathbb{R}}$ is ergodic on the space $(X \times Y, \mathcal{B} \otimes \mathcal{C}, \rho)$, is denoted by $J_e(\{T_t\}, \{S_t\})$. The set $J(\{T_t\}, \{S_t\})$ is non–empty, because $\mu \otimes \nu \in J(\{T_t\}, \{S_t\})$, and it is a simplex whose extreme points are the elements from $J_e(\{T_t\}, \{S_t\})$. Any joining $\rho \in J(\{T_t\}, \{S_t\})$ defines an operator $\Phi_\rho : L^2(X, \mathcal{B}, \mu) \to L^2(Y, \mathcal{C}, \nu)$ determined by

$$\int_{X\times Y} f(x)g(y)\, d\rho(x, y) = \int_Y \Phi_\rho(f)(y)g(y)\, d\nu(y)$$

for any $f \in L^2(X, \mathcal{B}, \mu)$ and $g \in L^2(Y, \mathcal{C}, \nu)$. The operator Φ_ρ is a Markov operator, i.e.

$$\Phi_\rho 1 = \Phi_\rho^* 1 = 1 \text{ and } \Phi_\rho f \geq 0, \text{ gdy } f \geq 0.$$

Moreover,

$$\Phi_\rho \circ T_t = S_t \circ \Phi_\rho \text{ for any } t \in \mathbb{R}. \tag{11}$$

In addition, there exists one-to-one correspondence between the set of Markov operators fulfilling (11) and the set $J(\{T_t\}, \{S_t\})$ (see [51]). For example, the product measure corresponds to the operator $\int$ given by the formula $\int(f) = \int_X f\, d\mu$. Thus, it is possible to introduce a weak topology derived from the weak operator topology on $J(\{T_t\}, \{S_t\})$. Together with this topology, $J(\{T_t\}, \{S_t\})$ becomes a compact metrizable space, where $\rho_n \to \rho$ iff $\langle \Phi_{\rho_n} f, g\rangle \to \langle \Phi_\rho f, g\rangle$ for any $f \in L^2(X, \mathcal{B}, \mu)$ and $g \in L^2(Y, \mathcal{C}, \nu)$. The joinings can be composed with the application of the following rule: if $\rho_1 \in J(\{T_t\}, \{S_t\})$ and $\rho_2 \in J(\{S_t\}, \{W_t\})$, then the joining $\rho_2 \circ \rho_1 \in J(\{T_t\}, \{W_t\})$ is determined by

$$\Phi_{\rho_2 \circ \rho_1} = \Phi_{\rho_2} \circ \Phi_{\rho_1}.$$

We say that flows $\{T_t\}_{t\in\mathbb{R}}$ and $\{S_t\}_{t\in\mathbb{R}}$ are *disjoint* when $J(\{T_t\}, \{S_t\}) = \{\mu \otimes \nu\}$. The concept of the disjointness of dynamical systems introduced by Furstenberg in [27] is a much stronger concept than the lack of isomorphism. The disjointness of dynamical systems implies the lack of common factors, and thus it testifies to the fundamental differences in the dynamics of the systems. Spectral disjointness is the strongest determinant of the lack of common features of dynamical systems, which means that the maximal spectral types of flows are mutually singular. It is not difficult to check that this property implies disjointness in the sense of Furstenberg.

For any $s \in \mathbb{R}$, Koopman operator $U_{T_s} : L^2(X, \mathcal{B}, \mu) \to L^2(X, \mathcal{B}, \mu)$, which we will briefly denote by T_s, is a Markov operator fulfilling (11), thus $T_s \in J(\{T_t\}, \{T_t\})$. Then, the joining corresponding to operator T_s is a measure concentrated on the graph of the automorphism T_s.

Definition 6. We say that an ergodic flow $\{T_t\}_{t\in\mathbb{R}}$ has the property of *ELF* when $\overline{\{T_s : s \in \mathbb{R}\}} \subset J_e(\{T_t\}, \{T_t\})$.

