MIXING AUTOMORPHISMS WHICH ARE MARKOV QUASI-EQUIVALENT BUT NOT WEAKLY ISOMORPHIC

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Abstract. Using Gaussian cocycles over a mixing Gaussian automorphism T, we construct two mixing extensions of T which are Markov quasi-equivalent and are not weakly isomorphic.

1. Introduction

Assume that $(X, \mathcal{B}, \mu)$ is a probability standard Borel space and let T be its automorphism. Then T induces a unitary Koopman operator U_T acting on $L^2(X, \mathcal{B}, \mu)$ by the formula $U_T f = f \circ T$. Note that U_T is an example of a Markov operator (i.e. of a continuous linear operator between L^2-spaces, doubly stochastic and preserving the cone of non-negative functions.

In [12], Vershik introduced the concept of Markov quasi-equivalence (MQ-equiv.) between automorphisms, namely, if T_i is an automorphism of $(X_i, \mathcal{B}_i, \mu_i)$, $i = 1, 2$, then T_1 and T_2 are said to be MQ-equiv. if there are Markov operators

$$\Phi : L^2(X_1, \mathcal{B}_1, \mu_1) \to L^2(X_2, \mathcal{B}_2, \mu_2),$$
$$\Psi : L^2(X_2, \mathcal{B}_2, \mu_2) \to L^2(X_1, \mathcal{B}_1, \mu_1)$$

both with dense range and satisfying

$$\Phi \circ U_{T_1} = U_{T_2} \circ \Phi, \quad \Psi \circ U_{T_2} = U_{T_1} \circ \Psi.$$

The concept of MQ-equiv. is closely related to the notion of joinings and we refer the reader to [2] and [12] for more information on this subject.

We recall also that the MQ-equiv. is related to classical notions equivalence in the theory of dynamical systems in the following manner:

$$\begin{aligned} &\text{Isomorphism} \Rightarrow \text{Weak isomorphism} \\ &\quad \Rightarrow \text{MQ-equiv.} \Rightarrow \text{Spectral isomorphism.} \end{aligned} \tag{1}$$

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Vershik in [12], asked whether MQ-equiv. implies weak isomorphism, and the negative answer was given in [2]. It follows that in (1) no reversed implication holds. The constructions in [2] yield ergodic automorphisms, but since some ideas from [3] are used, the automorphisms considered in [2] are extensions of discrete spectrum automorphisms, in particular they are not weakly mixing.

The aim of the present note is to extend the main result from [2] and provide mixing automorphisms which are MQ-equiv. but not weakly isomorphic. We will use a theory of so called GAG automorphisms developed in [5] (for the general theory of Gaussian automorphisms we refer the reader to [1]) and use Gaussian cocycles [4].

2. Gaussian automorphisms and Gaussian cocycles

We will recall now necessary facts from [4] and [5] needed for the sequel.

Assume that σ is a finite continuous symmetric Borel measure on $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Then, on the space $X_\sigma = \mathbb{R}^{\mathbb{Z}}$ endowed with the natural Borel structure there exists a probability measure μ_σ (called a Gaussian measure) such that the process $(P_n)_{n\in\mathbb{Z}}$ defined by

$$P_n : X_\sigma \to \mathbb{R}, \quad P_n(\omega) = \omega_n \quad \text{for} \quad n \in \mathbb{Z}$$

is a real stationary centered Gaussian process whose spectral measure is σ, i.e.

$$\widehat{\sigma}(n) = \int_{\mathbb{T}} z^n \, d\sigma(z) = \int_{X_\sigma} P_n P_0 \, d\mu_\sigma \quad \text{for all} \quad n \in \mathbb{Z}.$$

If we denote by T_σ the shift transformation on X_σ then the automorphism $T_\sigma : (X_\sigma, \mu_\sigma) \to (X_\sigma, \mu_\sigma)$ is a (standard) Gaussian automorphism with the real Gaussian space

$$H_\sigma = \overline{\operatorname{span}}\{P_n = P_0 \circ T_\sigma^n : n \in \mathbb{Z}\} \subset L^2(X_\sigma, \mu_\sigma).$$

