Ergod. Th. & Dynam. Sys. (2003), , 1–14
Printed in the United Kingdom

A class of special flows over irrational rotations which is disjoint from mixing flows

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Abstract. It is proved that special flows over irrational rotations and under functions whose Fourier coefficients are of order $\mathrm{O}(1/|n|)$ are disjoint in the sense of Furstenberg from all mixing flows. This is an essential strengthening of a classical result by Kočergin on the absence of mixing of special flows built over irrational rotations and under bounded variation roof functions.

1. *Introduction*

Assume that T is an ergodic automorphism of a probability standard space $(X, \mathcal{B}, \mu)$ and let f be a positive function on X of integral 1. Denote by T^f the special flow built from T and f. In 1972, it was shown by Kočergin in [**13**], that special flows built over an irrational rotation $Tx = x + \alpha$ and under the roof function f of bounded variation are not mixing. On the other hand a further weakening of regularity of f and some Diophantine restrictions on α may lead to mixing flows, see [**4**], [**12**], [**14**], [**15**] (most of special flows in these papers turn out to be special representations of some smooth flows on surfaces). The absence of mixing when f is of bounded variation is caused by the Denjoy–Koksma inequality, it makes the limit distributions of the set $\{(f_0^{(q_n)})_*\mu\}$ be concentrated on a finite interval (here $\{q_n\}$ stands for the sequence of denominators of α and $f_0 = f - \int_X f\,d\mu$). Kočergin's result on the absence of mixing is then generalized in [**18**] to the case of f whose Fourier coefficients are of order $\mathrm{O}(1/|n|)$, while in a recent paper [**16**] by Kočergin, it has been shown that a further weakening does not seem to be possible. Namely, if $\Psi : (0, \delta) \to \mathbb{R}$ is a positive concave function satisfying $\Psi(0+) = 0$ and $\Psi'(0+) = +\infty$ then there exists a positive function f whose modulus of continuity is of order Ψ and whose Fourier coefficients are of order $\mathrm{O}(\Psi(1/|n|))$ such that for some irrational rotation T, the special flow T^f is mixing. On the other hand, in

† Research partly supported by KBN grant 5 P03A 027 21(2001).

[**18**] it is shown that whenever T is an ergodic rotation and T^f is mixing then in the space $\mathcal{P}(\overline{\mathbb{R}})$ of Borel probability measures on $\overline{\mathbb{R}}$ the sequence of distributions

$$\{(f_0^{(n)})_*\mu\} \text{ converges to the Dirac measure at } \infty. \tag{1}$$

Recently, this latter result has been improved by K. Schmidt in [**25**]: (1) holds in an arbitrary mixing special flow over an ergodic T. The case $\{(f_0^{(q_n)})_*\mu\}$ tends to a measure concentrated on a finite interval (where $\{q_n\}$ is a rigidity time for T) excludes mixing and moreover it may be considered as an opposite condition to (1). It turns out that not only mixing is excluded in such a case but a much stronger result holds.

THEOREM 1.1. *If T is ergodic, $\{q_n\}$ its rigidity time, $f \in L^2(X,\mu)$ is a positive function for which there exists $c > 0$ such that $f^{(k)}(x) \geq ck$ a.s. for all $k \in \mathbb{N}$ large enough, and the sequence $\{f_0^{(q_n)}\}$ is bounded in $L^2(X,\mu)$ then the special flow T^f is disjoint in the sense of Furstenberg from all mixing flows.*

The main ingredient of the proof of Theorem 1.1 is to show that whenever $\{(f_0^{(q_n)})_*\mu\}$ converges to a distribution P (which is concentrated on $\mathbb{R}$) then the sequence of the operators $\{(T^f)_{q_n}\}$ on $L^2(X^f,\mu^f)$ converges (in the weak operator topology) to the operator $\int_{\mathbb{R}}(T^f)_{-t}\,dP(t)$ (see Proposition 4.1 below). Suppose now that $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}$ is a mixing flow on $(Y,\mathcal{C},\nu)$ and ρ is a joining between T^f and $\mathcal{S}$. Thus

$$\int_{X^f\times Y} g\left((T^f)_t x\right) h(y)\,d\rho(x,y) = \int_{X^f\times Y} g(x)h(S_{-t}y)\,d\rho(x,y)$$

for each $g \in L^2(X,\mathcal{B},\mu)$ and $h \in L^2(Y,\mathcal{C},\nu)$. Since $\mathcal{S}$ is mixing and $(T^f)_{q_n} \to \int_{\mathbb{R}}(T^f)_{-s}\,dP(s)$, by passing to the limits along the sequence $\{q_n\}$, we obtain

$$\int_{\mathbb{R}}\left(\int_{X^f\times Y} g\left((T^f)_{-s}x\right) h(y)\,d\rho(x,y)\right) dP(s) = \int_{X^f\times Y} g(x)h(y)\,d\mu^f(x)\,d\nu(y).$$

Since the product measure $\mu^f \otimes \nu$ is ergodic, for P-a.a. $s \in \mathbb{R}$ and each $g \in L^2(X,\mathcal{B},\mu)$ and $h \in L^2(Y,\mathcal{C},\nu)$ we have

$$\int_{X^f\times Y} g\left((T^f)_{-s}x\right) h(y)\,d\rho(x,y) = \int_{X^f\times Y} g(x)h(y)\,d\mu^f(x)\,d\nu(y).$$

By taking $s \in \mathbb{R}$ for which the above property holds and noticing that the product measure is invariant under $(T^f)_s \times Id_Y$ we conclude that $\rho = \mu^f \otimes \nu$.

In particular, Kočergin's flows under roof functions, which are uniformly separated from zero on the circle and whose Fourier coefficients are of order $\mathrm{O}(1/|n|)$ are disjoint from mixing flows. If additionally in Theorem 1.1, $f \in L^\infty$ and the sequence $\{\|f_0^{(q_n)}\|_\infty\}$ is bounded, the integral form of the limit of $\{(T^f)_{q_n}\}$ allows us to prove more: T^f is even spectrally disjoint from an arbitrary mixing flow. In particular, Kočergin's flows with roof function of bounded variation are spectrally disjoint from all mixing flows. If in Theorem 1.1, P is not a Dirac measure, then the integral operator is decomposable (that is, it corresponds to a non–ergodic self–joining of T^f) and this is quite opposite phenomenon to what is observed in Gaussian systems (see [**19**] and Section 3 below). In fact, we prove that once P is not a Dirac measure, our special flow is disjoint from all Gaussian flows. In particular, cocycles considered in [**6**], [**20**] and [**22**] lead to special flows which are disjoint from all Gaussian flows, whence we obtain some classes of smooth flows on the 2–torus (see [**3**] and [**5**] for further details) which are disjoint from all Gaussian flows. For other disjointness results from Gaussian systems see [**10**], [**27**].

