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# ON THE SELF-SIMILARITY PROBLEM FOR GAUSSIAN-KRONECKER FLOWS

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Abstract. It is shown that a countable symmetric multiplicative subgroup $G = -H \cup H$ with $H \subset \mathbb{R}_+^*$ is the group of self-similarities of a Gaussian-Kronecker flow if and only if $H$ is additively $\mathbb{Q}$-independent. In particular, a real number $s \neq \pm 1$ is a scale of self-similarity of a Gaussian-Kronecker flow if and only if $s$ is transcendental. We also show that each countable symmetric subgroup of $\mathbb{R}^*$ can be realized as the group of self-similarities of a simple spectrum Gaussian flow having the Foiaş-Stratila property.

## 1. Introduction

Assume that $\mathcal{T} = (T_t)_{t\in\mathbb{R}}$ is a (measurable) measure-preserving flow acting on a standard probability Borel space $(X, \mathcal{B}, \mu)$. Given $s \in \mathbb{R}^*$, one says that it is a *scale of self-similarity* of $\mathcal{T}$ if $\mathcal{T}$ is isomorphic to $\mathcal{T}_s := (T_{st})_{t\in\mathbb{R}}$. Denote by $I(\mathcal{T})$ the set of all scales of self-similarities of $\mathcal{T}$. Then $\mathcal{T}$ is called *self-similar* if $I(\mathcal{T}) \neq \{\pm 1\}$. Classical examples of self-similar flows are given by horocycle flows where $I(\mathcal{T})$ equals either $\mathbb{R}^*$ or $\mathbb{R}_+^*$ [19]. A systematic study of the problem of self-similarity has been done recently in [4] and [6]. In particular, $I(\mathcal{T})$ turns out to be a multiplicative subgroup of $\mathbb{R}^*$ ([6]) which is Borel ([4]), and one of the main problems in this domain is to classify all Borel subgroups of $\mathbb{R}^*$ that may appear as groups of self-similarities of ergodic flows; see also [3], [13], [24], [25] for a recent contribution to other aspects of the problem of self-similarity of ergodic flows. From this point of view the subclass of so called GAG flows [17][1] of the class of Gaussian flows is especially attractive since self-similarities appear there as natural invariants, see (1.1) below. By definition, GAG flows are those Gaussian flows whose ergodic self-joinings remain Gaussian. All Gaussian flows with simple spectrum are GAG flows [17]. If $\mathcal{T}^\sigma = (T_t^\sigma)_{t\in\mathbb{R}}$ denotes the Gaussian flow determined by a finite positive (continuous) measure $\sigma$ on $\mathbb{R}_+$ and the flow is GAG then

(1.1) $\qquad I(\mathcal{T}^\sigma)$ is equal to the (multiplicative) group $-I(\sigma) \cup I(\sigma)$,

where $I(\sigma) = \{s \in \mathbb{R}_+^* : \sigma_s \equiv \sigma\}$ and $\sigma_s = (R_s)_*(\sigma)$ denotes the image of $\sigma$ via the map $R_s : x \mapsto sx$ [17]. Recall that $-1$ is always a scale of self-similarity for a Gaussian flow.

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[1]In [15] as well as in [17] only Gaussian automorphisms are considered, however all results can be rewritten for Gaussian flows.

In this note we focus on the problem of self-similarities in some subclasses of simple spectrum Gaussian flows. We first recall already known results. Classically, if $\sigma$ is concentrated on an additively $\mathbb{Q}$-independent Borel set $A \subset \mathbb{R}_+$ then the Gaussian flow $\mathcal{T}^\sigma$ has simple spectrum, see [2]. Moreover, the subgroup $H := I(\mathcal{T}^\sigma) \cap \mathbb{R}_+^*$ is an additively $\mathbb{Q}$-independent set. Indeed, suppose that $H$ is not an additively $\mathbb{Q}$-independent set. That is, for some distinct $h_1, \ldots, h_m \in H$ we have

$$\sum_{i=1}^{m} k_i h_i = 0 \quad \text{with} \quad k_i \in \mathbb{Z},\ i = 1, \ldots, m \quad \text{and} \quad \sum_{i=1}^{m} k_i^2 > 0. \tag{1.2}$$

Denote by $H_0 \subset H$ the multiplicative subgroup generated by $h_1, \ldots, h_m$. Since $H_0 \subset I(\mathcal{T}^\sigma)$, we have $\sigma_h \equiv \sigma$ for $h \in H_0$, thus the Borel set $B = \bigcap_{h \in H_0} hA$ has full $\sigma$-measure, is $\mathbb{Q}$-independent, and is literally $H_0$-invariant. Take any non-zero $x \in B$. Then the elements $h_i x \in B$, $i = 1, \ldots, m$, are distinct. Now, (1.2) yields

$$\sum_{i=1}^{m} k_i (h_i x) = x \sum_{i=1}^{m} k_i h_i = 0,$$

so $B$ is not independent, a contradiction. On the other hand, in [6], it is shown that whenever a countable group $H \subset \mathbb{R}_+^*$ satisfies:

(1.3) For each polynomial $P \in \mathbb{Q}[x_1, \ldots, x_m]$ if there is a collection of distinct elements $h_1, \ldots, h_m$ in $H$ such that $P(h_1, \ldots, h_m) = 0$ then $P \equiv 0$,

then there exists a probability $\sigma$ concentrated on a Borel $\mathbb{Q}$-independent set such that $I(\mathcal{T}^\sigma) = -H \cup H$. It is not difficult to see that the condition (1.3) is equivalent to saying that $H$ is an additively $\mathbb{Q}$-independent set.

**Theorem 1.1** ([6]). *Assume that $G = -H \cup H$, where $H \subset \mathbb{R}_+^*$ is a countable multiplicative subgroup. Then $G$ can be realized as $I(\mathcal{T}^\sigma)$ for a Gaussian flow whose spectral measure $\sigma$ is concentrated on a Borel $\mathbb{Q}$-independent set if and only if $H$ is an additively $\mathbb{Q}$-independent set.*

Note that for $H$ cyclic generated by $s \in \mathbb{R}_+$, the $\mathbb{Q}$-independence condition is equivalent to saying that $s$ is transcendental. Hence, by Theorem 1.1, a real number $s$ can be realized as a scale of self-similarity of a Gaussian flow whose spectral measure is concentrated on a $\mathbb{Q}$-independent Borel set if and only if $s$ is transcendental.

On the other hand, there are no restrictions on $H$ in the class of all Gaussian flows having simple spectrum.

**Theorem 1.2** ([4]). *For each countable subgroup $H \subset \mathbb{R}_+^*$ there exists a simple spectrum Gaussian flow $\mathcal{T}^\sigma$ such that $I(\mathcal{T}^\sigma) = -H \cup H$.*

Note that, in particular, the above result of Danilenko and Ryzhikov brings the positive answer to the open problem [14] of existence of Gaussian flows $\mathcal{T}^\sigma$ with simple spectrum such that $\sigma$ is not concentrated on a $\mathbb{Q}$-independent set; indeed, whenever $H$ is not an additively $\mathbb{Q}$-independent set, by Theorem 1.1, the spectral measure $\sigma$ resulting from Theorem 1.2 cannot be concentrated on a Borel $\mathbb{Q}$-independent set. See also [3] for constructions of Gaussian flows with zero entropy and having uncountable groups of self-similarities.

Our aim is to continue investigations on realization of countable subgroups as the groups of self-similarities in further restricted classes of Gaussian flows whose spectral measures are classical from the harmonic analysis point of view. Recall some basic notions. For every $s \in \mathbb{R}$ let $\xi_s : \mathbb{R} \to \mathbf{S}^1$ be given by $\xi_s(t) = \exp(2\pi i s t)$. A finite positive Borel measure $\sigma$ on $\mathbb{R}$ is called Kronecker if for each $f \in L^2(\mathbb{R}, \sigma)$, $|f| = 1$ $\sigma$-a.e., there exists a sequence $(t_n) \subset \mathbb{R}$, $t_n \to \infty$, such that

$$\xi_{t_n} \to f \quad \text{in} \quad L^2(\mathbb{R}, \sigma). \tag{1.4}$$

Each measure $\sigma$ concentrated on a Kronecker set [12], [18] is a Kronecker measure. Indeed, Kronecker sets are compact subsets of $\mathbb{R}$ on which each continuous function of modulus one is a uniform limit of characters. Kronecker sets are examples of $\mathbb{Q}$-independent sets [18]. In general, as shown in [15], a Kronecker measure is concentrated on a Borel set which is the union of an increasing sequence of Kronecker sets, hence a Kronecker measure is concentrated on a Borel $\mathbb{Q}$-independent set, and the restriction on $H$ in Theorem 1.1 applies. This turns out to be the only restriction as the main result of the note shows.