The space H_σ corresponds to the subspace $\mathscr{H}_\sigma$ of $L^2(\mathbb{T}, \sigma)$ consisting of functions g satisfying $g(\overline{z}) = \overline{g(z)}$. In this representation, the action of U_{T_σ} on H_σ is given by $V(g)(z) = zg(z)$, while the variable P_0 corresponds to the constant function $\mathbf{1} = \mathbf{1}_{\mathbb{T}}$. If $g \in \mathscr{H}_\sigma (\simeq H_\sigma)$ is of modulus 1 (a.e.), then it determines a unitary operator W on $L^2(\mathbb{T}, \sigma)$ acting by the formula $W(f)(z) = g(z)f(z)$. Moreover, $W \circ V = V \circ W$. Then, there is a unique extension of W to a unitary operator U_S on $L^2(X_\sigma, \mu_\sigma)$, where $S : (X_\sigma, \mu_\sigma) \to (X_\sigma, \mu_\sigma)$ and S belongs to the Gaussian centralizer $C^g(T_\sigma)$ of T_σ (i.e. the set of all elements of centralizer $C(T_\sigma)$ which preserve the Gaussian space). Because of the continuity of σ, T_σ is ergodic, in fact, weakly mixing.

Following [5], T_σ is called GAG (or σ is a GAG measure) if for each $T_\sigma \times T_\sigma$-invariant and ergodic measure ρ on $\mathbb{R}^{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}}$ with marginals μ_σ

we have all non-zero variables $(\omega,\omega')\mapsto Q(\omega)+Q'(\omega')$ Gaussian whenever $Q,Q'\in H_\sigma$. All Gaussian automorphisms with simple spectrum are GAG (see [5]).

For the theory of cocycles we refer the reader to [10]. Fix T_σ and let G be a second countable locally compact Abelian group. Then each measurable $f:X_\sigma\to G$ is called a cocycle. Such a cocycle is said to be a coboundary if the equation $f=j-j\circ T_\sigma$ has a measurable solution $j:X_\sigma\to G$ (because of ergodicity of T_σ, j is unique up to a constant). Given a cocycle $f:X_\sigma\to G$ we can define the corresponding group extension T_f on $(X_\sigma\times G,\mu_\sigma\otimes\lambda_G)$ (with λ_G a Haar measure on G) by setting

$$T_f(x,g)=(Tx,f(x)+g).$$

Each variable $Q\in H_\sigma$ is called a (real) Gaussian cocycle. A Gaussian cocycle Q is called a Gaussian coboundary if it is a coboundary with $j\in H_\sigma$ [1]. The following result has been proved in [4]:

Proposition 1 ([4]). *Assume that $Q\in H_\sigma$. Then the following conditions are equivalent:*

(i) $Q:X_\sigma\to\mathbb{R}$ *is a coboundary;*
(ii) $Q:X_\sigma\to\mathbb{R}$ *is a Gaussian coboundary;*
(iii) $e^{2\pi iQ}:X_\sigma\to\mathbb{T}$ *is a coboundary;*
(iv) *there exists* $|c|=1$ *such that* $e^{2\pi iQ}=c\cdot\xi/\xi T$ *for some measurable* $\xi:X_\sigma\to\mathbb{T}$.

We will need the following properties of σ:

$$\frac{1}{1-z}\notin L^2(\mathbb{T},\sigma)\ ^2; \tag{2}$$

$$T_\sigma \text{ is mixing GAG.} \tag{3}$$

We describe how the two properties can be achieved. We start with T_η an arbitrary mixing GAG (for example simple spectrum mixing Gaussian) [5], then we translate the spectral measure η so that 1 belongs to the topological support of the translation and then symmetrize the measure to obtain a GAG measure σ_1 (see Proposition 11 in [5]) with 1 in the topological support, and still T_{σ_1} is mixing. In view of Lemma 5 [4] there is $0\neq h\in\mathscr{H}_{\sigma_1}$ so that h is not an $L^2(\mathbb{T},\sigma_1)$-coboundary and finally take $\sigma=|h|^2\sigma_1\ll\sigma_1$. Then 1 is not an $L^2(\mathbb{T},\sigma)$-coboundary, which yields (2). Since $\sigma\ll\sigma_1$, T_σ is both GAG and mixing.