2. *Preliminaries*

This section is mainly to fix notation. Moreover, we will briefly put together necessary definitions and some known facts about flows that will be needed in what follows.

2.1. *Joinings between flows*

The following basic information about joinings between flows can be either found or is an easy adaptation of the case of $\mathbb{Z}$–actions in [**8**], [**11**], [**19**], [**24**], [**28**].

Assume that $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}$ is a flow on $(X, \mathcal{B}, \mu)$. By a flow we mean always a so called measurable flow, *i.e.* the map $\mathbb{R} \ni t \to \langle f \circ S_t, g\rangle \in \mathbb{R}$ is continuous for each $f, g \in L^2(X, \mathcal{B}, \mu)$. Assume moreover that $\mathcal{S}$ is ergodic and let $\mathcal{T} = \{T_t\}_{t\in\mathbb{R}}$ be another ergodic flow defined on $(Y, \mathcal{C}, \nu)$. By a *joining* between $\mathcal{S}$ and $\mathcal{T}$ we mean any probability $\{S_t \times T_t\}_{t\in\mathbb{R}}$–invariant measure on $(X \times Y, \mathcal{B}\otimes\mathcal{C})$ whose projections on X and Y are equal to μ and ν respectively. The set of joinings between $\mathcal{S}$ and $\mathcal{T}$ is denoted by $J(\mathcal{S}, \mathcal{T})$. The subset of ergodic joinings is denoted by $J^e(\mathcal{S}, \mathcal{T})$ and we write $J(\mathcal{S})$ and $J^e(\mathcal{S})$ instead of $J(\mathcal{S}, \mathcal{S})$ and $J^e(\mathcal{S}, \mathcal{S})$ respectively. Ergodic joinings are exactly extremal points in the simplex $J(\mathcal{S}, \mathcal{T})$. Given $\rho \in J(\mathcal{S}, \mathcal{T})$ define an operator $\Phi_\rho : L^2(X, \mathcal{B}, \mu) \to L^2(Y, \mathcal{C}, \nu)$ by requiring that

$$\int_{X\times Y} f(x)g(y)\, d\rho(x, y) = \int_Y \Phi_\rho(f)(y)g(y)\, d\nu(y)$$

for each $f \in L^2(X, \mathcal{B}, \mu)$ and $g \in L^2(Y, \mathcal{C}, \nu)$. This operator has the following Markov property

$$\Phi_\rho 1 = \Phi_\rho^* 1 = 1 \text{ and } \Phi_\rho f \geq 0 \text{ whenever } f \geq 0. \tag{2}$$

Moreover,

$$\Phi_\rho \circ S_t = T_t \circ \Phi_\rho \text{ for each } t \in \mathbb{R}. \tag{3}$$

In fact, there is a one-to-one correspondence between the set of Markov operators $\Phi : L^2(X, \mathcal{B}, \mu) \to L^2(Y, \mathcal{C}, \nu)$ satisfying (3) and the set $J(\mathcal{S}, \mathcal{T})$, where the joining ρ given by Φ is determined by the formula

$$\rho(A \times B) = \int_B \Phi(\chi_A)\, d\nu$$

for each $A \in \mathcal{B}$ and $B \in \mathcal{C}$. Markov operators corresponding to ergodic joinings will be called *indecomposable*. Notice that the product measure corresponds to the

Markov operator denoted by $\int$, where $\int(f)$ equals the constant function $\int_X f\,d\mu$. Assume now that $\mathcal{U}=\{U_t\}_{t\in\mathbb{R}}$ is another ergodic flow on $(Z,\mathcal{D},\eta)$. If $\rho\in J(\mathcal{S},\mathcal{T})$ and $\kappa\in J(\mathcal{T},\mathcal{U})$, then $\Phi_\kappa\circ\Phi_\rho$ is a Markov operator from $L^2(X,\mathcal{B},\mu)\to L^2(Z,\mathcal{D},\eta)$ and it satisfies (3) (with $\{T_t\}_{t\in\mathbb{R}}$ replaced by $\{U_t\}_{t\in\mathbb{R}}$). It corresponds hence to a unique joining of $\mathcal{S}$ and $\mathcal{U}$ which will be denoted by $\kappa\circ\rho$. Even if κ and ρ are ergodic, $\kappa\circ\rho$ need not be ergodic. If $\rho=\int_Y\rho_y\,d\nu(y)$ and $\kappa=\int_Y\kappa_y\,d\nu(y)$ are disintegrations over Y of ρ and κ respectively then $\kappa\circ\rho$ is the projection on $X\times Z$ of the *relative product* $\rho\times_Y\kappa$ which is defined by

$$\rho\times_Y\kappa=\int_Y\rho_y\otimes\kappa_y\,d\nu(y).$$

If in the above construction $\mathcal{S}=\mathcal{T}=\mathcal{U}$, $\kappa=\rho$ and the above relative product is ergodic then the flow $(\{S_t\times S_t\}_{t\in\mathbb{R}},\rho)$ is called *relatively weakly mixing* over the first coordinate X. On $J(\mathcal{S})$ we consider the weak operator topology. In this topology $J(\mathcal{S})$ becomes a metrizable compact semitopological semigroup in which $\rho_n\to\rho$ iff $\langle\Phi_{\rho_n}f,g\rangle\to\langle\Phi_\rho f,g\rangle$ for each $f,g\in L^2(X,\mathcal{B},\mu)$. For each $t\in\mathbb{R}$, S_t can be considered as a Markov operator on $L^2(X,\mathcal{B},\mu)$. The corresponding self–joining is denoted by μ_{S_t} and it is exactly the joining concentrated on the graph of S_t.

Following [**7**], $\mathcal{S}$ and $\mathcal{T}$ are called *disjoint* if $J(\mathcal{S},\mathcal{T})=\{\mu\otimes\nu\}$. Equivalently, the operator $\int$ is the only Markov operator that intertwines S_t and T_t (for each $t\in\mathbb{R}$).

Each flow $\mathcal{S}=\{S_t\}_{t\in\mathbb{R}}$ determines a unitary action, still denoted by $\mathcal{S}$, of $\mathbb{R}$ on $L^2(X,\mathcal{B},\mu)$ by the formula

$$f\mapsto f\circ S_t.$$

The maximal spectral type of $\mathcal{S}$ on $L^2_0(X,\mathcal{B},\mu)$ we denote by $\tau_{\mathcal{S}}$, while spectral measure of $f\in L^2(X,\mathcal{B},\mu)$ is denoted by $\tau_{f,\mathcal{S}}$ or τ_f if $\mathcal{S}$ is understood. Classically, if $\tau_{\mathcal{S}}$ and $\tau_{\mathcal{T}}$ are mutually singular (*i.e.* $\mathcal{S}$ and $\mathcal{T}$ are *spectrally disjoint*), then $\mathcal{S}$ and $\mathcal{T}$ are disjoint.