**Theorem 1.3.** *Assume that $G = -H \cup H$, where $H \subset \mathbb{R}^*_+$ is a countable multiplicative subgroup. Then $G$ can be realized as $I(\mathcal{T}^\sigma)$* [2] *for a Gaussian-Kronecker flow if and only if $H$ is an additively $\mathbb{Q}$-independent set. In particular, $h \in \mathbb{R}_+$ can be a scale of self-similarity for a Gaussian-Kronecker flow if and only if $h$ is transcendental.*

An extremal case when two dynamical systems are non-isomorphic is the disjointness in the Furstenberg sense [7], see also [9], [11], [14], [23] for disjointness results in ergodic theory. We would like also to emphasize that the notion of disjointness turned out to be quite meaningful in the problem of non-correlation with the Möbius function of sequences of dynamical origin [1]: we need that an automorphism $T$ has the property that $T^p$ and $T^q$ are disjoint for any two different primes. In connection with that we will prove the following.

**Theorem 1.4.** *Assume that $\mathcal{T}^\sigma = (T^\sigma_t)_{t\in\mathbb{R}}$ is a Gaussian-Kronecker flow. If $s \in \mathbb{Q}\setminus\{\pm 1\}$ then $T^\sigma_s$ is disjoint from $T^\sigma_1$. For every Gaussian-Kronecker automorphism $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ the iterations $T^n$, $T^m$ are disjoint for any two distinct natural numbers $n, m$.*

*If $s$ is irrational then there exists a Gaussian-Kronecker flow $\mathcal{T}^\sigma$ such that $T^\sigma_s$ and $T^\sigma_1$ have a non-trivial common factor.*

An importance of Kronecker measures in ergodic theory follows from the following remarkable result of Foiaş and Stratila [5] (see also [2], and remarks on that result in [15] and [21]):

(1.5)
If $(S_t)_{t\in\mathbb{R}}$ is an ergodic flow of a standard probability
Borel space $(Y, \mathcal{C}, \nu)$, $f \in L^2(Y, \mathcal{C}, \nu)$ is real and the spectral measure
$\sigma_f$ of $f$ is the symmetrization of a Kronecker measure,
then the (stationary) process $(f \circ S_t)_{t\in\mathbb{R}}$ is Gaussian.

In [15], any measure $\sigma$ satisfying the assertion (1.5) of Foiaş-Stratila theorem is called an FS measure. Each Kronecker measure is a Dirichlet measure[3] [18], but as

[2] In a sense, we can also control the flows $\mathcal{T}^{\sigma_s}$ for $s \notin -H \cup H$; we will prove their disjointness from $\mathcal{T}^\sigma$, see the proof of this theorem.

[3] A probability Borel measure $\sigma$ on $\mathbb{R}$ is Dirichlet, if (1.4) is satisfied for $f = 1$. From the dynamical point of view, Dirichlet measures correspond to rigidity: a flow $\mathcal{T}$ is *rigid* if $T^{t_n} \to Id$ for some $t_n \to \infty$.

shown in [15], there are FS measures which are not Dirichlet measures (see Figure 1). Moreover, in [21], it is announced that each continuous measure concentrated on

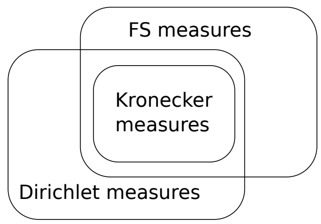


FIGURE 1. Different classes of measures

an independent Helson[4] set is a Kronecker measure (for some examples in [21], the resulting Gaussian flows have no non-trivial rigid factors). We will strenghten Theorem 1.2 to the following result.

**Theorem 1.5.** *Any symmetric countable group $G \subset \mathbb{R}^*$ can be realized as the group of self-similarities of a simple spectrum Gaussian flow $\mathcal{T}^\sigma$ with $\sigma$ being an FS measure.*

In particular, in connection with the forementioned question from [14], there is an FS measure for which the Gaussian flow has simple spectrum but $\sigma$ is not concentrated on a $\mathbb{Q}$-independent set. These are apparently the first examples of FS measures which are not concentrated on $\mathbb{Q}$-independent Borel sets but yield Gaussian flows with simple spectrum (cf. [15] and [21]).

At the end of the note we will discuss self-similarity properties of Gaussian flows arising from a "typical" measure or from the maximal spectral types of a "typical" flow (cf. the disjointness results from [4]).

**Theorem 1.6.** *Assume that $0 \leq a < b$. For a "typical" $\sigma \in \mathcal{P}([a,b])$ the flow $\mathcal{T}^\sigma$ is Gaussian-Kronecker such that for each $|r| \neq |s|$ the flows $\mathcal{T}^{\sigma_r}$ and $\mathcal{T}^{\sigma_s}$ are disjoint. In particular $I(\mathcal{T}^\sigma) = \{\pm 1\}$.*

*For a "typical" flow $\mathcal{T}$ of a standard probability Borel space $(X, \mathcal{B}, \mu)$, for its maximal spectral type $\sigma_\mathcal{T}$ we have: $\mathcal{T}^{\sigma_\mathcal{T}|_{\mathbb{R}_+}}$ has simple spectrum and for $|r| \neq |s|$ the flows $\mathcal{T}^{(\sigma_\mathcal{T}|_{\mathbb{R}_+})_r}$ and $\mathcal{T}^{(\sigma_\mathcal{T}|_{\mathbb{R}_+})_s}$ are disjoint. In particular $I(\mathcal{T}^{\sigma_\mathcal{T}|_{\mathbb{R}_+}}) = \{\pm 1\}$.*

## 2. NOTATION AND BASIC RESULTS

Assume that $\mathcal{T} = (T_t)_{t\in\mathbb{R}}$ is a measurable[5] measure-preserving flow acting on a standard probability Borel space $(X, \mathcal{B}, \mu)$. It then induces a (continuous) one-parameter group of unitary operators acting on $L^2(X, \mathcal{B}, \mu)$ by the formula $T_t f = f \circ T_t$. By Bochner's theorem, the function $t \mapsto \int_X T_t f \cdot \overline{f}\, d\mu$ determines the so called *spectral measure* $\sigma_f$ of $f$ for which $\widehat{\sigma}_f(t) = \int_X T_t f \cdot \overline{f}\, d\mu$, $t \in \mathbb{R}$. Usually, one only considers spectral measures of $f \in L^2_0(X, \mathcal{B}, \mu)$, that is, of elements with zero mean (the spectral measure of the constant function $c$ is equal to $|c|^2\delta_0$). Then $\sigma_f$ is a finite positive Borel measure on $\mathbb{R}$. Among spectral measures there are maximal ones with respect to the absolute continuity relation. Each such maximal

[4] $A \subset \mathbb{R}$ is called Helson if for some $\delta > 0$ and each complex Borel measure $\kappa$ concentrated on $A$ the $\sup_{t\in\mathbb{R}} \left|\int_\mathbb{R} e^{2\pi i t x}\, d\kappa(x)\right|$ is bounded away from the $\delta$-fraction of the total variation of $\kappa$.

[5] Measurability means that for each $f \in L^2(X, \mathcal{B}, \mu)$ the map $t \mapsto f \circ T_t$ is continuous.

measure is called a *maximal spectral type measure* and, by some abuse of notation, it will be denoted by $\sigma_{\mathcal{T}}$. We refer the reader to [11] and [14] for some basics about spectral theory of unitary representations of locally compact Abelian groups in the dynamical context.

Assume that $\mathcal{T}$ is ergodic and let $\mathcal{S} = (S_t)_{t\in\mathbb{R}}$ be another ergodic flow (acting on $(Y, \mathcal{C}, \nu)$). Any probability measure $\rho$ on $(X \times Y, \mathcal{B} \otimes \mathcal{C})$ which is $(T_t \times S_t)_{t\in\mathbb{R}}$-invariant and has marginals $\mu$ and $\nu$ respectively, is called a *joining* of $\mathcal{T}$ and $\mathcal{S}$. If, additionally, the flow $((T_t\times S_t)_{t\in\mathbb{R}}, \rho)$ is ergodic then $\rho$ is called an *ergodic joining* [6]. The ergodic joinings are extremal points in the simplex of all joinings. If the set of joinings is reduced to contain only the product measure then one speaks about *disjointness* of $\mathcal{T}$ and $\mathcal{S}$ [7] and we will write $\mathcal{T} \perp \mathcal{S}$. Similar notions appear when one considers automorphisms. Note that whenever for some $t \neq 0$, $T_t \perp S_t$ then $\mathcal{T} \perp \mathcal{S}$. Note also that whenever

$$\mathcal{T} \text{ is weakly mixing then } \mathcal{T} \perp \mathcal{S} \text{ if and only if } T_1 \perp S_1. \tag{2.1}$$

Indeed, if $T_1 \not\perp S_1$ then there exists an ergodic joining $\rho$ between them different than the product measure. Then, $\rho\circ(T_r\times S_r)$ for $0 \le r < 1$ has the same properties. By disjointness of $\mathcal{T}$ and $\mathcal{S}$, $\int_0^1 \rho \circ (T_r \times S_r)\, dr = \mu \otimes \nu$. But $T_1$ is weakly mixing, so $\mu\otimes\nu$ is an ergodic joining of $T_1$ and $S_1$, and therefore $\rho\circ(T_r \times S_r) = \mu\otimes\nu$. We refer the reader to [9] for the theory of joinings in ergodic theory.