3. Coalescence of two-sided cocycle extensions

Let us fix $T=T_\sigma$ a standard Gaussian automorphism which is GAG (and (2) are assumed to hold); its process representation is denoted

[1] Note that it means that if $f\in\mathscr{H}_\sigma$ corresponds to Q, then $f(z)=\xi(z)-V(\xi)(z)=\xi(z)(1-z)$ for some $\xi\in L^2(\mathbb{T},\sigma)$; equivalently $f(z)/(1-z)\in L^2(\mathbb{T},\sigma)$.

[2] This is equivalent to saying that $\mathbf{1}_\mathbb{T}$ is not an $L^2(\mathbb{T},\sigma)$-coboundary, or that P_0 is not a Gaussian coboundary.

by $(P_n)_{n\in\mathbb{Z}}$ and the Gaussian space $H_\sigma = \overline{\text{span}}\{P_n : n \in \mathbb{Z}\}$. Set $f = P_0$. As in [4], fix α which is a transcendental complex number of modulus 1 and define $W \in U(L^2(\mathbb{T},\sigma))$ by setting $(Wj)(z) = g(z)j(z)$, where $g(z) = \alpha$ on the upper half of the circle and $g(z) = \overline{\alpha}$ otherwise. This isometry extends in a unique way to $S \in C^g(T)$. We will consider now a class of automorphisms which are group extensions of T given by cocycles taking values in $\mathbb{T}^{\mathbb{Z}}$:

$$T_{\dots,i_{-1},i_0,i_1,\dots} := T_{\dots,\exp(2\pi i f\circ S^{i-1}),\exp(2\pi i f\circ S^{i_0}),\exp(2\pi i f\circ S^{i_1}),\dots}. \tag{4}$$

In view of [3] and [4] have the following:

(5) the automorphism (4) is ergodic for arbitrary sequence of integers $(i_k)_{k\in\mathbb{Z}}$, provided that $i_k \neq i_l$ whenever $k \neq l$.

Recall also that in [4] the following has been proved: for all $U \in C^g(T)$, $j \in H_\sigma$, $n_1,\dots,n_t, r \in \mathbb{Z}$ and pairwise distinct integers $p_1,\dots,p_t$

(6) if $n_1 f\circ S^{p_1} + \dots + n_t f\circ S^{p_t} - f\circ S^r \circ U = j - j\circ T$ then $t = 1$ and $n_1 = \pm 1$.

Indeed (the argument from [4]), we rewrite the above as

$$n_1(g(z))^{p_1} + \dots + n_t(g(z))^{p_t} - (g(z))^r u(z) = k(z)(1-z),$$

where $u \in \mathscr{H}_\sigma$ is of modulus 1 (and $k \in \mathscr{H}_\sigma$). If we put $Q(z) = n_1 z^{p_1} + \dots + n_t z^{p_t}$ and $l(z) = Q(g(z)) - (g(z))^r u(z)$ then

$$|l(z)| \geq \big||Q(g(z))| - 1\big| = \big||Q(\alpha)| - 1\big| \quad \text{for all} \quad z \in \mathbb{T}.$$

Suppose that $t \geq 2$ or $t = 1$ with $|n_1| \neq 1$. Since α is transcendental, the modulus of $Q(\alpha)$ cannot be equal to 1. Therefore there is a constant $A > 0$ such that $|l(z)| > A$ (σ–a.e.). Consequently, the function $z \mapsto 1/(1-z) = k(z)/l(z)$ is in $\mathscr{H}_\sigma$. Once more we obtain that P_0 is a coboundary.