Recall also that $\{q_n\}\subset\mathbb{R}$ is said to be a *rigidity* sequence for the flow $\mathcal{S}$ if $S_{q_n}\to\mathrm{Id}$.

2.2. *Special flows* Let T be an ergodic automorphism of a standard probability space $(X,\mathcal{B},\mu)$. Denote by λ Lebesgue measure on $\mathbb{R}$. Assume that $f:X\to\mathbb{R}$ is a measurable positive function such that $\int_X f\,d\mu=1$. The *special flow* $T^f=\{(T^f)_t\}_{t\in\mathbb{R}}$ built from T and f is defined on the space $X^f=\{(x,t)\in X\times\mathbb{R}:0\le t<f(x)\}$ (considered with $\mathcal{B}^f$ the restriction of the product σ–algebra and μ^f the restriction of the product measure $\mu\otimes\lambda$ of $X\times\mathbb{R}$). Under the action of the special flow each point in X^f moves vertically at unit speed, and we identify the point $(x,f(x))$ with $(Tx,0)$ (see *e.g.* [**3**], Chapter 11). Given $m\in\mathbb{Z}$ we put

$$f^{(m)}(x)=\begin{cases} f(x)+f(Tx)+\ldots+f(T^{m-1}x) & \text{if}\quad m>0\\ 0 & \text{if}\quad m=0\\ -\big(f(T^mx)+\ldots+f(T^{-1}x)\big) & \text{if}\quad m<0.\end{cases}$$

The action of T^f can be well understood when we consider the following actions on the space $(X\times\mathbb{R},\mu\otimes\lambda)$. First, let $S_{-f}:(X\times\mathbb{R},\mu\otimes\lambda)\to(X\times\mathbb{R},\mu\otimes\lambda)$ denote the skew product given by

$$S_{-f}(x,r) = (Tx, r - f(x)).$$

Notice that $(S_{-f})^k(x,r) = (T^k x, r - f^{(k)}(x))$ for each $k \in \mathbb{Z}$. Let us consider the quotient space $\Gamma_f = X \times \mathbb{R}/\sim$, where the relation $\sim$ is defined by $(x,r) \sim (x',r')$ iff $(x,r) = (S_{-f})^k(x',r')$ for an integer k. Since $f^{(k)}(x) \to +\infty$, μ-a.e., with no loss of generality we have that the set

$$\{(x,r) \in X \times \mathbb{R} : 0 \le r < f(x)\}$$

intersects equivalence class of $\sim$ in exactly one point (and hence can be identified with Γ_f). Let $\sigma = \{\sigma_t\}_{t\in\mathbb{R}}$ stand for the flow on $(X \times \mathbb{R}, \mu \otimes \lambda)$ given by

$$\sigma_t(x,r) = (x, r+t).$$

Notice that σ_t commutes with S_{-f}. Then the special flow T^f can be seen as the quotient flow of the action σ by the relation $\sim$. It follows that given $(x,r) \in X^f$ and $t \in \mathbb{R}$ there exists a unique $k \in \mathbb{Z}$ such that

$$(T^f)_t(x,r) = (S_{-f})^k \circ \sigma_t(x,r).$$

Let us recall that the Ambrose–Kakutani theorem says that under some natural conditions, each flow has a representation as a special flow.

2.3. *Gaussian flows* A flow $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}$ on $(X, \mathcal{B}, \mu)$ is called a *Gaussian* flow if there exists an infinite dimensional real space $H \subset L^2_0(X, \mathcal{B}, \mu)$ which generates $\mathcal{B}$, which is invariant under all S_t, $t \in \mathbb{R}$, and which all non–zero elements are Gaussian variables. A classical result (see *e.g.* [**3**], Chapter 8 for the case of $\mathbb{Z}$-actions) is that a Gaussian flow $\mathcal{S}$ is ergodic iff the spectral type of $\mathcal{S}$ on the Gaussian space H is continuous. Then the whole flow is weakly mixing. For a new joining theory of Gaussian systems we refer the reader to [**19**]. That paper is written only for $\mathbb{Z}$–actions, but the extension to $\mathbb{R}$–actions is straightforward.

3. *Flows with an ergodic weak closure property*

An ergodic flow $\mathcal{S} = \{S_t : t \in \mathbb{R}\}$ on a standard probability space $(X, \mathcal{B}, \mu)$ is said to have the ELF† property if $\overline{\mathcal{S}} := \overline{\{S_t : t \in \mathbb{R}\}} \subset J^e(\mathcal{S})$. Shortly, we will speak about ELF flows.

Remark. If $\mathcal{S}$ is mixing then $\overline{\mathcal{S}} = \{S_t\}_{t\in\mathbb{R}} \cup \{\int\}$, so it is an ELF flow. It is also easy to see that all ergodic flows with discrete spectrum have the ELF property. Below, we will show that Gaussian flows have the ELF property. More examples of ELF flows will be published elsewhere.

The following result can already be deduced from a description of so called Gaussian joinings from [**19**], but we will give a direct proof.

† The name ELF has been proposed to us by F. Parreau, it comes from the French abbreviation of *ergodicité des limites faibles*.

PROPOSITION 3.1. *Each ergodic Gaussian flow $\mathcal{S}$ is an ELF flow.*

Proof. Denote by $H \subset L^2_0(X,\mathcal{B},\mu)$ the Gaussian space for $\mathcal{S}$. Assume that $\mu_{S_{t_n}} \to \rho$ in $J(\mathcal{S})$. Take $f,g \in H$. Since $f \circ S_{t_n} + g \in H$, for each $r \in \mathbb{R}$ we have $\int_X e^{2\pi i r(f(S_{t_n}x)+g(x))}\,d\mu(x) = e^{-2\pi^2 r^2 \|f\circ S_{t_n}+g\|_2^2}$ and therefore

$$\begin{aligned}\int_{X\times X} e^{2\pi i r(f(x)+g(y))}\,d\rho(x,y) &= \lim_{n\to\infty}\int_X e^{2\pi i r(f(S_{t_n}y)+g(y))}\,d\mu(y) && (4)\\ &= e^{-2\pi^2 r^2(\|\Phi_\rho f+g\|_2^2+\|f\|_2^2-\|\Phi_\rho f\|_2^2)}. && (5)\end{aligned}$$

It follows that $f(x)+g(y) \in L^2(X\times X,\rho)$ is a Gaussian variable. Therefore the space $F = \overline{\{f(x)+g(y):\ f,g\in H\}}$ is a Gaussian space for the flow $(\{S_t\times S_t\}_{t\in\mathbb{R}},\rho)$ and since the spectral type on F is equal to the spectral type of $\mathcal{S}$ on H, ρ is ergodic. □

Recall that if a flow is ergodic then all but a countable subset of its time automorphisms are ergodic $\mathbb{Z}$–actions (*e.g.* [**3**], p. 326).