A flow $\mathcal{T}$ is called Gaussian if there is a $\mathcal{T}$-invariant subspace $\mathcal{H} \subset L^2_0(X, \mathcal{B}, \mu)$ of the zero mean real-valued functions such that all non-zero variables from $\mathcal{H}$ are Gaussian and the smallest $\sigma$-algebra making all these variables measurable equals $\mathcal{B}$. A Gaussian flow is ergodic if and only if the maximal spectral type on $\mathcal{H}$ is continuous (and then $\mathcal{T}$ is weakly mixing). Since Gaussian variables are real, it is not hard to see that their spectral measures are symmetric, that is, for $f \in \mathcal{H}$, $\sigma_f$ is invariant under the map $R_{-1} : x \mapsto -x$.

A standard way to obtain a (weakly mixing) Gaussian flow is to start with a finite positive continuous Borel measure $\sigma$ on $\mathbb{R}_+$. Consider the symmetrization $\widetilde{\sigma} = \sigma + (R_{-1})_*\sigma$ [7]. We let $\mathcal{V} = (V_t)_{t\in\mathbb{R}}$ denote the one-parameter group of unitary operators on $L^2(\mathbb{R}, \widetilde{\sigma})$ defined by $V_t(f)(x) = e^{2\pi itx}f(x)$. Then the correspondence

$$f(x) \mapsto f(-x) \tag{2.2}$$

yields the unitary conjugation of $\mathcal{V}$ and its inverse. Let $(X, \mathcal{B}, \mu)$ be a *Gaussian probability space*, that is, a standard probability space together with an infinite dimensional, closed, real and $\mathcal{B}$-generating subspace $\mathcal{H} \subset L^2(X, \mathcal{B}, \mu)$ whose all non-zero variables are Gaussian. We then consider $\mathcal{H} + i\mathcal{H}$, so called *complex Gaussian space*, and define an isomorphic copy of $\mathcal{V}$ on it. It is then standard to show (see e.g. [17], Section 2) that $\mathcal{V}$ has a unique extension to a (measurable) flow $\mathcal{T}^\sigma = (T^\sigma_t)$ of $(X, \mathcal{B}, \mu)$ for which $U_{T^\sigma_t}|_{\mathcal{H}+i\mathcal{H}} = V_t$, $t \in \mathbb{R}$. By the same token, the correspondence (2.2) extends to an isomorphism of $(X, \mathcal{B}, \mu)$ which conjugates the Gaussian flow and its inverse $(T^\sigma_{-t})_{t\in\mathbb{R}}$.

A Gaussian flow $\mathcal{T}^\sigma$ is called Gaussian-Kronecker (FS resp.) if $\sigma$ is a continuous Kronecker (FS resp.) measure. Following [17], a Gaussian flow $\mathcal{T}^\sigma$ (with the

[6] If $\mathcal{T} = \mathcal{S}$ then we speak about *self-joinings*.

[7] In general, when $f$ is a measurable map from $(X, \mathcal{B})$ to $(Y, \mathcal{C})$ and $\kappa$ is a probability measure on $X$ then $f_*(\kappa)$ is the measure on $Y$ defined by $f_*(\kappa)(C) = \kappa(f^{-1}(C))$.

Gaussian space $\mathcal{H}$) is called GAG if for each its ergodic self-joining $\eta$ the space

$$\{f(x)+g(y) : f,g \in \mathcal{H}\}$$

consists solely of Gaussian variables (the flow $(T_t^\sigma \times T_t^\sigma)_{t\in\mathbb{R}}$ is then a Gaussian flow as well). We have [17]

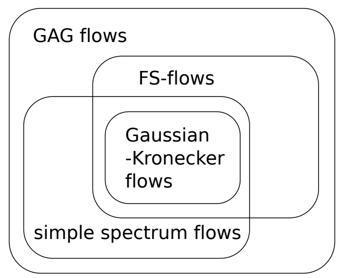


FIGURE 2. Different subclasses of GAG flows

For all these classes of flows we have that if $\mathcal{T}^\sigma$ is in the class, so is $\mathcal{T}^{\sigma_s}$ for $s \neq 0$.

In general, Gaussian flows given by equivalent measures are isomorphic. It follows from [17] that any isomorphism between a GAG flow $\mathcal{T}^\sigma$ and another Gaussian flow $\mathcal{T}^\nu$ is entirely determined by a unitary isomorphism of restrictions of the unitary actions $(T_t^\sigma)_{t\in\mathbb{R}}$ and $(T_t^\nu)_{t\in\mathbb{R}}$ to their Gaussian subspaces. That is, in the GAG situation, $\mathcal{T}^\sigma$ are $\mathcal{T}^\nu$ are isomorphic if and only if $\sigma \equiv \nu$. If we apply that to $\sigma$ and $\sigma_s$ for $s \in \mathbb{R}_+$ we will immediately get (1.1) to hold (in the GAG case).

We will now prove the following.

**Proposition 2.1.** *Assume that $\mathcal{T}^\sigma$ is GAG. Fix $s \neq 0$. Then the sets of self-joinings of $\mathcal{T}^\sigma$ and of self-joinings of $T_s^\sigma$ are the same. (Hence ergodic self-joinings are also the same.) In particular, the factors and the centralizer of the flow and of the time $s$-automorphism are the same.*

*Proof.* This follows from the proof of Theorem 6.1 in [10] which asserts that such an equality of the sets of self-joinings takes place whenever each ergodic self-joining of the flow is an ergodic self-joining for the time-$s$ automorphism. In the GAG case, by definition, such ergodic joinings for the flow $\mathcal{T}^\sigma$ are Gaussian joinings, so they are automatically ergodic for the $T_s^\sigma$ [17]. □

**Corollary 2.2.** *Assume that $\mathcal{T}^\sigma$ is GAG. Then $T_s^\sigma$ is a GAG automorphism for each $s \neq 0$.*

We will also make use of the following results.

**Theorem 2.3** ([17]). *Assume that $\mathcal{T}^\sigma$ is GAG and let $\mathcal{T}^\eta$ be an arbitrary Gaussian flow. Then $\mathcal{T}^\sigma \perp \mathcal{T}^\eta$ if and only if $\widetilde{\sigma} \perp \widetilde{\eta} * \delta_r$ for each $r \in \mathbb{R}$.*

**Proposition 2.4** ([15]). *If $\sigma_1$ and $\sigma_2$ are measures with the FS property and $\mathcal{T}^{\sigma_1} \perp \mathcal{T}^{\sigma_2}$ then $\sigma = \frac{1}{2}(\sigma_1+\sigma_2)$ is also an FS measure.*

## 3. Auxiliary lemmas

Given a compact subset $X \subset \mathbb{R}$ denote by $\mathcal{P}(X)$ the set of all Borel probability measures concentrated on $X$ endowed with the usual weak topology which is compact and metrizable: if $\{f_n : n \geq 1\}$ is a dense set in $C(X)$ then

$$d(\sigma, \eta) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\left|\int f_n \, d\sigma - \int f_n \, d\eta\right|}{1 + \left|\int f_n \, d\sigma - \int f_n \, d\eta\right|} \tag{3.1}$$

defines a metric compatible with the weak topology. Denote $\mathcal{U}(X) = \{f \in C(X) : |f| = 1\}$ which is a closed subset of $C(X)$ in the uniform topology, in particular $\mathcal{U}(X)$ is a Polish space.