Proposition 2. *Assume that $\bar{i} = (i_k)_{k\in\mathbb{Z}}$ is a strictly increasing sequence of integer numbers. If $(i_k)_{k\in\mathbb{Z}}$ is an arithmetic sequence, i.e. the sequence $(i_{k+1} - i_k)_{k\in\mathbb{Z}}$ is constant, then $T_{\bar{i}} = T_{\dots,i_{-1},i_0,i_1,\dots}$ is coalescent, that is, each endomorphism commuting with $T_{\bar{i}}$ is invertible.*

Proof. In view of (5), $T_{\bar{i}}$ is ergodic. Since T is GAG, it is a canonical factor of its group extension [5], therefore if $\widetilde{U} \in C(T_{\bar{i}})$ then

$$\widetilde{U} = U_{\xi,v}, \quad U_{\xi,v}(x,g) = (Ux, v(g)\cdot\xi(x)),$$

where $U \in C^g(T)$, $\xi : X_\sigma \to \mathbb{T}^{\mathbb{Z}}$ is measurable and $v : \mathbb{T}^{\mathbb{Z}} \to \mathbb{T}^{\mathbb{Z}}$ is a continuous algebraic epimorphism (see [7], [8]). Moreover, $v\circ\psi/\psi\circ U = \xi/\xi\circ T$, where

$$\psi = (\dots, \exp(2\pi i f\circ S^{i-1}), \exp(2\pi i f\circ S^{i_0}), \exp(2\pi i f\circ S^{i_1}), \dots).$$

Using Proposition 1 and the form of v we obtain that on each coordinate $r \in \mathbb{Z}$ we must have

$$n_1 f \circ S^{i_{p_1}} + \cdots + n_t f \circ S^{i_{p_t}} - f \circ S^{i_r} \circ U = j_r - j_r \circ T$$

with $n_1, \ldots, n_t \in \mathbb{Z}$, $j_r \in H_\sigma$. By (6), it follows that $t = 1$ and $n_1 = \pm 1$. Therefore, $v\big((z_r)_{r\in\mathbb{Z}}\big) = \big((z^{m_r}_{\pi(r)})_{r\in\mathbb{Z}}\big)$, where $\pi : \mathbb{Z} \to \mathbb{Z}$ and $m_r = \pm 1$ for $r \in \mathbb{Z}$, whence

$$m_r f \circ S^{i_{\pi(r)}} - f \circ S^{i_r} \circ U = j_r - j_r \circ T.$$

Since $S, U \in C^g(T)$, it follows that

$$m_r f \circ S^{i_{\pi(r)} - i_r} - f \circ U = cob.$$

and for $r \neq s$ we obtain that

$$m_r f \circ S^{i_{\pi(r)} - i_r} - m_s f \circ S^{i_{\pi(s)} - i_s} = cob.$$

However, because of ergodicity of $T_{\ldots,j_{-1},j_0,j_1,\ldots}$ for any choice of sequence (j_k) of distinct integer numbers (see (5)) we must have

$$i_{\pi(r)} - i_r = const \quad \text{and} \quad m_r = const.$$

Since the sequence $(i_k)_{k\in\mathbb{Z}}$ is arithmetic, it follows that π is a permutation (translation on $\mathbb{Z}$). Therefore, v is invertible, hence $\widetilde{U} = U_{\xi,v}$ is invertible and the result follows. □

Similar arguments to those above apply to show the following criterion for the isomorphism of skew products of the form $T_{\bar{i}}$.

Proposition 3. *Given two strictly increasing sequences $\bar{i} = (i_k)_{k\in\mathbb{Z}}$ and $\bar{j} = (j_k)_{k\in\mathbb{Z}}$ of integers, the two automorphisms $T_{\bar{i}}$ and $T_{\bar{j}}$ are isomorphic if and only if there exists $m \in \mathbb{Z}$ and a permutation $\pi : \mathbb{Z} \to \mathbb{Z}$ such that $j_{\pi(k)} - i_k = m$ for all $k \in \mathbb{Z}$.*

As an application, consider two extensions $T_{\bar{i}}$, $\bar{i} = (\ldots, -1, 0, 1, 2, \ldots)$ and $T_{\bar{j}}$, $\bar{j} = (\ldots, -1, 0, 2, 3, \ldots)$. They are not isomorphic. Indeed, otherwise there exists $m \in \mathbb{Z}$ and a permutation $\pi : \mathbb{Z} \to \mathbb{Z}$ such that $j_{\pi(k)} = m + i_k = m + k$ for all $k \in \mathbb{Z}$. Therefore, $j_{\pi(-m+1)} = 1$, which is a contradiction.