Assume now that $\mathcal{S}$ is an ELF flow. Hence $\overline{\mathcal{S}}$ is a (compact) semitopological semigroup. By the main result of [**2**] it follows that for every ρ such that $\Phi_\rho \in \overline{\mathcal{S}}$ the flow $(\{S_t\times S_t\}_{t\in\mathbb{R}},\rho)$ is relatively weakly mixing over its marginals. Indeed, since ρ and $\rho\circ\rho$ are ergodic, for some t_0, ρ and $\rho\circ\rho$ are ergodic for $S_{t_0}\times S_{t_0}$. By [**2**], the relative product $\rho\times_X\rho$ is ergodic for $(S_{t_0}\times S_{t_0})\times(S_{t_0}\times S_{t_0})$, so the more it is ergodic for the flow $\{(S_t\times S_t)\times(S_t\times S_t)\}_{t\in\mathbb{R}}$.

LEMMA 3.1. *Assume that $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}$ is an ELF flow and let $\rho \in \overline{\mathcal{S}}$. Let $\mathcal{T} = \{T_t\}_{t\in\mathbb{R}}$ be an ergodic flow on $(Y,\mathcal{C},\nu)$. Assume that λ is an ergodic joining of $\mathcal{T}$ and $\mathcal{S}$. Then $\rho\circ\lambda$ is still ergodic.*

Proof. We have: $(X_1\times X_2,\rho)$ (where $X_1 = X_2 = X$) is relatively weakly mixing over X_1 and $(Y\times X_1,\lambda)$ is relatively ergodic over X_1, hence the relative product $\lambda\times_{X_1}\rho$ is still relatively ergodic over X_1 (see [**28**]). It is then ergodic and the projection on $Y\times X_2$ so is which means that $\rho\circ\lambda$ is indeed ergodic. □

We will now show that some flows are disjoint from ELF flows. Let $J:\mathbb{R}\to J(\mathcal{T},\mathcal{S})$ be a continuous function. Given a Borel probability measure P on $\mathbb{R}$, $f\in L^2(Y,\mathcal{C},\nu)$ and $g\in L^2(X,\mathcal{B},\mu)$ consider

$$\langle\langle f,g\rangle\rangle = \int_{\mathbb{R}}\langle J(s)f,g\rangle\,dP(s).$$

In this way we obtain a bilinear map $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle : L^2(Y,\mathcal{C},\nu)\times L^2(X,\mathcal{B},\mu)\to\mathbb{C}$ for which $|\langle\langle f,g\rangle\rangle| \le \|f\|_2\|g\|_2$. Hence there exists a unique linear bounded operator denoted by $\int_{\mathbb{R}} J(s)\,dP(s)$ for which

$$\left\langle\left(\int_{\mathbb{R}} J(s)\,dP(s)\right)f,g\right\rangle = \int_{\mathbb{R}}\langle f\circ J(s),g\rangle\,dP(s) \tag{6}$$

for every $f\in L^2(Y,\mathcal{C},\nu)$ and $g\in L^2(X,\mathcal{B},\mu)$. Notice that $\int_{\mathbb{R}} J(s)\,dP(s)\in J(\mathcal{T},\mathcal{S})$. Now suppose that J is a Markov operator, $J\in J(\mathcal{T},\mathcal{S})$. Then the map

$$\mathbb{R}\ni t\mapsto J\circ T_t\in J(\mathcal{T},\mathcal{S}) \tag{7}$$

is continuous. Assume additionally that $J \neq \int$. If $J \circ T_r = J$ then T_{-r} is not ergodic and it follows that the map (7) is at most countable to one. Therefore the set $\{J \circ T_t : t \in \mathbb{R}\}$ is a Borel subset in $J(\mathcal{T}, \mathcal{S})$ (*e.g.* [**26**], Th. 4.12.4). If J is additionally indecomposable, then the map (7) takes values in $J^e(\mathcal{T}, \mathcal{S})$. If by P' we denote the image of P via (7) then

$$\int_{\mathbb{R}} J \circ T_s \, dP(s) = \int_{J^e(\mathcal{T},\mathcal{S})} \Phi \, dP'(\Phi),$$

in other words the latter integral represents the ergodic decomposition of the left-hand side operator.

PROPOSITION 3.2. *Suppose that $\mathcal{T} = \{T_t\}_{t \in \mathbb{R}}$ is an ergodic flow on $(Y, \mathcal{C}, \nu)$ such that for a sequence $\{t_n\} \subset \mathbb{R}$,*

$$T_{t_n} \to \int_{\mathbb{R}} T_s \, dP(s),$$

where P is a Borel probability measure on $\mathbb{R}$. Then
(i) $\mathcal{T}$ is disjoint from all mixing flows;
(ii) $\mathcal{T}$ is disjoint from all weakly mixing ELF flows whenever P is not a Dirac measure.

Proof. We first prove (ii). Assume that $\mathcal{S}$ is an ELF flow on $(X, \mathcal{B}, \mu)$. Let $J : L^2(Y, \mathcal{C}, \nu) \to L^2(X, \mathcal{B}, \mu)$ be a Markov operator corresponding to an ergodic joining of $\mathcal{T}$ and $\mathcal{S}$. We then have $J \circ T_t = S_t \circ J$ and by passing to a subsequence of $\{t_n\}$ if necessary, we have