**Lemma 3.1.** *Assume that $X = [a, b]$. Let $\{h_0, h_1, \ldots, h_m\} \subset \mathbb{R}^*$ be a $\mathbb{Q}$-independent set. Then for each $f \in \mathcal{U}\left(\bigcup_{j=0}^m h_j X\right)$ and $\varepsilon > 0$*

$$\begin{gathered} A_{f,\varepsilon}(h_1, \ldots, h_m) := \\ \left\{\sigma \in \mathcal{P}\left([a, b]\right) : \ (\exists t \in \mathbb{R}) \ \|f - \xi_t\|_{L^2\left(\mathbb{R}, \frac{1}{m+1} \sum_{j=0}^m \sigma_{h_j}\right)} < \varepsilon\right\} \end{gathered} \tag{3.2}$$

*is open and dense in $\mathcal{P}([a, b])$.*

*Proof.* The set $A_{f,\varepsilon}(h_1, \ldots, h_m)$ is clearly open, so we need to show its denseness in $\mathcal{P}(X)$. Since discrete measures with a finite number of atoms form a dense subset of $\mathcal{P}(X)$ we take $\nu = \sum_{s=1}^N a_s \delta_{y_s}$ with $y_s \in [a, b]$, $a_s > 0$, $s = 1, \ldots, N$ and $\sum_{s=1}^N a_s = 1$ and fix $\delta > 0$. All we need to show is to find a subset $\{x_1, \ldots, x_N\} \subset [a, b]$ such that $|x_s - y_s| < \delta$ for $s = 1, \ldots, N$ and such that the set

$$L := \bigcup_{j=0}^m \{h_j x_1, \ldots, h_j x_N\} \quad \text{is } \mathbb{Q}\text{-independent.}$$

Indeed, in this case by Kronecker's theorem, the set $L$ is a finite Kronecker set, so the measure $\frac{1}{m+1} \sum_{j=0}^m \left(\sum_{s=1}^N a_s \delta_{x_s}\right)_{h_j}$ is Kronecker, whence belongs to $A_{f,\varepsilon}(h_1, \ldots, h_m)$ and it $\delta$-approximates $\nu$. To show that $x_1, \ldots, x_N$ can be selected so that $L$ is $\mathbb{Q}$-independent, consider the algebraic varieties of the form

$$\left\{(z_1, \ldots, z_N) \in X^{\times N} : \ \sum_{j=0}^m \sum_{s=1}^N k_{js} h_j z_s = 0\right\}$$

for some non-zero integer matrix $(k_{js})$. Since

$$\sum_{j=0}^m \sum_{s=1}^N k_{js} h_j z_s = \sum_{s=1}^N \left(\sum_{j=0}^m k_{js} h_j\right) z_s$$

and $\sum_{j=0}^m k_{js} h_j \neq 0$ whenever $(k_{0s}, \ldots, k_{ms}) \neq (0, \ldots, 0)$ (and there are such vectors since the matrix $(k_{js})$ is not zero), each such variety has $N$-dimensional Lebesgue measure zero. Since there are only countably many such varieties involved, we may discard the union $S$ of them from $[a, b]^{\times N}$. Now, each choice of $(x_1, \ldots, x_N)$ from $(y_1 - \delta, y_1 + \delta) \times \ldots \times (y_N - \delta, y_N + \delta) \setminus S$ satisfies our requirements. □

**Lemma 3.2.** *Given $H \subset \mathbb{R}_+^*$ a countable subset which is a $\mathbb{Q}$-independent set, the set of continuous (Kronecker) measures $\sigma \in \mathcal{P}([a,b])$ for which the measure*

$$\sum_{h\in H} a_h\sigma_h \quad \text{is a Kronecker measure (on } \mathbb{R}) \tag{3.3}$$

*for each choice of $a_h \geq 0$, $\sum_{h\in H} a_h = 1$, is a $G_\delta$ and dense subset of $\mathcal{P}([a,b])$.*

*Proof.* Denote by $\mathcal{P}_c([a,b])$ the set of continuous measures which is a $G_\delta$ and dense subset of $\mathcal{P}([a,b])$. Let $H = \{h_0, h_1, h_2, \ldots\}$. For every $m \geq 0$ fix a countable dense family $\left\{g_i^{(m)} : i \geq 1\right\} \subset \mathcal{U}\left(\bigcup_{i=0}^m h_i[a,b]\right)$. Then, by Lemma 3.1, the set

$$\mathcal{P}_c([a,b]) \cap \bigcap_{m=1}^\infty \bigcap_{i=1}^\infty \bigcap_{p=1}^\infty A_{g_i^{(m)}, \frac{1}{p}}(h_1, \ldots, h_m)$$

is $G_\delta$ and dense in $\mathcal{P}([a,b])$ and it remains to show that this is precisely the set of measures satisfying (3.3). Indeed, given $m \geq 1$, the set

$$\mathcal{K}_m(H) := \mathcal{P}_c([a,b]) \cap \bigcap_{i=1}^\infty \bigcap_{p=1}^\infty A_{g_i^{(m)}, \frac{1}{p}}(h_1, \ldots, h_m)$$

is precisely the set of continuous Kronecker measures $\sigma \in \mathcal{P}([a,b])$ such that the measure $\frac{1}{m+1}\sum_{i=0}^m \sigma_{h_i}$ is a Kronecker measure (on the real line). Moreover, each measure absolutely continuous with respect to a Kronecker measure is also a Kronecker measure [15]. Therefore the set $\mathcal{K}_m(H)$ is equal to the set of all Kronecker measures $\sigma \in \mathcal{P}([a,b])$ such that $\sum_{i=0}^m b_i\sigma_{h_i}$ is Kronecker for arbitrary choice of $b_i \geq 0$, $\sum_{i=0}^m b_i = 1$. Finally, for each $m \geq 1$,

$$\frac{1}{\sum_{i=0}^m a_{h_i}} \sum_{i=0}^m a_{h_i}\sigma_{h_i} \ll \frac{1}{m+1}\sum_{i=0}^m \sigma_{h_i},$$

so if for each $m \geq 1$ the measure $\frac{1}{m+1}\sum_{i=0}^m \sigma_{h_i}$ is Kronecker, so is $\sum_{h\in H} a_h\sigma_h$. □

*Remark* 3.3. The idea of the above proofs is taken from a letter that has been sent to us by T.W. Körner. In this letter, T.W. Körner shows that given a transcendental number $h \in \mathbb{R}$, for a "typical" (in the Hausdorff metric) closed subset $K \subset [a,b]$ the set $K \cup hK$ is Kronecker and uncountable. The proofs are the same since finite sets are dense in the Hausdorff metric and if $h$ is transcendental then given distinct $y_1, \ldots, y_N \in [a,b]$ and $\delta > 0$ we can find $q_i \in \mathbb{Q}$ so that for $x_i := h^{2i}q_i$ we have $|x_i - y_i| < \delta$ for $i = 1, \ldots, N$ and clearly the set $\{x_1, \ldots, x_N, hx_1, \ldots, hx_N\}$ is $\mathbb{Q}$-independent. It only remains to notice that uncountable closed subsets are typical in the Hausdorff metric.

Note also that using the proofs of Lemmas 3.1 and 3.2, given $H \subset \mathbb{R}_+^*$ a countable multiplicative subgroup which is additively $\mathbb{Q}$-independent, we obtain that a typical (with respect to the Hausdorff distance) closed subset $K \subset [a,b]$ has the property that for each finite subset $C \subset H$ the set $\bigcup_{h\in C} hK$ is Kronecker, so the set $\bigcup_{h\in H} hK$ is a $\mathbb{Q}$-independent $F_\sigma$-set.

We will also need the following "compact $\mathbb{Q}$-independent set" version of Lemma 3.2.

**Lemma 3.4.** *Assume that $K \subset \mathbb{R}$ is a compact uncountable set. Assume that $H \subset \mathbb{R}_+^*$ is a countable set which is additively $\mathbb{Q}$-independent. Assume moreover* *that the set $\bigcup_{h\in H} hK$ is $\mathbb{Q}$-independent. Then the set of continuous (Kronecker) measures $\sigma$ concentrated on $K$ for which the measure*

$$\sum_{h\in H} a_h\sigma_h \quad \text{is a Kronecker measure} \tag{3.4}$$

*for each choice of $a_h \geq 0$, $\sum_{h\in H} a_h = 1$, is a $G_\delta$ and dense subset of $\mathcal{P}(K)$.*

*Proof.* This follows from the proofs of Lemmas 3.1 and 3.2, where in addition the proof of Lemma 3.1 is simplified; indeed, for any choice of $\{y_1,\dots,y_N\} \subset K$ the set $\bigcup_{j=0}^m \{h_jy_1,\dots,h_jy_N\}$ is $\mathbb{Q}$-independent by assumption (so we may take $x_i = y_i$ for $i = 1,\dots,N$). □

*Remark* 3.5. For any non-trivial compact interval $[a,b] \subset \mathbb{R}$ denote by $\mathcal{P}_c^{[a,b]}(\mathbb{R})$ the set of measures $\nu \in \mathcal{P}_c(\mathbb{R})$ such that $\nu([a,b]) > 0$. Since the map $\mathcal{P}_c(\mathbb{R}) \ni \nu \mapsto \nu([a,b]) \in \mathbb{R}$ is continuous, the set $\mathcal{P}_c^{[a,b]}(\mathbb{R})$ is open and dense in $\mathcal{P}_c(\mathbb{R})$. Let us consider the map $\Delta = \Delta^{[a,b]} : \mathcal{P}_c^{[a,b]}(\mathbb{R}) \to \mathcal{P}_c([a,b])$ such that $\Delta(\nu)$ is the conditional probability measure $\nu(\,\cdot\,|[a,b])$. This map is continuous and the preimage of any dense subset of $\mathcal{P}_c([a,b])$ is dense in $\mathcal{P}_c^{[a,b]}(\mathbb{R})$. Indeed, let $A \subset \mathcal{P}_c([a,b])$ be dense and take any $\nu \in \mathcal{P}_c^{[a,b]}(\mathbb{R})$. Then there exists a sequence $(\widetilde{\nu}_n)_{n\leq 1}$ in $A$ such that $\widetilde{\nu}_n \to \Delta(\nu)$ weakly. For every $n \geq 1$ define $\nu_n \in \mathcal{P}_c^{[a,b]}(\mathbb{R})$ so that the restriction of $\nu_n$ to $[a,b]$ is $\nu([a,b])\widetilde{\nu}_n$ and the measures $\nu_n$ and $\nu$ coincide on $\mathbb{R}\setminus[a,b]$. Then $\Delta(\nu_n) = \widetilde{\nu}_n \in A$ and $\nu_n \to \nu$ weakly. Consequently, the preimage $\Delta^{-1}A$ of any $G_\delta$ dense subset $A \subset \mathcal{P}_c([a,b])$ is $G_\delta$ dense in $\mathcal{P}_c^{[a,b]}(\mathbb{R})$.