Remark 1. It has been already noticed in [8] that whenever an automorphism R is coalescent and R is weakly isomorphic to R' then R is isomorphic to R'. By Proposition 2, $T_{\ldots,-1,0,1,2,\ldots}$ is coalescent. It follows that $T_{\ldots,-1,0,1,2,\ldots}$ and $T_{\ldots,-1,0,2,3,\ldots}$ are not weakly isomorphic as well.

Remark 2. Note that not every ergodic automorphism $T_{\ldots,i_{-1},i_0,i_1,\ldots}$ is coalescent. For example, the non-invertible map

$$(x, \underline{z}) \mapsto (S^2 x, \ldots, z_{-1}, z_0, \overset{0}{z_2}, z_3, z_4, \ldots)$$

is an element of the centralizer of $T_{\ldots,-6,-4,-2,0,1,2,3,\ldots}$.

4. Main result

Let T be an ergodic automorphism of $(X,\mathcal{B},\mu)$. We take $\varphi : X \to \mathbb{T}$ so that the group extension T_φ is ergodic. Then assume that we can find S acting on $(X,\mathcal{B},\mu)$, $S\circ T = T\circ S$ (that is, $S \in C(T)$), such that if we set $G = \mathbb{T}^{\mathbb{Z}}$ and define

$$\psi : X \to G, \quad \psi(x) = (\ldots, \varphi(S^{-1}x), \overset{0}{\varphi(x)}, \varphi(Sx), \varphi(S^2x), \ldots)$$

then T_ψ is ergodic as well. Put now $T_1 = T_\psi$ and let us take a factor T_2 of T_1 obtained by "forgetting" the first $\mathbb{T}$-coordinate. In other words on $(X\times\mathbb{T}^{\mathbb{Z}}, \mu\otimes\lambda_{\mathbb{T}^{\mathbb{Z}}})$ we consider two automorphisms

$$T_1(x,\underline{z}) = (Tx, \ldots, z_{-1}\cdot\varphi(S^{-1}x), z_0\cdot\overset{0}{\varphi(x)}, z_1\cdot\varphi(Sx), z_2\cdot\varphi(S^2x), \ldots),$$

$$T_2(x,\underline{z}) = (Tx, \ldots, z_{-1}\cdot\varphi(S^{-1}x), z_0\cdot\overset{0}{\varphi(x)}, z_1\cdot\varphi(S^2x), z_2\cdot\varphi(S^3x), \ldots),$$

where $\underline{z} = (\ldots, z_{-1}, \overset{0}{z_0}, z_1, z_2, \ldots)$. For $n\in\mathbb{Z}$ define $I_n : X\times\mathbb{T}^{\mathbb{Z}} \to X\times\mathbb{T}^{\mathbb{Z}}$ by setting

$$I_n(x,\underline{z}) = (S^nx, \ldots, z_{n-1}, \overset{0}{z_n}, z_{n+2}, z_{n+3}, \ldots).$$

Then I_n is measure-preserving and $I_n\circ T_1 = T_2\circ I_n$. Therefore

$$U_{T_1}\circ U_{I_n} = U_{I_n}\circ U_{T_2} \tag{7}$$

with U_{I_n} being an isometry (which is not onto) and

$$U^*_{I_n}F(x,\underline{z}) = \int_{\mathbb{T}} F(S^{-n}x, \ldots, \overset{0}{z_{-n}}, \ldots, \overset{n}{z_0}, z, z_1, \ldots)\,dz.$$

Denote by $l_0(\mathbb{Z})$ the subspace of $l^2(\mathbb{Z})$ of complex sequences $\bar{x} = (x_n)_{n\in\mathbb{Z}}$ such that $\{n\in\mathbb{Z} :\ x_n \neq 0\}$ is finite.