$$J \circ \int_{\mathbb{R}} T_s \, dP(s) = \Phi_\rho \circ J,$$

where $\rho = \lim_{n\to\infty} \mu_{S_{t_n}}$. In view of Lemma 3.1, $\Phi_\rho \circ J$ remains indecomposable. Now $J \circ \int_{\mathbb{R}} T_s \, dP(s) = \int_{\mathbb{R}} J \circ T_s \, dP(s)$ by (6). It follows that $\int_{\mathbb{R}} J \circ T_s \, dP(s)$ is indecomposable which is possible iff $J \circ T_s =$ const. P-a.e. If P is not a Dirac measure we must have $J = J \circ T_r$ for some $r \neq 0$. Hence $T_{-r} \circ J^* = J^*$. We obtain that $\mathrm{Im}\, J^*$ is contained in $L^2(\mathcal{I})$, where $\mathcal{I}$ stands for the σ–algebra of T_{-r}–invariant sets. Clearly, $\mathcal{I}$ is a factor of $\mathcal{T}$. However, T_{-r} acts on $\mathcal{I}$ as the identity map, so the unitary action on $L^2(\mathcal{I})$ associated to $\mathcal{T}$ on $\mathcal{I}$ is a unitary representation of the circle. It follows that the flow $\mathcal{T}$ restricted to $\mathcal{I}$ has discrete spectrum. But J^* settles a joining between $\mathcal{S}$ and $\mathcal{T}$ restricted to $\mathcal{I}$. Since $\mathcal{S}$ is weakly mixing, J^* must be trivial and the result follows.
(i) Assume now that $\mathcal{S}$ is mixing. By repeating the beginning of the proof of (ii) we obtain that $J \circ \int_{\mathbb{R}} T_s \, dP(s) = \int$. Therefore, $J \circ T_s = \int$ for P-a.a. $s \in \mathbb{R}$ and we must have $J = \int$. □

4. *Special flows which have integral operators in the weak closure of their times*

Assume that $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is an ergodic automorphism. Let $\{q_n\}$ be a rigidity time for T. Suppose that $f \in L^2(X, \mu)$ is a positive function such that

$\int_X f(x)\,d\mu(x) = 1$ and

$$\sup_n \|f_0^{(q_n)}\|_{L^2} =: C < +\infty. \tag{8}$$

By passing to a further subsequence if necessary we can assume that

$$(f_0^{(q_n)})_*\mu \to P$$

weakly in $\mathcal{P}(\mathbb{R})$ the set of Borel probability measures on $\mathbb{R}$. Let $C(\overline{\mathbb{R}})$ denote the set of all continuous functions $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\lim_{x\to-\infty} \varphi(x) = \lim_{x\to+\infty} \varphi(x) \in \mathbb{R}$.

LEMMA 4.1. *For every $\varphi \in C(\overline{\mathbb{R}})$, $g \in L^1(X, \mathcal{B}, \mu)$ and any measurable function $h : X \to \mathbb{R}$ we have*

$$\int_X \varphi\left(f_0^{(q_n)}(x) + h(x)\right) g(x)\,d\mu(x) \tag{9}$$

$$\to \int_X \int_{\mathbb{R}} \varphi\left(t + h(x)\right) g(x)\,dP(t)\,d\mu(x). \tag{10}$$

Proof. We first recall that

$$\int_X \left(\varphi \circ f_0^{(q_n)}\right) \cdot g\,d\mu \to \int_{\mathbb{R}} \varphi\,dP \int_X g\,d\mu$$

whenever $\varphi \in C(\overline{\mathbb{R}})$ and $g \in L^1(X, \mathcal{B}, \mu)$ (see the proof of Proposition 8 in [**20**]). This gives (9) in the case where h takes only finitely many values. Indeed, suppose that $h = \sum_{j=1}^k h_j \cdot \chi_{A_j}$. Then

$$\begin{aligned}
&\int_X \varphi\left(f_0^{(q_n)}(x) + h(x)\right) g(x)\,d\mu(x)\\
&= \sum_{j=1}^k \int_X \varphi\left(f_0^{(q_n)}(x) + h_j\right) (g \cdot \chi_{A_j})(x)\,d\mu(x)\\
&\to \sum_{j=1}^k \int_{\mathbb{R}} \varphi(t + h_j)\,dP(t) \int_X (g \cdot \chi_{A_j})(x)\,d\mu(x)\\
&= \int_X \int_{\mathbb{R}} \varphi\left(t + h(x)\right) g(x)\,dP(t)\,d\mu(x).
\end{aligned}$$

Therefore it suffices to show that for every $\varepsilon > 0$ we can find a measurable function $h_\varepsilon : \mathbb{R} \to \mathbb{R}$ taking finitely many values such that for every natural n

$$\left\{\begin{array}{l} |\int_X \varphi\left(f_0^{(q_n)}(x) + h(x)\right) g(x)\,d\mu(x) \\ -\int_X \varphi\left(f_0^{(q_n)}(x) + h_\varepsilon(x)\right) g(x)\,d\mu(x)| < \varepsilon \end{array}\right. \tag{11}$$

and

$$\left\{\begin{array}{l} |\int_X \int_{\mathbb{R}} \varphi(t + h(x)) g(x)\,dP(t)\,d\mu(x) \\ -\int_X \int_{\mathbb{R}} \varphi(t + h_\varepsilon(x)) g(x)\,dP(t)\,d\mu(x)| < \varepsilon. \end{array}\right. \tag{12}$$

Fix $\varepsilon > 0$. Since φ is uniformly continuous we can find $\delta > 0$ such that $|s - t| < \delta$ implies $|\varphi(s) - \varphi(t)| < \varepsilon/(2\|g\|_{L^1})$. Let η be a positive real number such that

$\int_A |g(x)|\,d\mu(x) < \varepsilon/(4\|\varphi\|_\infty)$, whenever $\mu(A) < \eta$. Finally we can choose a measurable function $h_\varepsilon : \mathbb{R} \to \mathbb{R}$ which takes finitely many values and such that

$$\mu(\{x \in X : |h_\varepsilon(x) - h(x)| \geq \delta\}) < \eta.$$

Then

$$\begin{aligned} &|\int_X \varphi\left(f_0^{(q_n)}(x) + h(x)\right) g(x)\,d\mu(x) - \int_X \varphi\left(f_0^{(q_n)}(x) + h_\varepsilon(x)\right) g(x)\,d\mu(x)| \\ \leq\ & 2\int_{\{x\in X:|h_\varepsilon(x)-h(x)|\geq\delta\}} \|\varphi\|_\infty |g(x)|\,d\mu(x) \\ &+ \int_{\{x\in X:|h_\varepsilon(x)-h(x)|<\delta\}} \frac{\varepsilon}{2\|g\|_{L^1}} |g(x)|\,d\mu(x) < \varepsilon \end{aligned}$$

for every natural n. Similarly, we can show (12). □

We will still need some more auxiliary facts about special flows.