Before we prove a certain disjointness property of Kronecker measures, we will need the following general observation.

**Lemma 3.6.** *Let $(X,\mathcal{B})$ be a standard Borel space and let $\varphi : X \to X$ be a measurable map. Let $\sigma$ be a finite positive continuous Borel measure on $X$ such that the map $\varphi : (X,\sigma) \to (X,\varphi_*\sigma)$ is almost everywhere invertible. Assume that $\sigma(\{x \in X : \varphi(x) = x\}) = 0$ and that the measures $\sigma$ and $\varphi_*\sigma$ are not mutually singular. Then there exists a measurable set $A \in \mathcal{B}$ such that $\sigma(A) > 0$, $\sigma(A\cap\varphi^{-1}A) = 0$ and the measures $\sigma$ and $\varphi_*\sigma$ restricted to $A$ are equivalent.*

*Proof.* By assumption, there exists $Y \in \mathcal{B}$ such that $\sigma(Y) > 0$ and the measures $\sigma$ and $\varphi_*\sigma$ restricted to $Y$ are equivalent. It follows that if $A \in \mathcal{B}$, $A \subset Y$, $\sigma(A) > 0$, then the measures $\sigma$ and $\varphi_*\sigma$ restricted to $A$ are also equivalent.

*Case 1.* Suppose that there exists $B \in \mathcal{B}$ such that $B \subset Y$ and $\sigma(B\setminus\varphi(B)) > 0$. Set $A := B\setminus\varphi(B)$. Then $\sigma(A) > 0$ and $A\cap\varphi^{-1}A = (B\setminus\varphi(B))\cap(\varphi^{-1}B\setminus B) = \emptyset$. Since $A \subset B \subset Y$, our claim follows.

*Case 2.* Suppose that for every $B \in \mathcal{B}$ with $B \subset Y$ we have $\sigma(B\setminus\varphi(B)) = 0$. As $\sigma$ and $\varphi_*\sigma$ restricted to $Y$ are equivalent, it follows that

$$0 = \varphi_*\sigma(B\setminus\varphi(B)) = \sigma(\varphi^{-1}B\setminus B) \quad \text{for every} \quad B \subset Y. \tag{3.5}$$

We now show that there exists $A \in \mathcal{B}$ such that $A \subset Y$, $\sigma(A) > 0$ and $\sigma(A\cap\varphi^{-1}A) = 0$, which gives our assertion. Suppose that, contrary to our claim, for every $A \in \mathcal{B}$ with $A \subset Y$ the condition $\sigma(A) > 0$ implies $\sigma(A\cap\varphi^{-1}A) > 0$. It follows that

$$\sigma(B\setminus\varphi^{-1}B) = 0 \quad \text{for every measurable} \quad B \in \mathcal{B} \text{ with } B \subset Y. \tag{3.6}$$

Indeed, otherwise for some $B$ as above, $A := B \setminus \varphi^{-1}(B) \subset Y$ would be of positive $\sigma$-measure and since

$$(B \setminus \varphi^{-1}B) \cap \varphi^{-1}(B \setminus \varphi^{-1}B) = (B \setminus \varphi^{-1}B) \cap (\varphi^{-1}B \setminus \varphi^{-2}B) = \emptyset,$$

and we would get $\sigma(A \cap \varphi^{-1}A) = 0$, a contradiction.

Now, (3.5) combined with (3.6) gives $\sigma(B \triangle \varphi^{-1}B) = 0$ for every $B \in \mathcal{B}$ with $B \subset Y$. It follows that $\varphi(x) = x$ for $\sigma$-a.e. $x \in Y$, contrary to assumption. □

For any real $s$ let $\theta_s : \mathbb{R} \to \mathbb{R}$, $\theta_s(t) = t + s$. Recall that for every $n \in \mathbb{Z}$ and $z_1, z_2 \in \mathbf{S}^1$ we have

$$|z_1^n - z_2^n| \leq |n||z_1 - z_2|. \tag{3.7}$$

**Lemma 3.7.** *Let $\sigma$ be a continuous Kronecker measure on $\mathbb{R}$. Then for every $s \in \mathbb{Q}^* \setminus \{1\}$ and $r \in \mathbb{R}$ we have $\sigma \perp \sigma_s * \delta_r$.*

*Proof.* Suppose that, contrary to our claim, there exists $s \in \mathbb{Q}^* \setminus \{1\}$ and $r \in \mathbb{R}$ such that $\sigma \not\perp \sigma_s * \delta_r$. Let $\varphi := \theta_r \circ R_s$. Then $\varphi : \mathbb{R} \to \mathbb{R}$ is an invertible map with one fixed point and $\sigma_s * \delta_r = \varphi_*\sigma$. By Lemma 3.6, there exists a Borel set $A_0 \subset \mathbb{R}$ such that $\sigma(A_0) > 0$, $\sigma(A_0 \cap \varphi^{-1}A_0) = 0$ and the measures $\sigma$, $\varphi_*\sigma$ restricted to $A_0$ are equivalent. Thus $\sigma(\varphi^{-1}A_0) > 0$. Let $A_1, A_2 \subset A_0$ be disjoint Borel subsets such that $\sigma(\varphi^{-1}A_1) > 0$ and $\sigma(\varphi^{-1}A_2) > 0$.

Let $s = q/p$ with $p$ and $q$ relatively prime integer numbers. Choose $z_0 \in \mathbf{S}^1$ such that $z_0^q \neq 1$. Let us consider the measurable map $f : \mathbb{R} \to \mathbf{S}^1$ such that $f(x) = z_0$ if $x \in \varphi^{-1}A_2$ and $f(x) = 1$ otherwise. Since $\sigma$ is a Kronecker measure, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ of real numbers such that $\xi_{t_n} \to f$ in $L^2(\mathbb{R}, \sigma)$. Thus $\xi_{t_n} \circ \varphi^{-1} \to f \circ \varphi^{-1}$ in $L^2(\mathbb{R}, \varphi_*\sigma)$. Since

$$\begin{aligned}
g_n^0(x) &:= \chi_{A_0}(x)\,|\exp(2\pi i t_n x) - 1| \leq |\xi_{t_n}(x) - f(x)| \\
g_n^1(x) &:= \chi_{A_1}(x)\left|\exp(2\pi i t_n s^{-1}(x - r)) - 1\right| \leq \left|\xi_{t_n}(\varphi^{-1}x) - f(\varphi^{-1}x)\right| \\
g_n^2(x) &:= \chi_{A_2}(x)\left|\exp(2\pi i t_n s^{-1}(x - r)) - z_0\right| \leq \left|\xi_{t_n}(\varphi^{-1}x) - f(\varphi^{-1}x)\right|,
\end{aligned}$$

it follows that $(g_n^0)$ tends to zero in measure $\sigma$ and the sequences $(g_n^1)$, $(g_n^2)$ tend to zero in measure $\varphi_*\sigma$. As $\sigma \equiv \varphi_*\sigma$ on $A_0$ and $A_1, A_2 \subset A_0$, the sequences $(g_n^1)$, $(g_n^2)$ tend to zero in measure $\sigma$, as well. Fix

$$0 < \varepsilon < \frac{|z_0^q - 1|}{2(|p| + |q|)}. \tag{3.8}$$

Then there exist measurable sets $A'_k \subset A_k$, $k = 0, 1, 2$ and $n \in \mathbb{N}$ such that for $k = 0, 1, 2$

$$\sigma(A_k \setminus A'_k) < \frac{1}{4}\min(\sigma(A_1), \sigma(A_2)) \text{ and } g_n^k(x) < \varepsilon \text{ for all } x \in A'_k.$$