Proposition 4 ([2])**.** *There exists a nonnegative sequence $\bar{a} = (a_n)_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$ such that $\sum_{n\in\mathbb{Z}} a_n = 1$ and*

$$\text{for every } \bar{x} = (x_n)_{n\in\mathbb{Z}} \in l^2(\mathbb{Z}) \text{ if } \bar{a}*\bar{x} \in l_0(\mathbb{Z}) \text{ then } \bar{x} = \bar{0}. \tag{8}$$

Let $\bar{a} = (a_n)_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$ be a nonnegative sequence such that $\sum_{n\in\mathbb{Z}} a_n = 1$ and (8) holds. Let $J : L^2(X\times\mathbb{T}^{\mathbb{Z}}, \mu\otimes\lambda_{\mathbb{T}^{\mathbb{Z}}}) \to L^2(X\times\mathbb{T}^{\mathbb{Z}}, \mu\otimes\lambda_{\mathbb{T}^{\mathbb{Z}}})$ stand for the Markov operator defined by

$$J = \sum_{n\in\mathbb{Z}} a_n U_{I_n}.$$

In view of (7), J intertwines U_{T_1} and U_{T_2}.

Denote by $Fin = \mathbb{Z}^{\oplus\mathbb{Z}}$ which is naturally identified with the dual of $\mathbb{T}^{\mathbb{Z}}$. Let us consider the following two operations on Fin. For $A = (A_s)_{s\in\mathbb{Z}} \in Fin$ (only finitely many $A_s \neq 0$) we set

$$\widehat{A} = (\widehat{A}_s)_{s\in\mathbb{Z}} = \begin{cases} \widehat{A}_s = A_s & \text{if } \ s \le 0 \\ \widehat{A}_s = A_{s-1} & \text{if } \ s > 1 \\ \widehat{A}_1 = 0 & \end{cases}$$

and given $B = (B_s)_{s\in\mathbb{Z}} \in Fin$ such that $B_1 = 0$ we put

$$\widetilde{B} = (\widetilde{B}_s)_{s\in\mathbb{Z}} = \begin{cases} \widetilde{B}_s = B_s & \text{if} \quad s \le 0 \\ \widetilde{B}_s = B_{s+1} & \text{if} \quad s > 0. \end{cases}$$

Of course,

$$\widetilde{\widehat{A}} = A \quad \text{and} \quad \widehat{\widetilde{B}} = B.$$

For $A = (A_s)_{s\in\mathbb{Z}} \in Fin$ and $n \in \mathbb{Z}$ let

$$A + n = ((A+n)_s)_{s\in\mathbb{Z}},$$

where $(A+n)_s = A_{s-n}$ for $s \in \mathbb{Z}$. We have

$$\left(\widehat{A} + n\right)_{n+1} = \widehat{A}_{n+1-n} = \widehat{A}_1 = 0. \tag{9}$$

Assume that $B = (B_s)_{s\in\mathbb{Z}} \in Fin$ and $B_{n+1} = 0$; then the element

(10) $\widetilde{B-n}$ is the unique element $C \in Fin$ such that $\widehat{C} + n = B$.

Let $\sim$ stand for the equivalence relation in Fin defined by $A \sim B$ if $A = B + n$ for some $n \in \mathbb{Z}$. Denote by Fin_0 a fundamental domain for this relation.

Lemma 5 (cf. [2])**.** *J has trivial kernel.*

Proof. Each $F \in L^2(X \times \mathbb{T}^{\mathbb{Z}}, \mu \otimes \lambda_{\mathbb{T}^{\mathbb{Z}}})$ can be written as

$$F(x, \underline{z}) = \sum_{A\in Fin} f_A(x) A(\underline{z}),$$

where

$$A(\underline{z}) = \Pi_{s\in\mathbb{Z}} z_s^{A_s} \quad \text{whenever} \quad A = (A_s)_{s\in\mathbb{Z}} \quad \text{and} \quad f_A \in L^2(X, \mu).$$

Note that $\sum_{A\in Fin} \|f_A\|^2_{L^2(X,\mu)} = \|F\|^2_{L^2(X\times\mathbb{T}^{\mathbb{Z}},\mu\otimes\lambda_{\mathbb{T}^{\mathbb{Z}}})}$. Since

$$U_{I_n}\left(f_A \otimes A\right)(x, \underline{z}) = \left(f_A \otimes A\right)\left(I_n(x, \underline{z})\right) = f_A(S^n x)(\widehat{A} + n)(\underline{z}),$$

we have

$$JF(x, \underline{z}) = \sum_{n\in\mathbb{Z}} \sum_{A\in Fin} a_n f_A(S^n x)(\widehat{A} + n)(\underline{z}).$$