It is obvious that mixing flows are ELFs, because then $\overline{\{T_s : s \in \mathbb{R}\}} = \{T_s : s \in \mathbb{R}\} \cup \{\int\}$. Paper [D1] includes a simple proof of the fact that every ergodic Gaussian flow possesses properties of ELF (the proof of this fact also follows directly from the results obtained in [43]). In addition to Gaussian flows, the following major flows of probabilistic origin possess the property of ELF: ergodic Poisson suspension and ergodic flows derived from symmetric α–stable processes (see [10]). One of the fundamental ELF properties of flows is described by the following result proved in [D2], which is a direct conclusion drawn from the main result in paper [2].

Theorem 19 ([D1]). *Let us assume that $\{T_t\}_{t\in\mathbb{R}}$ is a flow fulfilling the property of ELF and $\rho \in \overline{\{T_s : s \in \mathbb{R}\}}$. Let $\{S_t\}_{t\in\mathbb{R}}$ be another ergodic flow and $\lambda \in J_e(\{S_t\}, \{T_t\})$. Then, $\rho \circ \lambda \in J_e(\{S_t\}, \{T_t\})$.*

This result makes it possible to prove the following criterion, which suggests how to prove disjointness with mixing flows or flows possessing the property of ELF.

Theorem 20 ([D1]). *Let $\{S_t\}_{t\in\mathbb{R}}$ be an ergodic flow on $(Y,\mathcal{C},\nu)$ such that for a certain sequence divergent to infinity $\{t_n\}_{n\in\mathbb{N}}$ of real numbers we have*

$$S_{t_n} \to \int_{\mathbb{R}} S_s \, dP(s),$$

where P is a probability Borel measure on $\mathbb{R}$. Then, the flow $\{S_t\}_{t\in\mathbb{R}}$ is disjoint from all mixing flows. If Fourier transformation of measure P is an analytic function, then $\{S_t\}_{t\in\mathbb{R}}$ is spectrally disjoint from all mixing flows. If the measure P is not a Dirac measure (concentrated in one point), then the flow $\{S_t\}_{t\in\mathbb{R}}$ is disjoint from all weakly mixing flows of ELF.

This criterion proves to be "effective" when the flow can be represented as the special flow over a rigid automorphism.

B.3 Special flows

Let T be a measure–preserving automorphism of a standard probability Borel space $(X,\mathcal{B},\mu)$. We say that T is *rigid* if there exists a sequence divergent to infinity $\{q_n\}_{n\in\mathbb{N}}$ (known as the time of the rigidity) such that $T^{q_n} \to \mathrm{Id}$ in the weak operator topology. Let $f : X \to \mathbb{R}$ be an integrable function with positive values. Then, the set $X^f = \{(x,t) \in X \times \mathbb{R} : 0 \le t < f(x)\}$ is the state space of the special flow $T^f = \{(T^f)_t\}_{t\in\mathbb{R}}$. The flow T^f acts on the point by moving it vertically upwards with unique speed and using the identification of points $(x, f(x))$ and $(Tx, 0)$. The flow T^f preserves the measure μ^f, which is the restriction of the product of measure μ with the Lebesgue measure on $\mathbb{R}$ to the set X^f.

Let us remember that any ergodic Hamiltonian C^r–flow on the torus (see Chapter A.4) without fixed points is C^r–conjugate with a certain special flow T^f, where T is an irrational rotation on the circle, while $f : \mathbb{T} \to \mathbb{R}$ is a function of C^r class, $r \in \mathbb{N} \cup \{\infty\}$ (see [8]). Moreover, every such special flow comes from a certain Hamiltonian flow (see [15]).