Therefore for $k = 1, 2$ we have

$$\sigma(A_k \setminus A'_0) \leq \sigma(A_0 \setminus A'_0) < \frac{1}{4}\sigma(A_k) \text{ and } \sigma(A_k \setminus A'_k) < \frac{1}{4}\sigma(A_k),$$

so $\sigma(A'_0 \cap A'_k) > \sigma(A_k)/2 > 0$. Choose two real numbers $x_1 \in A'_0 \cap A'_1$ and $x_2 \in A'_0 \cap A'_2$. Then

$$|\exp(2\pi i t_n x_1) - 1| = g_n^0(x_1) < \varepsilon, \quad \left|\exp(2\pi i t_n \frac{p}{q}(x_1 - r)) - 1\right| = g_n^1(x_1) < \varepsilon,$$
$$|\exp(2\pi i t_n x_2) - 1| = g_n^0(x_2) < \varepsilon, \quad \left|\exp(2\pi i t_n \frac{p}{q}(x_2 - r)) - z_0\right| = g_n^2(x_2) < \varepsilon.$$

In view of (3.7),

$$|\exp(2\pi i t_n p\, x_1) - 1| < |p|\varepsilon, \quad |\exp(2\pi i t_n p(x_1 - r)) - 1| < |q|\varepsilon,$$
$$|\exp(2\pi i t_n p\, x_2) - 1| < |p|\varepsilon, \quad |\exp(2\pi i t_n p(x_2 - r)) - z_0^q| < |q|\varepsilon.$$

Hence

$$|\exp(2\pi i t_n p r) - 1| < (|p| + |q|)\varepsilon, \quad |\exp(2\pi i t_n p r) z_0^q - 1| < (|p| + |q|)\varepsilon,$$

so

$$|1 - z_0^q| < 2(|p| + |q|)\varepsilon,$$

contrary to (3.8). □

Let us now consider the space $\mathcal{P}(\mathbb{R})$ of all Borel probability measures on $\mathbb{R}$ endowed with the weak topology.

By $\mathrm{supp}(\sigma)$ we always mean the topological support of the measure $\sigma$. Let us recall that

$$\text{if } \sigma \in \mathcal{P}(\mathbb{R}) \text{ has } \mathrm{supp}(\sigma) = \mathbb{R} \\ \text{then the set } \{\nu \in \mathcal{P}(\mathbb{R})\colon \nu \ll \sigma\} \text{ is dense in } \mathcal{P}(\mathbb{R}). \tag{3.9}$$

Denote by $\mathcal{P}_c(\mathbb{R})$ the set of all continuous members of $\mathcal{P}(\mathbb{R})$ (this is a $G_\delta$ and dense subset of $\mathcal{P}(\mathbb{R})$).

The proof of the lemma below is a slight modification of the proof of Lemma 3.1 from [4].

**Lemma 3.8.** *The set*

$$\mathcal{S} = \{\sigma \in \mathcal{P}_c(\mathbb{R})\colon\ \sigma_s \perp \sigma * \delta_t \quad \textit{for each } 1 \neq s \in \mathbb{R}^*,\ t \in \mathbb{R}\}$$

*is $G_\delta$ and dense in $\mathcal{P}(\mathbb{R})$.*

*Proof.* Denote by $\mathcal{I}$ the family of open subset of $\mathbb{R}$ which are finite unions of open intervals. Recall that for two measures $\sigma, \nu \in \mathcal{P}(\mathbb{R})$

$$\sigma \perp \nu \iff \forall_{n\in\mathbb{N}}\ \exists_{\mathcal{O}\in\mathcal{I}}\ \sigma(\mathcal{O}) < 1/n \quad \text{and} \quad \nu(\mathcal{O}) > 1 - 1/n. \tag{3.10}$$

For any compact rectangle $I \times J \subset (\mathbb{R}^* \setminus \{1\}) \times \mathbb{R}$ denote by $\mathcal{V}(I \times J)$ the set of all finite covers of $I \times J$ by compact rectangles contained in $(\mathbb{R}^* \setminus \{1\}) \times \mathbb{R}$. Notice that for each open subset $\mathcal{O} \in \mathcal{I}$ the map

$$\mathcal{P}_c(\mathbb{R}) \times \mathbb{R}^* \times \mathbb{R} \ni (\sigma, s, r) \mapsto \sigma_s * \delta_r(\mathcal{O}) \in \mathbb{R} \tag{3.11}$$

is continuous. Therefore, given a compact rectangle $F \subset (\mathbb{R}^* \setminus \{1\}) \times \mathbb{R}$ and an open subset $\mathcal{O} \in \mathcal{I}$ the map

$$f_{F,\mathcal{O}}\colon \mathcal{P}_c(\mathbb{R}) \ni \sigma \mapsto \left(\sigma(\mathcal{O}), \max_{(s,r)\in F} \sigma_s * \delta_r(\mathcal{O})\right) \in \mathbb{R}^2$$

is continuous. Let

$$\widetilde{\mathcal{S}} = \bigcap_{I\not\ni 1}\bigcap_{J}\bigcap_{n\in\mathbb{N}}\bigcup_{\kappa\in\mathcal{V}(I\times J)}\bigcap_{F\in\kappa}\bigcup_{\mathcal{O}\in\mathcal{I}} f^{-1}_{F,\mathcal{O}}\left((1-1/n,\infty)\times(-\infty,1/n)\right),$$

where $I$ and $J$ run over closed intervals with rational endpoints. Then $\widetilde{\mathcal{S}}$ is a $G_\delta$ set.

We claim that $\widetilde{\mathcal{S}} = \mathcal{S}$. Indeed, let $\sigma\in\mathcal{S}$. Let $I\not\ni 1$ and $J\subset\mathbb{R}$ be compact intervals and $n\in\mathbb{N}$. By assumption and (3.10), for every $(s_0,r_0)\in I\times J$ there exists an open set $\mathcal{O}_{s_0,r_0}\in\mathcal{I}$ such that

$$\sigma(\mathcal{O}_{s_0,r_0}) > 1-1/n \quad \text{and} \quad \sigma_{s_0}*\delta_{r_0}(\mathcal{O}_{s_0,r_0}) < 1/n.$$

Since the map (3.11) is continuous, there exist open rectangles $U'_{s_0,r_0}\subset U_{s_0,r_0}\subset\mathbb{R}^2$ such that $(s_0,r_0)\in U'_{s_0,r_0}$ and a compact rectangle $F_{s_0,r_0}\subset(\mathbb{R}^*\setminus\{1\})\times\mathbb{R}$ satisfying $U'_{s_0,r_0}\subset F_{s_0,r_0}\subset U_{s_0,r_0}$ such that

$$\sigma_s*\delta_r(\mathcal{O}_{s_0,r_0}) < 1/n \quad \text{for all } (s,r)\in U_{s_0,r_0}.$$

Since $I\times J$ is compact and $\{U'_{s,r}:(s,r)\in I\times J\}$ is its open cover, there exists a finite cover $\kappa := \{F_{s_1,r_1},\ldots,F_{s_k,r_k}\}$ of $I\times J$. It follows that

$$f_{F_{s_j,r_j},\mathcal{O}_{s_j,r_j}}(\sigma)\in(1-1/n,\infty)\times(-\infty,1/n) \quad \text{for all} \quad j=1,\ldots,k,$$

thus $\sigma\in\widetilde{\mathcal{S}}$.

Suppose that $\sigma\in\widetilde{\mathcal{S}}$ and fix $s_0\in\mathbb{R}^*\setminus\{1\}$, $r_0\in\mathbb{R}$ and $n\in\mathbb{N}$. Next choose $I\not\ni 1$ and $J\subset\mathbb{R}$ compact intervals such that $(s_0,r_0)\in I\times J$. By assumption, there exists a finite cover $\kappa\in\mathcal{V}(I\times J)$ such that for every $F\in\kappa$ there exists $\mathcal{O}_F\in\mathcal{I}$ with

$$\sigma(\mathcal{O}_F) > 1-1/n \quad \text{and} \quad \sigma_s*\delta_r(\mathcal{O}_F) < 1/n \quad \text{for all } (s,r)\in F.$$

Choosing $F\in\kappa$ for which $(s_0,r_0)\in F$ and applying (3.10) we have that $\sigma$ and $\sigma_{s_0}*\delta_{r_0}$ are orthogonal, so $\sigma\in\mathcal{S}$.