LEMMA 4.2. *For all measurable $A, B \subset X^f$ we have*

$$\mu^f\left((T^f)_t A \cap B\right) = \sum_{k\in\mathbb{Z}} \mu \otimes \lambda\left((S_{-f})^k \sigma_t A \cap B\right).$$

Proof. As we have already noticed, given $t \in \mathbb{R}$ and $(x,r) \in X^f$, $(T^f)_t(x,r) = (S_{-f})^k \sigma_t(x,r)$ for a unique $k \in \mathbb{Z}$. Thus given $t \in \mathbb{R}$, the space X^f can be partitioned into countably many subsets X_k^f ($k \in \mathbb{Z}$) where $(T^f)_t$ on X_k^f acts as $(S_{-f})^k\sigma_t$. Moreover, since $(T^f)_t$ is an automorphism, the images $(S_{-f})^k\sigma_t(X_k^f)$ are pairwise disjoint. The result follows from this observation. □

LEMMA 4.3. *Suppose that $A, B \subset X \times \mathbb{R}$ are measurable rectangles of the form $A = A_1 \times A_2$, $B = B_1 \times B_2$. Then*

$$\mu \otimes \lambda\left((S_{-f})^k A \cap B\right) = \int_{T^k A_1 \cap B_1} \lambda\left((A_2 + f^{(-k)}(x)) \cap B_2\right) d\mu(x).$$

Proof. We have $(x,t) \in (S_{-f})^k(A_1 \times A_2) \cap (B_1 \times B_2)$ iff $(x,t) = (T^k y, r - f^{(k)}(y))$, where $(y,r) \in A_1 \times A_2$ and $(x,t) \in B_1 \times B_2$. Thus $(x,t) \in (S_{-f})^k(A_1 \times A_2) \cap (B_1 \times B_2)$ iff $x \in T^k A_1 \cap B_1$ and $t \in (A_2 - f^{(k)}(T^{-k}x)) \cap B_2$. Since $f^{(k)}(T^{-k}x) = -f^{(-k)}(x)$, the result follows. □

Suppose that $f \in L^2(X,\mu)$ is a positive function for which (8) holds and moreover, there exist $c > 0$ and $k_0 \in \mathbb{N}$ such that $f^{(k)}(x) \geq ck$ for a.e. $x \in X$, $k \geq k_0$.

LEMMA 4.4. *For every pair of bounded sets $A_2, B_2 \subset \mathbb{R}$ there exists a sequence $\{a_k\}$ of positive numbers such that*
(i) $\sum_{k\in\mathbb{Z}} a_k < +\infty$
and
(ii) $\int_X \lambda\left((A_2 - f_0^{(q_n)}(x) + f^{(k)}(x)) \cap B_2\right) d\mu(x) \leq a_k$ *for each $n \in \mathbb{N}$ and each $k \in \mathbb{Z}$.*

Proof. Set $s := \operatorname{diam}(A_2 \cup B_2)$. Let k be an integer such that $|k| > k_1 := \max(k_0, s/c)$. Then

$$\begin{aligned}
&\int_X \lambda\left((A_2 - f_0^{(q_n)}(x) + f^{(k)}(x)) \cap B_2\right) d\mu(x) \\
&= \int_{\{x\in X:|f^{(k)}(x)-f_0^{(q_n)}(x)|\le s\}} \lambda\left((A_2 - f_0^{(q_n)}(x) + f^{(k)}(x)) \cap B_2\right) d\mu(x) \\
&\le s\,\mu\left(\{x \in X : |f^{(k)}(x) - f_0^{(q_n)}(x)| \le s\}\right) \\
&\le s\,\mu\left(\{x \in X : |f_0^{(q_n)}(x)| \ge c|k| - s\}\right) \\
&\le \frac{s\,C^2}{(c|k|-s)^2}
\end{aligned}$$

by Chebyshev's inequality. Putting $a_k := s\,C^2/(c|k|-s)^2$ whenever $|k| > k_1$ and $a_k := s$ otherwise we obtain our claim. □

Here is the main result of this section.

PROPOSITION 4.1. *Let $\{q_n\}$ be a rigidity sequence for T. Suppose that $f \in L^2(X,\mu)$ is a positive function with $\int_X f(x)\,d\mu(x) = 1$. Moreover, suppose that the sequence $\{f_0^{(q_n)}\}$ is bounded in $L^2(X,\mu)$, $(f_0^{(q_n)})_*\mu \to P$ weakly in $\mathcal{P}(\mathbb{R})$ and there exists $c > 0$ such that $f^{(k)}(x) \ge ck$ for a.a. $x \in X$ and for all $k \in \mathbb{N}$ large enough. Then*

$$(T^f)_{q_n} \to \int_{\mathbb{R}} (T^f)_{-t}\, dP(t).$$

Proof. First notice that all we need to show is that

$$\mu^f\left((T^f)_{q_n} A \cap B\right) \to \int_{\mathbb{R}} \mu^f\left((T^f)_{-t} A \cap B\right) dP(t)$$

for any pair of measurable rectangles $A, B \subset X^f$ of the form $A = A_1 \times A_2$, $B = B_1 \times B_2$ such that $A_2, B_2 \subset \mathbb{R}$ are bounded. By Lemma 4.2,

$$\mu^f\left((T^f)_{q_n} A \cap B\right) = \sum_{k\in\mathbb{Z}} \mu \otimes \lambda\left((S_{-f})^k (S_{-f})^{q_n} \sigma_{q_n} A \cap B\right).$$

Using Lemma 4.3 we obtain

$$\begin{aligned}
&\mu^f\left((T^f)_{q_n} A \cap B\right) \\
&= \sum_{k\in\mathbb{Z}} \int_{T^{q_n+k}A_1 \cap B_1} \lambda\left((A_2 + q_n + f^{(-q_n-k)}(x)) \cap B_2\right) d\mu(x) \\
&= \sum_{k\in\mathbb{Z}} \int_{T^{q_n+k}A_1 \cap B_1} \lambda\left((A_2 - f_0^{(q_n)}(T^{-q_n}x) + f^{(-k)}(T^{-q_n}x)) \cap B_2\right) d\mu(x) \\
&= \sum_{k\in\mathbb{Z}} \int_{T^k A_1 \cap T^{-q_n} B_1} \lambda\left((A_2 - f_0^{(q_n)}(x) + f^{(-k)}(x)) \cap B_2\right) d\mu(x).
\end{aligned}$$

By Lemma 4.4 and the rigidity of T along $\{q_n\}$ we have

$$\mu^f\left((T^f)_{q_n} A \cap B\right) - \sum_{k\in\mathbb{Z}} \int_{T^k A_1 \cap B_1} \lambda\left((A_2 - f_0^{(q_n)}(x) + f^{(-k)}(x)) \cap B_2\right) d\mu(x) \to 0.$$

Furthermore, by Lemma 4.1, for each $k \in \mathbb{Z}$,

$$\begin{aligned}&\int_{T^k A_1 \cap B_1} \lambda\Big((A_2 - f_0^{(q_n)}(x) + f^{(-k)}(x)) \cap B_2 \Big)\, d\mu(x) \\ &\quad\to \int_{T^k A_1 \cap B_1} \int_{\mathbb{R}} \lambda\Big((A_2 - t + f^{(-k)}(x)) \cap B_2 \Big)\, dP(t)\, d\mu(x).\end{aligned}$$

Using again Lemma 4.4 and then Lemmas 4.3 and 4.2, we conclude that

$$\begin{aligned}&\mu^f\big((T^f)_{q_n} A \cap B\big) \\ &\quad\to \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} \int_{T^k A_1 \cap B_1} \lambda\Big((A_2 - t + f^{(-k)}(x)) \cap B_2 \Big)\, d\mu(x)\, dP(t) \\ &\quad= \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} \mu \otimes \lambda\big((S_{-f})^k \sigma_{-t} A \cap B \big)\, dP(t) \\ &\quad= \int_{\mathbb{R}} \mu^f\big((T^f)_{-t} A \cap B \big)\, dP(t).\end{aligned}$$

The proof is now complete. □

Remark. The assertion of Proposition 4.1 also holds for the sequence $\{c \cdot q_n\}_{n\geq 1}$ if $\int_X f\, d\mu = c$.