Let us assume that $f \in L^2(X,\mathcal{B},\mu)$ and let us denote $f_0(x) = f(x) - \int_X f \, d\mu$. Let us denote by $f_0^{(\,\cdot\,)}(\,\cdot\,)$ the additive cocycle over the automorphism

T associated with the function $f_0 : X \to \mathbb{R}$ (see (5)). Let us assume that $\{q_n\}_{n\in\mathbb{N}}$ is a time of the rigidity for T and

$$\sup_{n\in\mathbb{N}} \|f_0^{(q_n)}\|_{L^2} < \infty.$$

Then, when pass to the subsequence, we can assume that

$$\left(f_0^{(q_n)}\right)_* \mu \to P$$

weakly in the space of probability Borel measures on $\mathbb{R}$. The following result was proved in [D1]:

Theorem 21 ([D1])**.** *If the function f is separated from zero, then*

$$(T^f)_{cq_n} \to \int_{\mathbb{R}} (T^f)_{-t} \, dP(t)$$

weakly, where $c = \int_X f \, d\mu$.

This result, together with Theorem 20 and Koksma–Denjoy type inequality proved in [1], makes it possible to generalize a classical result of Kochergin [35], which says that any special flow T^f such that T is an irrational rotation on the circle, while f is a function of bounded variation, is not mixing.

Theorem 22 ([D1])**.** *If T is an irrational rotation on the circle and $f \in L^2(\mathbb{T}, \lambda_1)$ is a positive function separated from zero and such that $\widehat{f}(n) \in O(1/|n|)$, then the flow T^f is disjoint from all mixing flows. If, in addition, f is of bounded variation, then T^f is spectrally disjoint from all mixing flows.*

Moreover, Theorems 20 and 21 make it possible construct Hamiltonian C^r–flows on the torus which are disjoint with ELFs for any $r \in \mathbb{N}$ (see the end of [D1]). The next examples of special flows over rotations which are disjoint with ELFs are considered in [D2] and these are flows built under such functions as

$$f(x) = -a(\log\{x\} + \log\{-x\}) + h(x), \tag{12}$$

where $a > 0$ and $h : \mathbb{T} \to \mathbb{R}$ is an absolutely continuous function. The main results of paper [D2] can be formulated in the following way:

Theorem 23 ([D2]). *Let $T : \mathbb{T} \to \mathbb{T}$ be an irrational rotation by α. If $f > 0$ has the form of* (12)*, then the special flow T^f is weakly mixing and disjoint from ELFs* (*even more so for mixing flows*)*. If, in addition, α is approximated fast enough by rational numbers, then T^f is spectrally disjoint from all mixing flows.*

This result makes it possible to provide at least a partially positive answer to the question whether, on compact surface with a positive genus, one can find weakly mixing (but not mixing) C^∞–flows. The following result was proved in paper [D2]:

Theorem 24 ([D2]). *On any compact and connected C^∞–surface with a negative even Euler characteristic, there exists a C^∞–flow preserving a positive C^∞–measure, which is weakly mixing, spectrally disjoint from all mixing flows and disjoint from ELFs in the sense of Furstenberg.*

The subject matter discussed in article [D1] and [D2] is continued and further results were included in papers [25], [10] and [26] sent for publication. All these papers were enclosed with the dissertation and denoted as [D3], [D4] and [D5] respectively.

In article [D3], which was accepted for publication in Fundamenta Mathematicae, Theorem 21 was generalized for the case where the automorphism in the base of the special flow is not rigid, but in a certain sense it is locally rigid. It makes it possible to determine the disjointness of certain special flows constructed over automorphisms which are more complicated than rotations on the circle. In paper [D3], a classical Katok result was generalized (see [33]). It says that special flows constructed over ergodic exchanges of intervals and under functions of bounded variation are not mixing. In [D3], it was proved that such flows are disjoint from mixing flows in the sense of Furstenberg. This paper also considers ergodic smooth measure–preserving flows on surfaces, which had been earlier examined by Kochergin in [38]. In [D3], it was proved that if a flow possesses a finite number of critical points, all of the non–degenerated saddle type, and which admit a "good" transversal curve, then it is disjoint from ELF flows.

However, it seems that the most important result in these papers is the proof of the thesis (included in [D5]) that special flows constructed over certain irrational rotations and under piecewise smooth functions with a non–zero sum of the jumps possess the property imitating the Ratner property (see [49] and [54]). This result makes it possible to construct mildly mixing flows, which are derived from differential equations with singular points of the simple pole type.

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