It remains to show that $\mathcal{S}$ is dense. To this end we use the proof of Proposition 3.4 in [4]. Namely, in this proposition there is a construction of a weakly mixing flow $\mathcal{T}$ such that for a certain sequence of real numbers $u_k\to\infty$ we have: for each $l\in\mathbb{N}$

$$T_{-du_k}\to 10^{-l} \quad \text{for } d=1-10^{-l} \text{ and} \tag{3.12}$$

$$T_{-cu_k}\to 0 \quad \text{uniformly in } c\in[1,10^l] \tag{3.13}$$

(the convergence takes place in the weak operator topology). It follows that

$$\sigma_{\mathcal{T}_d}\perp\sigma_{\mathcal{T}_c}*\delta_t \tag{3.14}$$

for all $t\in\mathbb{R}$; indeed, (3.12) and (3.13) mean respectively

$$\xi_{u_k}\to 10^{-l} \quad \text{weakly in } L^2(\mathbb{R},\sigma_{\mathcal{T}_d}),$$

and

$$\xi_{u_k}\to 0 \quad \text{weakly in } L^2(\mathbb{R},\sigma_{\mathcal{T}_c}).$$

It is easy to see that the latter condition implies

$$\xi_{u_k}\to 0 \quad \text{weakly in } L^2(\mathbb{R},\sigma_{\mathcal{T}_c}*\delta_t)$$

for each $t\in\mathbb{R}$, and the mutual singularity (3.14) follows.

Now, in view of (3.14), $\sigma_{\mathcal{T}} \perp \sigma_{\mathcal{T}_{c/d}} * \delta_{t/d}$, and since in (3.13) $c$ can be replaced by $-c$, it follows that $\sigma_{\mathcal{T}} \in \mathcal{S}$. It is also clear that $\mathcal{S}$ is closed under taking absolutely continuous measures. Since $\operatorname{supp} \sigma_{\mathcal{T}} = \mathbb{R}$ [8], the result follows from (3.9). □

Recall also the following basic observation.

**Lemma 3.9.** *Let $\overline{s} = (s_j)_{j \geq 1}$ be a sequence of positive numbers and let $\overline{g} = (g_j)_{j \geq 1}$ be a sequence of uniformly bounded continuous functions. Then the set*

$$\mathcal{W}_{\overline{s},\overline{g}} = \left\{ \nu \in \mathcal{P}(\mathbb{R}) \colon (\exists\ t_n \to \infty)\,(\forall\ j \geq 1) \quad \xi_{s_j t_n} \to g_j \text{ weakly in } L^2(\mathbb{R}, \nu) \right\}$$

*is $G_\delta$ in $\mathcal{P}(\mathbb{R})$.*

*Proof.* Let $(f_m)_{m \geq 1}$ be a sequence of continuous functions on $\mathbb{R}$ uniformly bounded by 1, which is linearly dense in $L^2(\mathbb{R}, \nu)$ for every $\nu \in \mathcal{P}(\mathbb{R})$. Set

$$\mathcal{R}(n, \varepsilon) = \left\{ \mu \in \mathcal{P}(\mathbb{R}) : \sum_{m,j \geq 1} \frac{1}{2^{m+j}} \left| \int_{\mathbb{R}} \left( e^{2\pi i s_j n x} - g_j(x) \right) f_m(x)\ d\mu(x) \right| < \varepsilon \right\}.$$

The set $\mathcal{R}(n, \varepsilon)$ is open. To complete the proof it suffices to notice that

$$\mathcal{W}_{\overline{s},\overline{g}} = \bigcap_{\mathbb{Q} \ni \varepsilon > 0} \bigcap_{m \geq 1} \bigcup_{n \geq m} \mathcal{R}(n, \varepsilon).$$

□

**Lemma 3.10.** *Let $H \subset \mathbb{R}^*_+$ be a countable multiplicative subgroup. Then for a typical $\nu \in \mathcal{P}(\mathbb{R})$ the measure $\eta := \sum_{h \in H} a_h \nu_h$ (with $a_h > 0$ and $\sum_{h \in H} a_h = 1$) yields a Gaussian flow $\mathcal{T}^{\eta|_{\mathbb{R}_+}}$ with simple spectrum.*

*Proof.* Set $G = -H \cup H$ and let $H = \{s_i : i \geq 0\}$ ($s_0 = 1$). In [4], Danilenko and Ryzhikov constructed a rank-1 flow $\mathcal{T}$ preserving a $\sigma$-finite measure $\mu$ (the flow acts on $(X, \mathcal{B}, \mu)$) such that if $\sigma = \sigma_{\mathcal{T}}$ denotes its maximal spectral type on $L^2(X, \mathcal{B}, \mu)$ then the Gaussian flow

$$\mathcal{T}^{\left(\sum_{i \geq 1} \frac{1}{2^i} \sigma_{s_i}\right)|_{\mathbb{R}_+}} \text{ has simple spectrum.} \tag{3.15}$$

To prove this, they used the following properties of $\mathcal{T}$:

a) $T_{\sqrt{2}s} \in WCP(T_s)$ [9] for each $s \in H$,

b) $\frac{1}{q} I + \frac{q-1}{q} T_s \in WCP(T_s)$ for each $s \in H$ and $q \in \mathbb{N}$,

c) for each finite sequence $s_1 < s_2 < \cdots < s_k$ of elements of $H$ and each $1 \leq l_0 \leq k$ there exists $t_j \to \infty$ such that
   (i) $T_{t_j s_j} \to \frac{1}{2k} I$ if $1 \leq l \leq k, l \neq l_0$,
   (ii) $T_{t_j s_{l_0}} \to \frac{1}{2k} T_{s_{l_0}}$.

Notice that the conditions a), b) and c) can be expressed as follows in terms of weak convergence of continuous and bounded functions in $L^2(\mathbb{R}, \sigma)$:

a') for each $s \in H$ there exists a sequence $n_k \to \infty$ such that

$$\xi_{s n_k} \to \xi_{\sqrt{2}s},$$

[8] This fact is well known for $\mathbb{Z}$-actions, e.g. [20], Chapter 3, and can be easily rewritten using special representation of flows. See also the proof of Theorem A in [22].

[9] An operator $Q$ belongs to the weak closure of powers WCP($R$) if for an increasing sequence $(m_j)$ of integers, $R^{m_j} \to P$ in the weak operator topology.

b') for each $s \in H$ and $q \in \mathbb{N}$ there exists a sequence $n_k \to \infty$ such that

$$\xi_{sn_k} \to \frac{1}{q} + \frac{q-1}{q}\xi_s,$$

c') for each finite sequence $s_1 < s_2 < \cdots < s_k$ of elements of $H$ and each $1 \le l_0 \le k$ there exists $t_j \to \infty$ such that
 (i) $\xi_{t_j s_j} \to \frac{1}{2k}$ if $1 \le l \le k, l \neq l_0$,
 (ii) $\xi_{t_j s_{l_0}} \to \frac{1}{2k}\xi_{s_{l_0}}$.

The arguments used in the proof of Theorem 4.4 in [4] show that for each continuous probability measure $\sigma$ on $\mathbb{R}$ conditions a'), b') and c') imply the simplicity of spectrum of the flow $\mathcal{T}^{(\sum_{k\ge1}\frac{1}{2^k}\sigma_{s_k})|_{\mathbb{R}_+}}$. Moreover, by Lemma 3.9, the set of measures $\nu \in \mathcal{P}(\mathbb{R})$ satisfying these conditions is $G_\delta$. We will show now that it is also dense in $\mathcal{P}(\mathbb{R})$. Notice that conditions a'), b') and c') hold also in $L^2(\mathbb{R},\nu)$ for any $\nu \ll \sigma$. Since $\sigma_{\mathcal{T}}$ is the maximal spectral type of a rank-1 infinite measure-preserving flow $\mathcal{T}$, the Gelfand spectrum of the corresponding Koopman representation is equal to $\mathbb{R}$. It follows that the topological support of $\sigma_{\mathcal{T}}$ is full and therefore the result follows from (3.9). □

## 4. Proofs of theorems

*Proof of Theorem 1.3.* (based on Lemmas 3.2 and 3.8.) Using these two lemmas, for a "typical" (continuous, Kronecker) measure $\sigma \in \mathcal{P}([a,b])$ we have (with $a_h > 0$, and $\sum_{h\in H} a_h = 1$)

$$-H \cup H \subset I(\mathcal{T}^\eta),$$

where $\eta := \sum_{h\in H} a_h\sigma_h$ is a Kronecker measure and moreover

$$\sigma_s \perp \sigma * \delta_t \tag{4.1}$$

for each non-zero real $s \neq 1$ and arbitrary $t \in \mathbb{R}$. All we need to show is that when $s \notin -H \cup H$ then $\eta_s \not\equiv \eta$. However if $s \notin H$ then even more is true: $\eta \perp \eta_s * \delta_t$ for arbitrary $t \in \mathbb{R}$ and $s \notin \{0,1\}$. It follows that

$$\widetilde{\eta}_s \perp \widetilde{\eta} * \delta_t$$

for each $s \notin -H\cup H$ and $t \in \mathbb{R}$. In view of Theorem 2.3, it follows that $\mathcal{T}^\eta$ is disjoint from $\mathcal{T}^{\eta_s}$ (isomorphic to $\mathcal{T}^\eta_s$) for $s \notin -H \cup H$. In particular, $-H \cup H = I(\mathcal{T}^\eta)$ and the result follows. □