5. *Proof of Theorem 1.1 and other consequences*

Proof of Theorem 1.1. The proof follows directly from Proposition 3.2 (i) and Proposition 4.1. □

We will now derive some other corollaries of Proposition 3.2.

In [**1**] it has been proved that the sequence $\{\|f_0^{(q_n)}\|_{L^2}\}$ is bounded provided that T is an irrational rotation and $f \in L^2(\mathbb{T})$ is a function for which $\hat{f}(n) = \mathrm{O}(1/|n|)$ $(n \in \mathbb{Z})$. We hence obtain the following.

COROLLARY 5.1. *Assume that $Tx = x + \alpha$ is an irrational rotation and $\hat{f}(n) = O(1/|n|)$. Assume moreover that $f(x) \geq c > 0$ a.e. Then the special flow T^f is disjoint from all mixing flows.* □

It turns out that the integral form of the limit joining in Proposition 4.1 allows us to strengthen the assertion of Theorem 1.1 in some special cases.

COROLLARY 5.2. *Under the assumptions of Theorem 1.1, assume additionally that the sequence $\{f_0^{(q_n)}\}$ is bounded in L^∞. Then the special flow T^f is spectrally disjoint from all mixing flows. In particular, all Kočergin's flows with the roof function of bounded variation are spectrally disjoint from all mixing flows.*

Proof. Take $g \in L_0^2(X, \mathcal{B}, \mu)$ such that τ_g is a Rajchman measure (*i.e.* the Fourier transform of τ_g vanishes at ∞). We then have

$$0 = \lim_{n\to\infty} \langle g \circ (T^f)_{q_n}, g \rangle = \int_{\mathbb{R}} \langle g \circ (T^f)_{-s}, g \rangle\, dP(s).$$

Therefore,

$$\int_{\mathbb{R}} \hat{P}(u)\, d\tau_g(u) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{-isu}\, d\tau_g(u) \right) dP(s) = 0.$$

By taking any h in the cyclic space of g and repeating the above reasoning we obtain that $\int_{\mathbb{R}} \hat{P}(u)\, d\nu = 0$ for every finite measure ν absolutely continuous with respect to τ_g. Therefore, $\hat{P}(\cdot) = 0$, τ_g-a.e. On the other hand the measure P has a compact topological support, because the sequence $\{f_0^{(q_n)}\}$ is bounded in L^∞. Thus its Fourier transform extends to an analytic function on the whole complex plane. Since the measure τ_g is continuous we deduce that τ_g must be zero measure, so g is constant equal to zero and the proof of the first part is complete.

If we assume that $Tx = x + \alpha$ is an irrational rotation and f is of bounded variation then f is Riemann integrable and in particular the ergodic theorem in L^∞ holds for f. Therefore all assumptions of the general case are satisfied and the result follows. □

As a direct consequence of Proposition 3.2 (ii) we obtain the following.

COROLLARY 5.3. *Under the assumptions of Proposition 4.1, assume additionally that the limit measure P is not a Dirac measure. Then the special flow T^f is disjoint from all ELF flows. In particular such flows are disjoint from all Gaussian flows.* □

Let us consider now special flows over an irrational rotation $Tx = x + \alpha$. As the analysis in Section 4 of [**20**] shows, for each piecewise absolutely continuous cocycle whose sum of jumps does not vanish, i.e. $\int_{\mathbb{T}} Df(x)dx \neq 0$, the limit measures (along the sequence of denominators of α) P are absolutely continuous. As the proof of Theorem 3 in [**20**] shows, even in case of sufficiently small perturbations (in the variation norm) of the above functions, the limit measures are not discrete. Therefore, such examples give rise to special flows which are disjoint from all Gaussian flows.

If the function f is absolutely continuous, then $\{(f_0^{(q_n)})_*\mu\}$ goes to Dirac measure at zero. However, for some functions $f \in C^{k-1}(\mathbb{T}) \setminus C^k(\mathbb{T})$ and α's satisfying some Diophantine condition the limit measures for the sequence $\{(f_0^{(q_n^{k+1})})_*\mu\}$ (for $k \geq 1$) are not Dirac measures. More precisely, suppose that $\alpha \in \mathbb{T}$ is an irrational number such that

$$\liminf_{n\to\infty} q_n^{k+1}\|q_n\alpha\| = 0.$$

Let us denote by $C_{1,+}^{k+PAC}(\mathbb{T})$ the space of all $(k-1)$–differentiable positive functions $f : \mathbb{T} \to \mathbb{R}$ of integral 1 such that $D^{k-1}f$ is absolutely continuous and $D^k f$ is piecewise absolutely continuous. By passing to a further subsequence if necessary we can assume that $\{q_n^{k+1}\}$ is a rigidity sequence for T (precisely that $q_n^{k+1}\|q_n\alpha\| \to 0$) and the sequence $\{f_0^{(q_n^{k+1})}\}$ is uniformly bounded (see Lemma 2.2.6 in [**22**]). Let us consider the following two subsets of $C_{1,+}^{k+PAC}(\mathbb{T})$. By C_1 denote the set of all $C_{1,+}^{k+PAC}$–functions f such that the sum of jumps of $D^k f$ does not vanish. By

C_2 denote the set of all $C^{k+PAC}_{1,+}$–functions f for which the sum of jumps of $D^k f$ vanishes and such that

$$\lim_{n\to\infty}\{q_n\beta_i\} = \gamma_i \text{ for } i = 1,\ldots,d$$

and $\gamma_i, i = 1,\ldots,d$ are pairwise distinct, where $\beta_i, i = 1,\ldots,d$ are all discontinuities of $D^k f$. Suppose that $f \in C_1 \cup C_2$. Then there exist constants $0 < C < 1$, $M > 0$ and there exists the collection of pairwise disjoint closed intervals $\{J_j^{(n)}\}_{j=0}^{q_n-1}$ such that for $j = 0, ..., q_n - 1$ we have

$$|J_j^{(n)}| \geq \frac{C}{q_n} \text{ and } x \in J_j^{(n)} \Rightarrow |Df_0^{(q_n^{k+1})}(x)| \geq Mq_n$$