By (9), $(\widehat{A} + n)_{n+1} = 0$, so by changing "the index": substituting $\widehat{A} + n =: B$ and using (10) (from which it follows that $A = \widetilde{B-n}$) we obtain

$$JF(x, \underline{z}) = \sum_{B\in Fin} \sum_{n\in\mathbb{Z}, B_{n+1}=0} a_n f_{\widetilde{B-n}}(S^n x) B(\underline{z}) = \sum_{B\in Fin} \widetilde{F}_B(x) B(\underline{z}),$$

where $\widetilde{F}_B(x) = \sum_{n\in\mathbb{Z}, B_{n+1}=0} a_n f_{\widetilde{B-n}}(S^n x)$. For every $B \in Fin_0$ and $x \in X$ we define $\xi^B(x) = (\xi^B_n(x))_{n\in\mathbb{Z}}$ by setting

$$\xi^B_{-n}(x) = \begin{cases} f_{\widetilde{B-n}}(S^n x) & \text{if} \quad B_{n+1} = 0 \\ 0 & \text{if} \quad B_{n+1} \neq 0. \end{cases}$$

Therefore, for $k \in \mathbb{Z}$

$$\begin{aligned}\widetilde{F}_{B+k}(x) &= \sum_{n\in\mathbb{Z},(B+k)_{n+1}=0} a_n f_{\widetilde{B-n+k}}(S^n x)\\ &= \sum_{n\in\mathbb{Z},B_{(n-k)+1}=0} a_n f_{\widetilde{B-(n-k)}}(S^{-(k-n)}(S^k x))\\ &= \sum_{n\in\mathbb{Z}} a_n \xi^B_{k-n}(S^k x) = [\bar{a} * \big(\xi^B(S^k x)\big)]_k.\end{aligned}$$

Suppose that $J(F) = 0$. It follows that for all $k \in \mathbb{Z}$ and $B \in Fin_0$ we have $[\bar{a} * \big(\xi^B(S^k x)\big)]_k = \widetilde{F}_{B+k}(x) = 0$ for μ-a.e. $x \in X$, whence a.s. we also have $[\bar{a} * \big(\xi^B(x)\big)]_k = 0$. Letting k run through $\mathbb{Z}$ we obtain that $\bar{a} * \big(\xi^B(x)\big) = \bar{0}$ for μ-a.e. $x \in X$. On the other hand $\xi^B(x) \in l^2(\mathbb{Z})$ for almost every $x \in X$. In view of (8), $\xi^B(x) = \bar{0}$ for every $B \in Fin_0$ and for a.e. $x \in X$, hence $f_{\widetilde{A}} = 0$ for every $A \in Fin$ with $A_1 = 0$. It follows that $f_A = 0$ for every $A \in Fin$, consequently $F = 0$. □

Lemma 6 (cf. [2]). *J^* has trivial kernel.*

Proof. Let

$$F(x, \underline{z}) = \sum_{A\in Fin} f_A(x)A(\underline{z}).$$

Then

$$U^*_{I_n}(f_A \otimes A)(x, \underline{z}) = f_A(S^{-n}x)\int_{\mathbb{T}} A(\dots, z_{-n}, \dots, \overset{n}{z_0}, \overset{n+1}{z}, \overset{n+2}{z_1}, \dots)\, dz.$$

It follows that

$$U^*_{I_n}(f_A \otimes A)(x, \underline{z}) = \begin{cases} f_A(S^{-n}x)\widetilde{A-n}(\underline{z}) & \text{if} \quad A_{n+1} = 0\\ 0 & \text{if} \quad A_{n+1} \neq 0.\end{cases}$$

It follows that

$$\begin{aligned}J^*F(x, \underline{z}) &= \sum_{A\in Fin}\sum_{n\in\mathbb{Z},A_{n+1}=0} a_n f_A(S^{-n}x)\widetilde{A-n}(\underline{z})\\ &= \sum_{B\in Fin}\sum_{n\in\mathbb{Z}} a_n f_{\widehat{B}+n}(S^{-n}x)B(\underline{z})\\ &= \sum_{A\in Fin,A_1=0}\sum_{n\in\mathbb{Z}} a_n f_{A+n}(S^{-n}x)\widetilde{A}(\underline{z}).\end{aligned}$$