*Proof of Theorem 1.3.* (based on Lemma 3.4.) Given $H \subset \mathbb{R}^*_+$ a multiplicative subgroup which is an additively $\mathbb{Q}$-independent set, in [6], there is a construction of a perfect compact set $K$ such that $\widehat{K} := \bigcup_{h\in H} hK$ is independent and for $\widetilde{K} := -\widehat{K}\cup\widehat{K}$ the following holds: $(r\widetilde{K}+t)\cap\widetilde{K}$ is countable whenever $|r| \notin H$ and $t \in \mathbb{R}$ is arbitrary. Using Lemma 3.4 find a (continuous, Kronecker) measure $\sigma \in \mathcal{P}(K)$ such that $\eta := \sum_{h\in H} a_h\sigma_h$ is a Kronecker measure. Then $\eta$ is concentrated on $\widehat{K}$. All we need to show is that if $|r| \notin H$, then the symmetrization of $\eta_r$ is not equivalent to the symmetrization of $\eta$. This is however clear, since the symmetrization of $\eta_r$ is a continuous measure concentrated on $r\widetilde{K}$. As in the previous proof we deduce that for $s \notin -H \cup H$ we obtain disjointness of the corresponding flows. □

*Proof of Theorem 1.4.* First notice that directly from Lemma 3.7, it follows that whenever $\sigma$ is a Kronecker measure then for each $r_1, r_2 \in \mathbb{Q}^*$, $r_1 \neq r_2$, we have

$$\sigma_{r_1} \perp \sigma_{r_2} * \delta_t \quad \text{for each} \quad t \in \mathbb{R}.$$

It follows that $\widetilde{\sigma}_{r_1} \perp \widetilde{\sigma}_{r_2} * \delta_t$ for all $t \in \mathbb{R}$, so by Theorem 2.3, the Gaussian-Kronecker flows $\mathcal{T}^{\sigma_{r_1}}$ and $\mathcal{T}^{\sigma_{r_2}}$ are disjoint. In view of (2.1), it follows that $T_1^{\sigma_{r_1}} \perp T_1^{\sigma_{r_2}}$, thus $T^\sigma_{r_1} \perp T^\sigma_{r_2}$.

Now suppose that $T = T_\sigma : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is a Gaussian-Kronecker automorphism, i.e. $\sigma = \sigma_0 + \overline{\sigma}_0$ for a continuous Kronecker measure $\sigma_0 \in \mathcal{P}(\mathbb{T})$.[10] Denote by $\sigma'$ the image of $\sigma_0$ via the map $\mathbb{T} \ni z \mapsto \operatorname{Arg}(z)/2\pi \in [0, 1)$. Then $\sigma'$ is a continuous Kronecker measure on $\mathbb{R}$ such that $(\xi_1)_* \widetilde{\sigma}' = \sigma$ and $\widetilde{\sigma}' * \delta_m \perp \widetilde{\sigma}'$ for all $m \in \mathbb{N}$. Denote by $\mathcal{H}$ the Gaussian space of the flow $\mathcal{T}^{\sigma'}$. Then the Koopman operator of $T_1^{\sigma'}$ has simple spectrum on $\mathcal{H}$ and its spectral type is $(\xi_1)_* \widetilde{\sigma}' = \sigma$, see Appendix in [16]. Since the spectral type of $\zeta_1$ (with respect to $T_1^{\sigma'}$) is $(\xi_1)_* \widetilde{\sigma}' = \sigma$, it follows that $\zeta_1 \circ (T_1^{\sigma'})^n$, $n \in \mathbb{Z}$, span the space $\mathcal{H}$. Thus $T_1^{\sigma'}$ is isomorphic to $T_\sigma$. By the first assertion of the theorem, it follows that $T_\sigma^n$ is disjoint from $T_\sigma^m$ for any pair of distinct natural numbers.

In order to prove the second part of the theorem note that if $s$ is irrational then the set $\{1, s\}$ is $\mathbb{Q}$-independent, so by Lemma 3.2 we can find a (continuous, Kronecker) measure $\sigma \in \mathcal{P}([a, b])$ such that $\eta := \frac{1}{2}(\sigma + \sigma_s)$ is a Kronecker measure. Since $\sigma_s \ll \eta$ and $\sigma_s \ll \eta_s$ the Gaussian-Kronecker flows $\mathcal{T}^\eta$ and $\mathcal{T}^{\eta_s}$ have a common non-trivial (Gaussian) factor. Its time one map is a common non-trivial factor of $T_1^\eta$ and $T_1^{\eta_s}$ and it remains to notice that the Gaussian automorphism $T_1^{\eta_s}$ is isomorphic to $T_s^\eta$. □

*Proof of Theorem 1.5.* Let $H = G \cap \mathbb{R}_+^*$ and let $(a_h)_{h \in H}$ be positive numbers such that $\sum_{h \in H} a_h = 1$. By Lemmas 3.8, 3.10 and Lemma 3.2 (applied to $H = \{1\}$) combined with Remark 3.5, there exists $\nu' \in \mathcal{P}_c(\mathbb{R})$ such that

(i) $\nu'_s \perp \nu' * \delta_t$ for all $s \in \mathbb{R}^* \setminus \{1\}$ and $t \in \mathbb{R}$;

(ii) the Gaussian flow $\mathcal{T}^{(\sum_{h \in H} a_h \nu'_s)|_{\mathbb{R}_+}}$ has simple spectrum

(iii) $\nu := \Delta(\nu') \in \mathcal{P}_c([a, b])$ is a Kronecker measure

(in fact, for a "typical" $\nu' \in \mathcal{P}_c(\mathbb{R})$ the properties (i)-(iii) hold). Since the conditions (i) and (ii) hold also for any measure absolutely continuous with respect to $\nu$, the Kronecker measure $\nu$ satisfies (i) and (ii) as well. Therefore, setting $\sigma := \sum_{h \in H} a_h \nu_s$, by (ii), the Gaussian flow $\mathcal{T}^\sigma$ has simple spectrum. The same argument as in the proof of Theorem 1.3 shows that (i) together with (ii) imply $I(\mathcal{T}^\sigma) = -H \cup H$ and $\mathcal{T}^{\nu_s} \perp \mathcal{T}^{\nu_r}$ whenever $|r| \neq |s|$. Each Kronecker measure $\nu_h$, $h \in H$ is an FS measure so, by Proposition 2.4, it follows that $\sigma = \sum_{h \in H} a_h \nu_s$ is an FS measure [11], which completes the proof. □

*Proof of Theorem 1.6.* The first part follows from Lemma 3.8 along the same lines as the first proof of Theorem 1.3 (for $H = \{1\}$).

In view of Corollary 2 in [16], a typical flow $\mathcal{T}$ has the SC property,[12] which is equivalent to the fact that $\mathcal{T}^{\sigma_\mathcal{T}}$ has simple spectrum. In particular, it implies that $\mathcal{T}^{\sigma_\mathcal{T}}$ is GAG.

In order to prove that $\sigma_\mathcal{T} \perp (\sigma_\mathcal{T})_s * \delta_r$, $s \in \mathbb{R}^* \setminus \{1\}$, $r \in \mathbb{R}$ for a typical flow $\mathcal{T}$ we follow the proof of Theorem 3.2 from [4] (using Lemma 3.8 and the existence of a

[10] $\mathbb{T}$ stands for $\{z \in \mathbb{C} \colon |z| = 1\}$.

[11] We use here the elementary fact that the $L^2$-limit of a sequence of Gausian variables remains Gaussian.

[12] The SC property means that if we set $\sigma = \sigma_\mathcal{T}$ then for each $n \geq 2$ the conditional measures of the disintegration of $\sigma^{\otimes n}$ over $\sigma^{*n}$ via the map $\mathbb{R}^n \ni (x_1, \ldots, x_n) \mapsto x_1 + \cdots + x_n \in \mathbb{R}$ are purely atomic with $n!$ atoms.

flow satisfying (3.14)). Since $\mathcal{T}^{\sigma\mathcal{T}}$ is GAG for a typical flow $\mathcal{T}$, by Proposition 2.3, it follows that $\mathcal{T}^{(\sigma\mathcal{T})_s}$ and $\mathcal{T}^{(\sigma\mathcal{T})_r}$ are disjoint wherever $|r| \neq |s|$. $\square$

Question. Is there a Kronecker measure $\sigma \in \mathcal{P}(\mathbb{R}_+)$ such that $I(\mathcal{T}^\sigma)$ is uncountable?
This question is to be compared with Ryzhikov's question whether there is a weakly mixing, non-mixing flows with uncountable group of self-similarities, see [3], Problem (1).

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