(see Lemma 2.2.9 in **[22]** and Corollary 3.2 in **[6]**). By the proof of Theorem 1.1 in **[6]**, it follows that

$$\limsup_{n\to\infty} |\int_{\mathbb{T}} e^{2\pi i l f_0^{(q_n^{k+1})}(x)} dx| \leq c < 1$$

for all l large enough. Now by the proof of Proposition 12 in **[20]**, we conclude that every limit measure of the sequence $\{(f_0^{(q_n^{k+1})})_*\mu\}$ is not a Dirac measure. Consequently, the functions from $C_1 \cup C_2$ give rise to examples of some smooth flows on $\mathbb{T}^2$ (see **[5]**) which are disjoint from Gaussian flows.

We conjecture however that all Kočergin's flows with the roof function of bounded variation are disjoint from all Gaussian flows.

Remark. It follows from a general theory of loosely Bernoulli (LB) transformations and flows (see **[21]**), the result of de la Rue (**[23]**) on the existence of zero entropy Gaussian systems which are LB, and the fact that each Gaussian automorphism is embedable in a measurable flow that for each irrational rotation $Tx = x + \alpha$ we can find $f \in L^1(\mathbb{T})$ so that T^f is a Gaussian flow.

Acknowledgements. The authors would like to thank Jon. Aaronson for fruitful discussions on the subject. In fact, the problems solved in the present paper have been formulated during his visit at N. Copernicus University in July 2002. Moreover, we would like to thank the referee for some remarks that improved the presentation of the results of the paper.

References

[1] J. Aaronson, M. Lemańczyk, C. Mauduit, H. Nakada, *Koksma's inequality and group extensions of Kronecker transformations*, in Algorithms, fractals, and dynamics (Okayama/Kyoto, 1992), 27–50, Plenum, New York, 1995.

[2] Y. Ahn, M. Lemańczyk, *An algebraic property of joinings*, Proc. Amer. Math. Soc. **131** (2003), 1711–1716.

[3] I.P. Cornfeld, S.V. Fomin, Ya.G. Sinai, *Ergodic Theory*, Springer-Verlag, New York, 1982.

[4] B. Fayad, *Analytic mixing reparametrizations of irrational flows*, Ergodic Theory Dynam. Systems **22** (2002), 437–468.

[5] B. Fayad, A.B. Katok, A. Windsor, *Mixed spectrum reparameterization of linear flows on* $\mathbb{T}^2$, Mosc. Math. J. **1** (2001), 521-537.

[6] K. Frączek, *On ergodicity of some cylinder flows*, Fund. Math. **163** (2000), 117-130.

[7] H. Furstenberg, *Disjointness in ergodic theory, minimal sets and diophantine approximation*, Math. Syst. Th. **1** (1967), 1–49.

[8] E. Glasner, *Ergodic Theory via Joinings*, Math. Surveys Monogr. 101, AMS, Providence, 2003.

[9] M. Herman, *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations*, Publ. Mat. IHES **49** (1979), 5–234.

[10] A. del Junco, M. Lemańczyk, *Simple systems are disjoint from Gaussian systems*, Studia Math. **133** (1999), 249–256.

[11] A. del Junco, D. Rudolph, *On ergodic actions whose self-joinings are graphs*, Erg. Th. Dynam. Syst. **7** (1987), 531–557.

[12] K.M. Khanin, Y.G. Sinai, *Mixing of some classes of special flows over rotations of the circle*, Funktsional. Anal. i Prilozhen. **26** (1992), 1–21.

[13] A.V. Kočergin, *On the absence of mixing in special flows over the rotation of a circle and in flows on a two-dimensional torus*, Dokl. Akad. Nauk SSSR **205** (1972), 949–952.

[14] A.V. Kočergin, *Change of velocity in flows and mixing*, Izw. Akad. Nauk SSSR **37** (1973), 1275–1298.

[15] A.V. Kočergin, *On mixing in special flows over interval exchange transformations and in smooth flows on surfaces*, Matem. Sbornik **96** (1975), 471–502.

[16] A.V. Kočergin, *Mixing special flow over a rotation of the circle with almost Lipschitz roof function*, Matem. Sbornik **193** (2002), 51–78.

[17] L. Kuipers, H. Niederreiter, *Uniform Distribution of Sequences*, John Wiley & Sons, New York, 1974.

[18] M. Lemańczyk, *Sur l'absence de mélange pour des flots spéciaux au dessus d'une rotation irrationnelle*, Coll. Math. **84/85** (2000), 29-41.

[19] M. Lemańczyk, F. Parreau, J.-P. Thouvenot, *Gaussian automorphisms whose ergodic self–joinings are Gaussian*, Fund. Math. **164** (2000), 253-293.

[20] M. Lemańczyk, F. Parreau, D. Volný, *Ergodic properties of real cocycles and pseudo–homogeneous Banach spaces*, Trans. Amer. Math. Soc. **348** (1996), 4919-4938.

[21] D.S. Ornstein, D.J. Rudolph, B. Weiss, *Equivalence of measure-preserving transformations*, Mem. Amer. Math. Soc. **37** (1982), no. 262.

[22] D.A. Pask, *Ergodicity of certain cylinder flows*, Isr. J. Math. **76** (1991), 129-152.

[23] T. de la Rue, *Systèmes dynamiques gaussiens d'entropie nulle, lachement et non lachement Bernoulli*, Ergodic Theory Dynam. Systems **16** (1996), 379–404.

[24] V.V. Ryzhikov, *Polymorphisms, joinings and the tensor simplicity of dynamical systems*, Funct. Analysis and its Applications, **31** (1997), 109-118.

[25] K. Schmidt, *Dispersing cocycles and mixing flows under functions*, Fund. Math. **173** (2002), 191-199.

[26] S.M. Srivastava, *A Course on Borel Sets*, Grad. Texts in Math. 180, Springer–Verlag, New York, 1998.

[27] J.-P. Thouvenot, *Some properties and applications of joinings in ergodic theory*, in: Ergodic Theory and its Connections with Harmonic Analysis, London Math. Soc. Lecture Notes Ser. 205, Cambridge Univ. Press, 1995, 207-235.

[28] R. Zimmer, *Extensions of ergodic group actions*, Illinois J. Math. **20** (1976), 373–409.