Furthermore,

$$\begin{aligned}J^*F(x, \underline{z}) &= \sum_{A\in Fin_0}\sum_{k\in\mathbb{Z},(A-k)_1=0}\sum_{n\in\mathbb{Z}} a_n f_{A+n-k}(S^{-n}x)\widetilde{A-k}(\underline{z})\\ &= \sum_{A\in Fin_0}\sum_{k\in\mathbb{Z},(A-k)_1=0} [\bar{a} * \big(\zeta^A(S^{-k}x)\big)]_k\widetilde{A-k}(\underline{z}),\end{aligned}$$

where $\zeta^A(x) = (\zeta^A_l(x))_{l\in\mathbb{Z}}$ is given by $\zeta^A_l(x) = f_{A-l}(S^l x)$.

Suppose that $J^*(F) = 0$. It follows that $[\bar{a} * \zeta^A(S^{-k}x)]_k = 0$ for every $A \in Fin_0$ and $k \in \mathbb{Z}$ with $A_{k+1} = 0$ and for a.e. $x \in X$. Hence $\bar{a} * \left(\zeta^A(x)\right) \in l_0(\mathbb{Z})$ for μ-a.e. $x \in X$ (the only possibly non-zero terms of the convolved sequence have indices belonging to $\{s \in \mathbb{Z} : (A-1)_s \neq 0\}$). Since $\zeta^A(x) \in l^2(\mathbb{Z})$, in view of (8), $\zeta^A(x) = \overline{0}$ for every $A \in Fin_0$ and for μ-a.e. $x \in X$. Thus $f_A = 0$ for all $A \in Fin$ and consequently $F = 0$. □

Theorem 7. *Automorphisms $T_{\dots,-1,0,1,2,\dots}$ and $T_{\dots,-1,0,2,3,\dots}$ are mixing and Markov quasi-equivalent but are not weakly isomorphic.*

Proof. By assumption (3), T is mixing. In view of (5) both its skew product extensions $T_{\dots,-1,0,1,2,\dots}$ and $T_{\dots,-1,0,2,3,\dots}$ are ergodic, hence they are also mixing. By Lemmas 5 and 6, there exists an operator with dense range and trivial kernel intertwining the Koopman operators associated to $T_{\dots,-1,0,1,2,\dots}$ and $T_{\dots,-1,0,2,3,\dots}$. It follows that $T_{\dots,-1,0,1,2,\dots}$ and $T_{\dots,-1,0,2,3,\dots}$ are Markov quasi-equivalent. Finally, by Remark 1, they are not weakly isomorphic. □

Remark 3. Since a Gaussian mixing automorphism is mixing of all orders (see [6]), from the result of Rudolph about multiple mixing of isometric extensions (see [9]), it follows that automorphisms $T_{\dots,-1,0,1,2,\dots}$ and $T_{\dots,-1,0,2,3,\dots}$ are also mixing of all orders.

Remark 4. In Section 2 the measure σ was chosen to satisfy (2) and (3). Here is another way of specifying it. For a mixing GAG T_η let $\sigma = \eta * \eta$. Then T_σ is also both mixing and GAG (the latter is unpublished result of F. Parreau). Since the Fourier coefficients of σ are non-negative, $T_{e^{2\pi i P_0}}$ has countable Lebesgue spectrum in the orthocomplement of $L^2(X_\sigma, \mu_\sigma)$ (see Corollary 4 in [4]). Hence P_0 is not a Gaussian coboundary and the conditions (2) and (3) hold. Moreover, $\|P_0^{(n)}\|^2_{L^2(X_\sigma,\mu_\sigma)}$ grows linearly with $|n|$ (where $P_0^{(1)} = P_0$, $P_0^{(n+1)} = P_0^{(n)} + P_0 \circ T^n$ for all $n \in \mathbb{Z}$). Therefore using the same arguments as in [11, Lemma 4.2] we obtain automorphisms $T_{\dots,-1,0,1,2,\dots}$ and $T_{\dots,-1,0,2,3,\dots}$ in Theorem 7 with countable Lebesgue spectrum in the orthocomplement of $L^2(X_\sigma, \mu_\sigma)$.

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